NuHAG Numerical Harmonic Analysis Group

Applications of Gabor Analysis to Engineering Problems

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ISIAM CONFERENCE

OVERVIEW over this lecture **40 MINUTES**

- Gabor analysis is an essential part of time-frequency analysis, initiated by the seminal paper of Denis Gabor in 1946 ([10]);
- Foundations of time-frequency analysis and the STFT;
- Basic facts about (regular) Gabor Analysis: sampling the STFT or non-orthogonal expansions;
- Foundations of Gabor Analysis ([7, 8, 11]);
- Applications to signal processing, audio compression (MP3) and mobile communication.
- Indications about 2D Gabor analysis;

My personal background: from abstract to applied HA

- Trained as an **abstract harmonic analyst** I have turned into an application oriented harmonic analyst, covering the wide range from "abstract" HA to "computational HA";
- I have been working on function spaces on LC (Wiener amalgams), distribution theory (modulation spaces), coorbit theory, Banach frame theory, irregular sampling and above all in Gabor Analysis;
- As the group leader of **NuHAG** (= the Numerical Harmonic Analysis Group) in Vienna (AUSTRIA) I keep pushing towards an integrated view of Harmonic Analysis, which I call

CONCEPTUAL HARMONIC ANALYSIS

- Goal: connecting theory with **applications** through efficient algorithms based on theoretical foundations.

Overall goal of this presentation and other talks in India

- Gabor analysis is a core part of time-frequency analysis
- it has many applications (implicit and explicit) in signal processing, mobile communication and other areas;
- one can understand the important principles of Gabor analysis starting from linear algebra (see [6]) combined with a few facts from group theory ([5]).
- although it is important to include (aside from e.g. $\mathbf{L}^p$-spaces) also generalized functions (distributions) in the exposition, basic facts from Banach space theory resp. functional analysis suffice, e.g. the new concept of Banach Gelfand Triples;
- we propagate a view-point of allowing to integrate abstract aspects (unifying, simplifying) with concrete (computational, approximation theoretic) methods resp. algorithms: Conceptual Harmonic Analysis

What are our goals when doing Fourier analysis?

- find relevant "harmonic components" in [almost] periodic functions;
- define the Fourier transform (first $\mathbf{L}^1(\mathbb{R}^d)$, then $\mathbf{L}^2(\mathbb{R}^d)$, etc.);
- describe time-invariant linear systems as convolution operators;
- describe such system as Fourier multipliers (via transfer functions);
- deal with (slowly) time-variant channels (communication) ;
- describe changing frequency content ("musical transcription");
- define operators acting on the spectrogram (e.g. for denoising) or perhaps pseudo-differential operators using the Wigner distribution;

CLAIM: What is really needed!

So what, according to our view, students in (application oriented) mathematics or (electrical) engineering (or geophysics, astronomy, etc.) should really learn is this:

- refresh their linear algebra knowledge (ONB, **SVD**!!!, linear independence, generating set of vectors), and matrix representations of linear mappings between finite dimensional vector spaces;
- Banach spaces, bd. operators, dual spaces norm and w^*-convergence;
- Hilbert spaces, orthonormal bases and unitary operators;
- frames and Riesz basis (resp. matrices of maximal rank);

FOURIER Transform, Inversion, Poisson

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{1}$$

The inverse Fourier transform then has the form

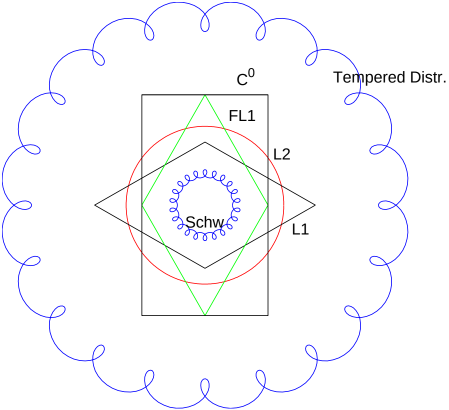
$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{2}$$

POISSON's formula:

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n). \tag{3}$$

Function Spaces for the Classical Fourier Transform 1

$\mathbf{L}^1(\mathbb{R}^d), \mathbf{L}^2(\mathbb{R}^d)$, Schwartz space $\mathcal{S}(\mathbb{R}^d)$, tempered distributions $\mathcal{S}'(\mathbb{R}^d)$, ...



Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

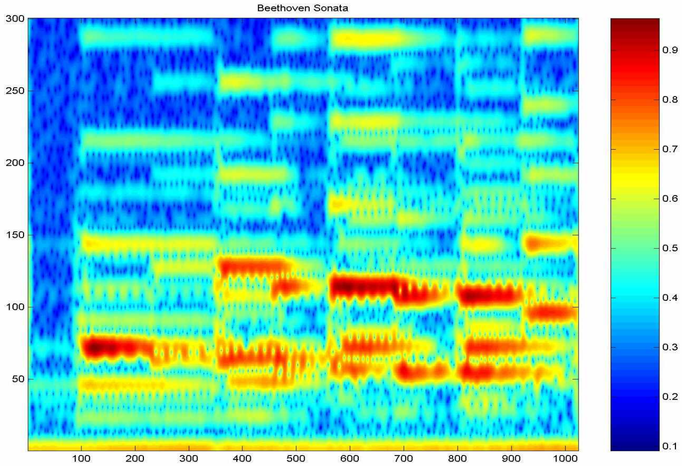
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

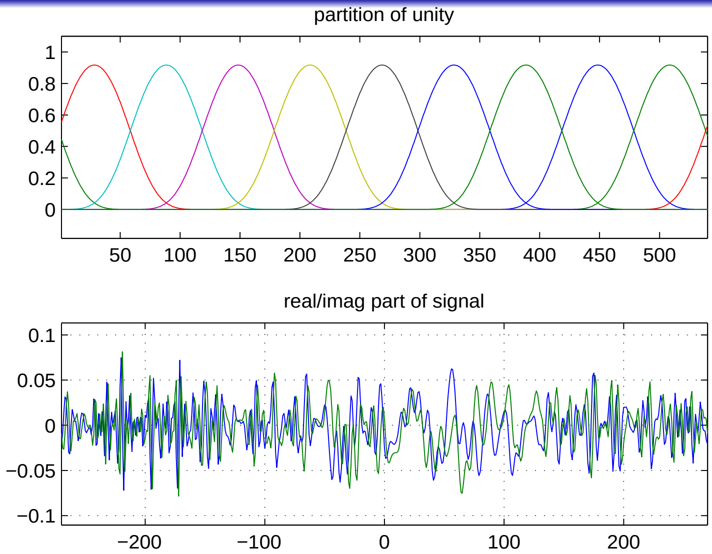
The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t,\omega) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \quad \lambda = (t,\omega);$$

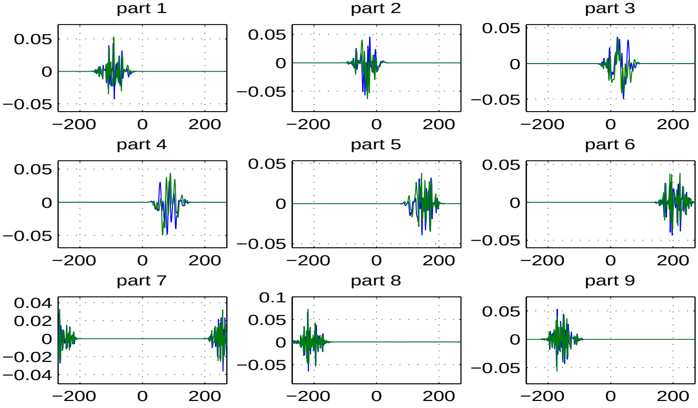
A Typical Musical STFT



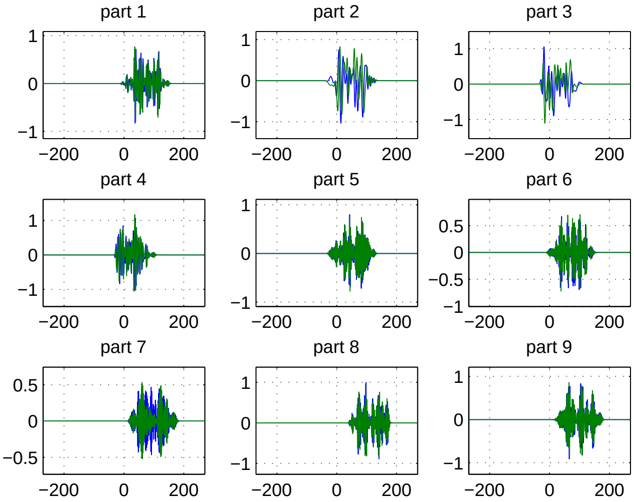
The idea of a “localized Fourier Spectrum” 1



The localized Fourier transform (spectrogram) 2



Spectral decomposition: variable bandwidth



Spectrogram == Local Fourier Transform

Traditionally we extract the frequency information of a signal f by means of the Fourier transform $\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t)e^{-2\pi it\omega}dt$.
If we know $\hat{f}(\omega)$ for all frequencies ω, then our signal f can be reconstructed by the inversion formula $f(t) = \int_{\widehat{\mathbb{R}^d}} \hat{f}(\omega)e^{2\pi it\omega}d\omega$ (valid pointwise or in the quadratic mean).
However, in many situations it is of relevance to know, *when and how long* each frequency appears in the signal f, e.g., for a pianist playing a piece of music.
Mathematically this leads to the study of different functions $S(f)(t,\omega)$ of the signal f, which describe the time-frequency content of f near"time" t. In the following we mention the most prominent time-frequency representations.

Wigner versus (linear) STFT

In 1932 Wigner introduced the first time-frequency representation of a function $f \in L^2(\mathbb{R}^d)$ by

$$W(f)(x,\omega) = \int_{\mathbb{R}^d} f(x+\frac{t}{2})\overline{f(x-\frac{t}{2})}e^{-2\pi i\omega t}dt, \qquad (4)$$

the so called *Wigner distribution* of f.

Nowadays, the linear *short-time Fourier transform* (STFT) has become the standard tool for (linear) time-frequency analysis. It is used as a measure of the time-frequency content of a signal f (energy distribution), but it also establishes a connection to the representation theory of the Heisenberg group.

STFT = Short-time Fourier Transform: Basic Properties

The STFT (or windowed FT) provides information about local (smoothness) properties of the signal f by multiplication with some *window function* g and a subsequent Fourier transform. Typically g is Schwartz function (from $\mathcal{S}(\mathbb{R}^d)$) concentrated around the origin, such as the Gaussian, and $f \mapsto V_g f$ is linear:

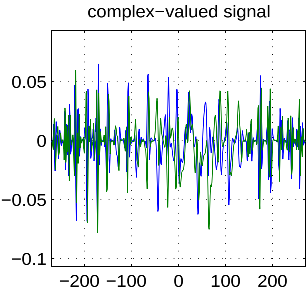
$$V_g f(x,\omega) = \int_{\mathbb{R}^d} f(t)\overline{g(t-x)} e^{-2\pi i t\omega} dt, \quad \text{for} \quad (x,\omega) \in \mathbb{R}^{2d}, \quad (5)$$

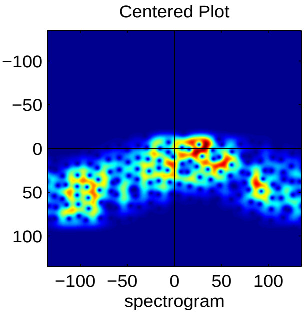
In 1927 Weyl pointed out that the translation and modulation operator satisfy the following commutation relation

$$T_x M_\omega = e^{-2\pi i x\omega} M_\omega T_x, \quad (x,\omega) \in \mathbb{R}^{2d}. \quad (6)$$

$\{T_x : x \in \mathbb{R}^d\}$ and $\{M_\omega : \omega \in \mathbb{R}^d\}$ are Abelian groups.

STFT of a function of "variable band-width"





Time-Frequency Shifts and the Heisenberg group

We have for $\pi(x,\omega) := M_\omega T_x$ the following composition law:

$$\pi(x,\omega)\pi(y,\eta) = e^{-2\pi i x\cdot\eta}\pi(x+y,\omega+\eta), \tag{7}$$

for $(x,\omega),(y,\eta)$ in the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. i.e. the mapping $(x,\omega) \mapsto \pi(x,\omega)$ defines (only) a projective representation of the time-frequency plane (viewed as an Abelian group) $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$. By adding a toral component, i.e. $\tau \in \mathbb{C}$ with $|\tau| = 1$ one can augment the phase space $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ to the so-called *Heisenberg group* $\mathbb{H}^d := \mathbb{R}^d \times \widehat{\mathbb{R}^d} \times \mathbb{T}$ and the mapping $(x,\omega,\tau) \mapsto \tau M_\omega T_x$ defines a (true) unitary representation of the Heisenberg group [9], the so-called *Schrödinger representation*. From this point of view the definition of $V_g f$ can be interpreted as representation coefficients:

$$V_g f(x,\omega) = \langle f, M_\omega T_x g\rangle, \qquad f,g \in \mathbf{L}^2(\mathbb{R}^d).$$

Fourier and Covariance Property of STFT

By Parseval's theorem and an application of the commutation relations (6) we derive the following relation

$$V_g f(x,\omega) = e^{-2\pi i x\omega} V_{\hat{g}} \hat{f}(\omega, -x), \tag{8}$$

which is sometimes called the*fundamental identity of time-frequency analysis* [11]. The equation (8) expresses the fact that the STFT is a joint time-frequency representation and that the Fourier transform amounts to a rotation of the time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}^d}$ by an angle of $\frac{\pi}{2}$ whenever the window g is Fourier invariant. Another important consequence of the definition of STFT (5) and the commutation relations (6) is the *covariance property* of the STFT:

$$V_g(T_u M_\eta f)(x,\omega) = e^{-2\pi i u\omega} V_g f(x-u, \omega-\eta). \tag{9}$$

MOYAL's FORMULA

As for the Fourier transform there is also a Parseval's equation for the STFT which is referred to as *Moyal's formula*.

Lemma

(Moyal's Formula) Let $f_1, f_2, g_1, g_2 \in \mathbf{L}^2(\mathbb{R}^d)$ then $V_{g_1} f_1$ and $V_{g_2} f_2$ are in $\mathbf{L}^2(\mathbb{R}^{2d})$ and the following identity holds:

$$\langle V_{g_1} f_1, V_{g_2} f_2 \rangle_{\mathbf{L}^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle \overline{\langle g_1, g_2 \rangle} . \qquad (10)$$

Moyal's formula implies that orthogonality of windows g_1, g_2 resp. of signals f_1, f_2 implies orthogonality of their STFT's. Most importantly we observe that one has for normalized $g \in \mathbf{L}^2(\mathbb{R}^d)$ (i.e. with $\|g\|_2 = 1$) for all $f \in \mathbf{L}^2(\mathbb{R}^d)$

$$\| V_g f \|_{\mathbf{L}^2(\mathbb{R}^{2d})} = \| f \|_{\mathbf{L}^2(\mathbb{R}^d)},$$

i.e., the STFT is an isometry from $\mathbf{L}^2(\mathbb{R}^d)$ to $\mathbf{L}^2(\mathbb{R}^{2d})$.

STFT: continuous inversion formula

The most important consequence of Moyal's formula is an inversion formula for the STFT. Assume that the analysis window $g \in \mathbf{L}^2(\mathbb{R}^d)$ and the synthesis window $\gamma \in \mathbf{L}^2(\mathbb{R}^d)$ satisfy $\langle g, \gamma \rangle \neq 0$. Then for $f \in \mathbf{L}^2(\mathbb{R}^d)$

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^{2d}} \langle f, \pi(x, \omega)\gamma \rangle \, \pi(x, \omega) g \, dx d\gamma. \qquad (11)$$

We observe that in contrast to the Fourier inversion the building blocks of the STFT inversion formula are just time-frequency shifts of a square- integrable function. Therefore also the Riemannian sums corresponding to this inversion integral are functions in $\mathbf{L}^2(\mathbb{R}^d)$ and are even norm convergent in $\mathbf{L}^2(\mathbb{R}^d)$ for nice windows (from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$).

Gabor Analysis: sampling the STFT

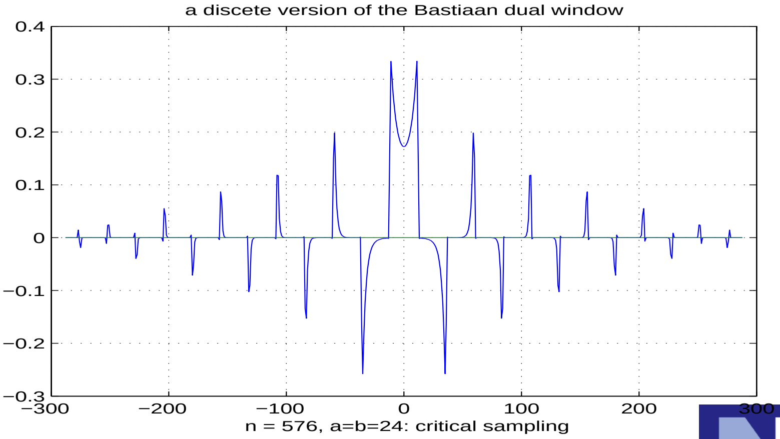
Clearly there is a lot of redundancy built into $V_g f$ (and the range of V_g is a rather small subspace of $\mathbf{L}^2(\mathbb{R}^{2d})$). The contribution of D. Gabor to the field is two-fold:
FIRST of all he argued that one should use in fact the Gauss function, because it is both Fourier invariant *and* minimizes the Heisenberg's uncertainty relation.
SECONDLY he suggested – in a modern reading of his work – to discretize the continuous reconstruction formulat (11) and come up with a Riemannian-type representation of the form

$$f = \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g \qquad (12)$$

respectively (more in the spirit of M. Bastiaans [1]):

$$f = C_\Lambda \cdot \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\gamma \rangle \pi(\lambda) g = C_\Lambda \cdot \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda)\gamma \qquad (13)$$

Bastiaan's dual window (critical case)

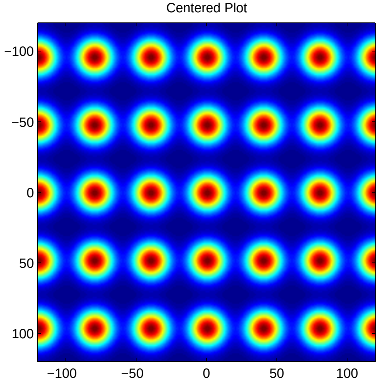


Gabor Analysis from the Linear Algebra view-point

We are looking at the problem of Gabor expansion in the setting of finite sequences of length n. In fact, we consider a sequence rather as a periodic (infinite) sequence, or alternatively as a function on the unit-roots of order n, and use as translation (or shift) operators shifts by multiples of some a (which should divide n, so we have n/a shifted copies of some template (Gabor Atom) g).
The same can be done "on the Fourier transform side", i.e. we choose some divisor b of n and apply the n/b frequency shift operators. Iin fact multiplications of the signal with samples of the corresponding pure frequency

$$e^{2\pi ikj/n} = cos(2\pi kj/n) + i * sin(2\pi kj/n).$$

A Gabor family



It has $n/a \cdot n/b$ different sequences ($red := n/ab$).

Gabor Frames: stable expansions

Definition

A family $(g_i)_{i \in I}$ in a Hilbert space **H** is called a **frame** if there exist constants $A, B > 0$ such that for all $f \in \mathbf{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, g_i \rangle|^2 \leq B\|f\|^2 \quad (14)$$

For a general frame one defines the **frame operator** S

Definition

$$Sf = \sum_{i \in I} \langle f, g_i \rangle g_i$$

and uses $f = S^{-1}Sf = SS^{-1}f$ to expand every $f \in \mathbf{H}$

$$f = \sum_{i \in I} \langle f, g_i \rangle \widetilde{g_i} = \sum_{i \in I} \langle f, \widetilde{g_i} \rangle g_i$$

Commutation Relation

It is now one of the crucial facts of Gabor analysis, that one does NOT have to compute the dual frame, which consists of the elements

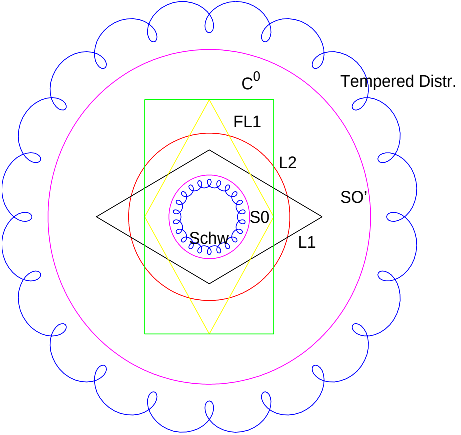
$$\tilde{g}_i = S^{-1} g_i$$

individually, but because $g_i = \pi(\lambda) g$ for some $\lambda \in Lambda$, with Λ being a lattice (discrete subgroup) of TF-space, and the same *Gabor atom* g on also has $\tilde{g}_i = \pi(\lambda)\tilde{g}$ for all $\lambda \in \Lambda$. One has

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

Moreover, for "good" atoms/windows g one can also guarantee good properties of $V_g f$ and from that the same good properties for $\tilde{g}$. This is the case, for example, for $g = g_0$, the Gauss function, and lattices of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$.

A Banach space of Test functions



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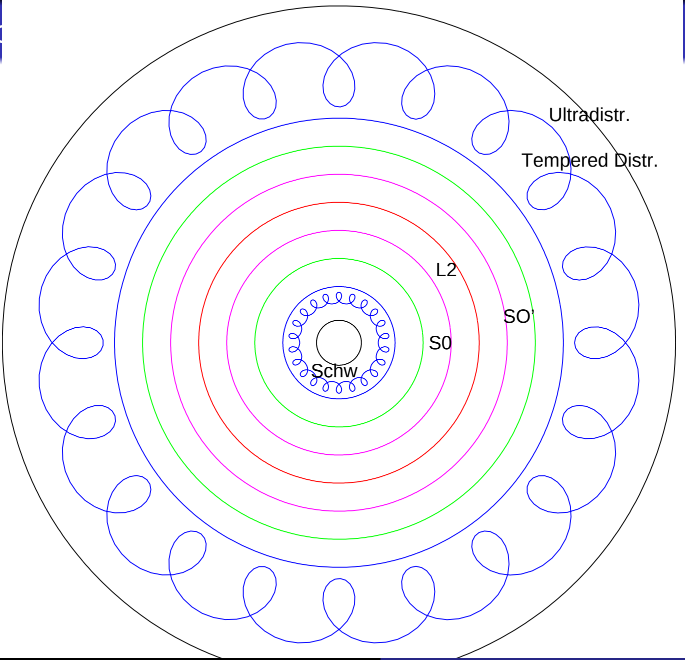
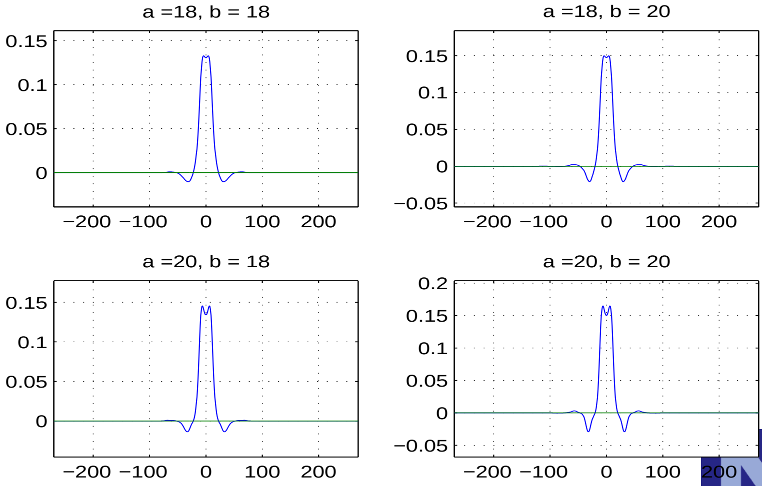


Dual Windows and their Fourier Transforms

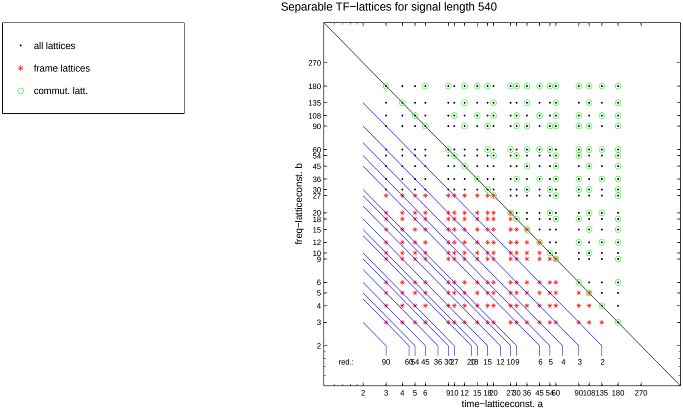


Main Questions in Gabor Analysis

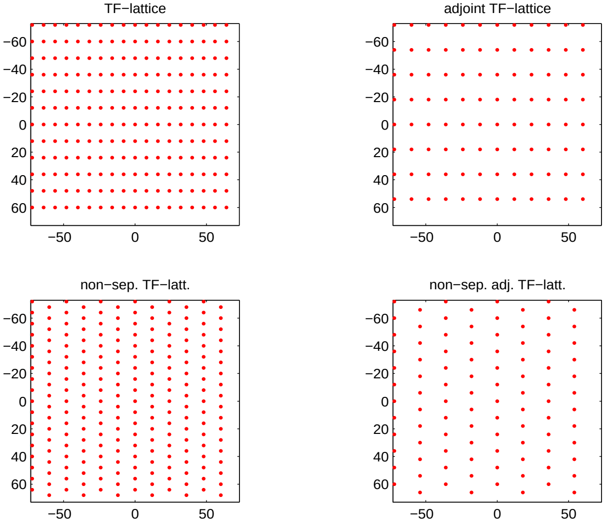
- For which pairs (Λ, g) of a lattice and some "Gabor atom" does one have a Gabor frame (i.e. stable reconstruction);
- What are the possible (non-unique!) pairs of functions (g, γ) (the relationship is symmetric!) which allow expansion of any $f \in \mathbf{L}^2(\mathbb{R}^d)$;
- what are the properties (in terms of growth and decay) of the Gabor coefficients in relation to decay and smoothness of the (distributions or) functions expanded (e.g. characterization of modulation spaces, $\mathcal{S}(\mathbb{R}^d)$, etc.).

There is by now a comprehensive theory, and a number of papers on the foundations of Gabor analysis.

The landscape of possible lattices



The landscape of possible lattices



The Wexler Raz Relationship

One the basic facts of Gabor analysis, which in turn is mathematically based on a version of Poisson's summation formula for the *symplectic Fourier transform*, is the so-called Wexler-Raz biorthogality relationship, which has been formalized in a more precise and general manner to the so-called Ron-Shen principle.
It shows that a problem concerning the question whether one has a Gabor frame of the form $(M_{\beta n} T_{k\alpha} g)$ for suitable divisors α, β of the signal length N can be translated into the equivalent (!) question whether the family $(M_{l/\alpha} T_{r/\beta} g)$ is a Riesz basis.
Clearly the first is possible only if the redundancy factor $N/(\alpha\beta)$ is larger than 1, or equivalently the second family contains at most N signals (cf. [**?**, 4]).

The Ron-Shen duality principle

Theorem

Let Λ be a lattice in $\mathbb{R}^{2d}$ with adjoint lattice Λ°. Then, for g, h satisfying eq:propA, the following holds.

1. *(Fundamental Identity of Gabor Analysis)*

$$\sum_{\lambda\in\Lambda}\langle f,\pi(\lambda)\gamma\rangle\langle\pi(\lambda)g,h\rangle = |\Lambda| \sum_{\lambda^\circ\in\Lambda^\circ}\langle g,\pi(\lambda^\circ)\gamma\rangle\langle\pi(\lambda^\circ)f,h\rangle \tag{15}$$

   for all $f, h \in \mathbf{L}^2(\mathbb{R}^d)$, where both sides converge absolutely.

2. *(Wexler-Raz Identity)*

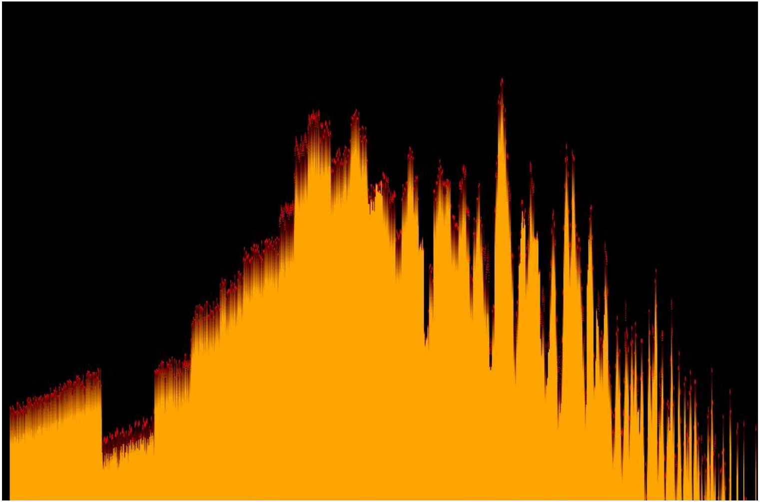
$$\langle\pi(\lambda^\circ)g,\pi(\mu^\circ)\gamma\rangle = \delta_{\lambda^\circ,\mu^\circ}, \quad \textit{for} \quad \lambda^\circ,\mu^\circ \in \Lambda^\circ. \tag{16}$$

The Ron-Shen duality principle

Theorem

[Ron-Shen duality principle] (see ([4])):
The family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame and only if $(\pi(\lambda^\circ)g)_{\lambda^\circ\in\Lambda^\circ}$ is a Gaborian Riesz basis (for its closed linear span), where Λ° is the ajoint lattice to the given lattice Λ. The condition number of the Gabor frame and of the corresponding Riesz basis are equal. Moreover, the canonical dual frame $(\pi(\lambda)\widetilde{g})_{\lambda\in\Lambda}$ is generated (up to some fixed constant) by the same element $\tilde{g}$ as the biorthogonal system, which is therefore of the form $(C_\Lambda\pi(\lambda^\circ)\widetilde{g})_{\lambda^\circ\in\Lambda^\circ}$.

Another (Standard) representation of a Musical STFT



Audio Signal Processing: Frequency Bands 1

Audio Signal Processing and Gabor Multipliers

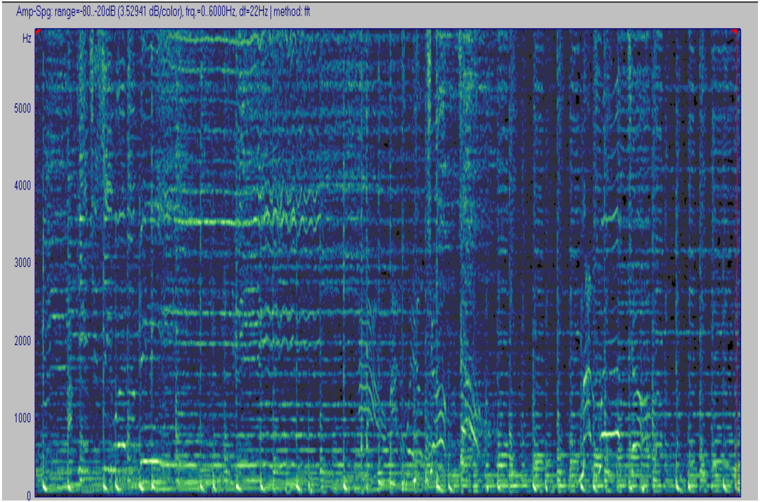
Nowadays everybody knows how to use a HIFI-device, change (over time) the loudness in the difference frequency bands, to give the music a certain "feeling", such as Classic, Rock, Jazz, and so on. Putting a high amplitude (amplification) to the high frequencies makes the sound fell a bit sharp, while the lack of high frequencies may make the sound a bit dull, like an old telephone line.
STFT thus gives an intuitive way, comparable to image processing methods, to manipulate sounds using visual tools.
There is however a warning in place: while image analysis tools are 2D translation invariant, and are therefore convolution operators (egde detection, etc.) there are complex phase factors coming up, and one has to handle twisted convolution operators.

Audio Signal Processing: Frequency Bands 2

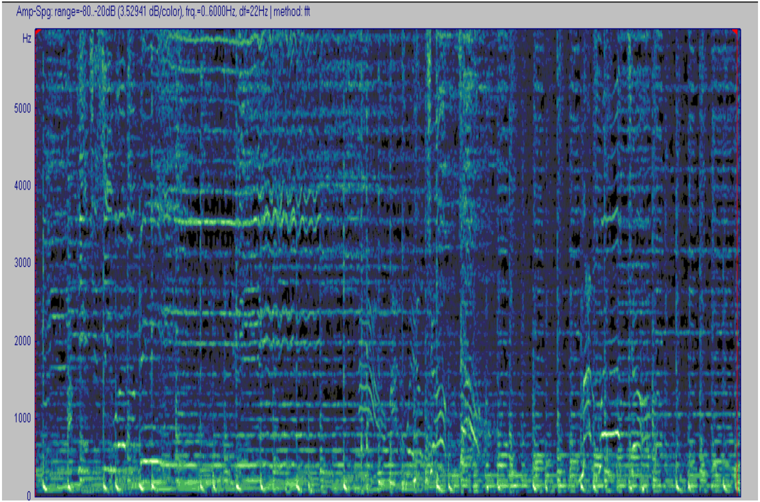
MP3: based on MASKING in the STFT domain

At typical MP3-player: Lossy compression based on time-frequency considerations: Closeby pure frequencies at lower amplitudes are not relevant and consequently need not be stored.

Picture provided by Peter Balazs (ARI, OEAW):



Picture provided by Peter Balazs (ARI, OEAW): MASKED



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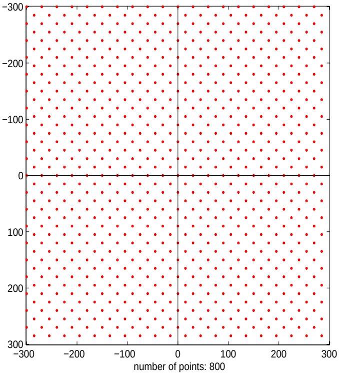
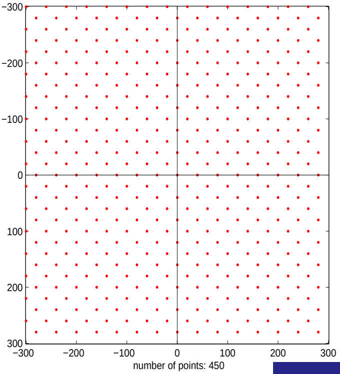
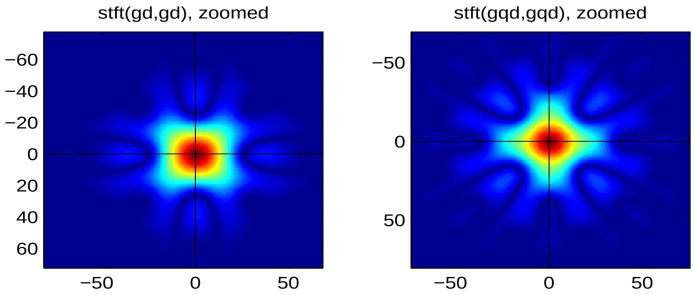
- P. Balazs, *Regular and Irregular Gabor Multipliers with Application to Psychoacoustic Masking*, PhD Thesis, Universität Wien (2005)
- P. Balazs, *Basic Definition and Properties of Bessel Multipliers*, Journal of Mathematical Analysis and Applications (2007)
- P. Balazs, B. Laback, G. Eckel and W. A. Deutsch, *Introducing Time-Frequency Sparsity by Removing Perceptually Irrelevant Components Using a Simple Model of Simultaneous Masking*, submitted

Non-standard Lattices: Quincux





Non-standard Lattices: dual windows



Mobile Communication

Mobile Communication 1

The goal of mobile communication is of course the transmission of (digital resp. digitized) information. We skip aspects of A/D - D/A conversion, (source and channel) coding, helping to make the communication reliable in the presence of disturbances, but concentrate on the mathematical core problem, which can be modeled (roughly) as follows:

The base station (provider) sends some information which is received by the user's mobile phone, who tries to convert it back to the original data (voice, image) which he can hear/see ("as if there was nothing in between the sender and the receiver").

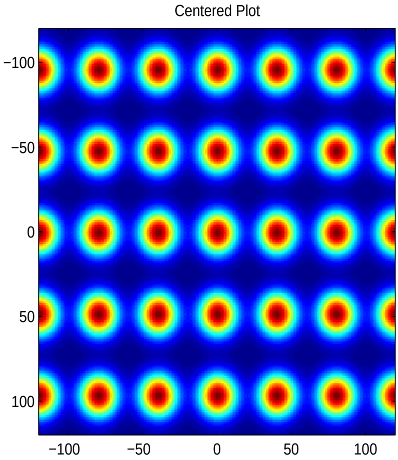
There are various practically relevant (and realistic) assumptions made: The channel T (based on the propagation of electromagnetic waves) can be understood approximately as a linear operator. The "ideal channel" is the identity.

Mobile Communication 2

Information transmitted are the coefficients with respect to some "linear independent" set, practically a Gaborian Riesz basis, e.g. a set of Gaussians centered at the lattice points of $\Lambda_1 = a_1\mathbb{Z} \times b_1\mathbb{Z}$, with $a_1 \cdot b_1 > 1$. Then it is possible to recover (at least in the ideal case) the coefficients (c_λ) from the sum $Tf = \sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) Tg$. In practice we have multipath-propagation (some smearing, expressed as convolution) and a (mild) Doppler-effect (due to moving receivers, in a car or train, for example). Overall these are limited effects (at least on Gabor atoms), described by the concept of underspread operators[1].

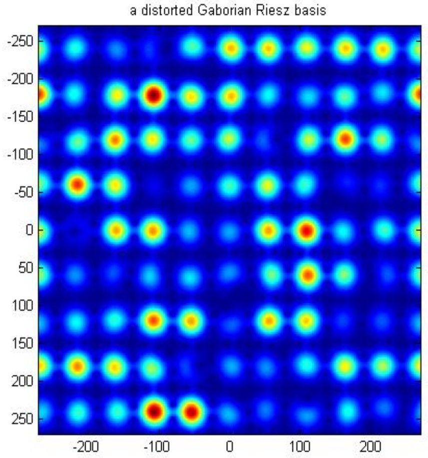
[1] This is like communicating by playing a piano, instead of a simple Morse alphabet: the receiver has to do transcription on the music signal that s/he receives in order to recover the amplitudes and time locations of the played musical score, which carries the relevant information.

Mobile Communication: Gabor Riesz Bases



Mobile Communication: Distorted Gabor Bases

GERALD



Mobile Communication 5

The advantage of this setting is that the family of well-shaped TF-pulses are - for all these channels joint approximate eigenvectors, i.e. up to some (complex-valued) diminishing factor one can hope that $T(g_\lambda)$ equals g_λ.
Simple demodulation schemes (such as OFDM, the orthogonal frequency division multiplexing scheme) are based on estimates of those diminishing factors, trying to invert their action.
Unfortunately this is not as simple as inverting some Fourier multiplier, because the set of building blocks is non-orthogonal.

The Spreading Representation 1

For the *mathematical modeling* of the transmission between sending station and receiver again time-frequency motivated description are useful.
One can describe such mappings as linear operators which "do not contain to strong TF-frequency components" in the same was as bandlimited functions do not contain high frequencies beyond the Nyquist rate.
And again linear algebra is a good starting point to understand the spreading representation. We look at the set of *all linear* mappings form $\mathbb{C}^n$ to $\mathbb{C}^n$. These linear mappings are exactly the matrix multiplications by (complex) $n \times n$ matrices. Obviously these matrices for an n^2 dimensional (complex) vector space.

The Spreading Representation 2

Since we have altogether n^2 TF-shift operators (each of the n cyclic shift operators can be followed by any of the n modulation operators) one may hope that this family forms a basis for the $n \times n$ matrices.

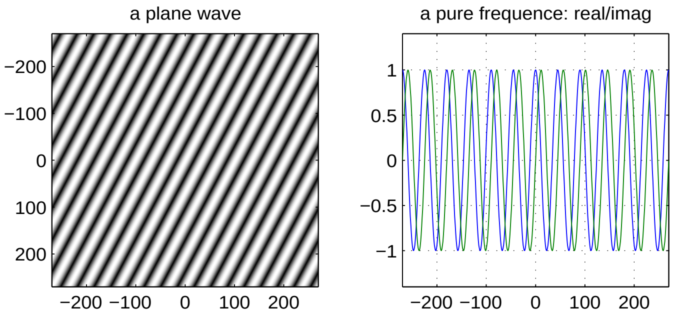
In fact, much more is true: they form an *orthnormal basis* with respect to the Frobenius scalar product (corresponding to the Frobenius norm, also known as Hilbert Schmidt):

$$\langle A, b \rangle_{HS} := \sum_{k,l} a_{k,l} \bar{b}_{k,l}.$$

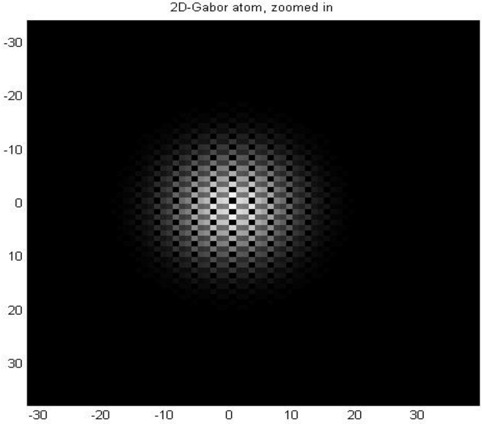
The corresponding orthonormal expansion of general matrices is called the spreading function, and it has a Hilbert Schmidt analogue in the Euclidean setting.

Consequences: If an operator commutes with a given lattice of TF-shift operators $\pi(\lambda), \lambda \in \Lambda$, then the support of its spreading function is in Λ°, the adjoint group.

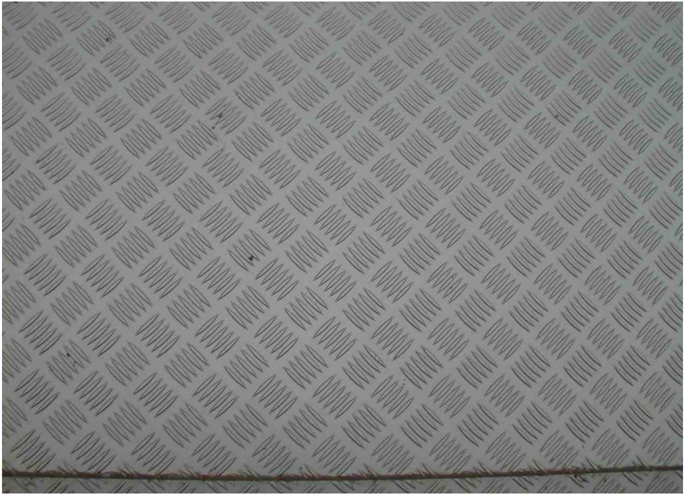
2D-Gabor Transform: Plane Waves



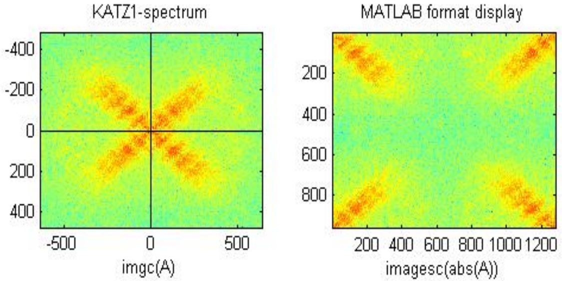
2D-Gabor Transform: Plane Waves



2D-Gabor Transform: an example



2D-Gabor Transform: the 2D spectrum



2D-Gabor Transform

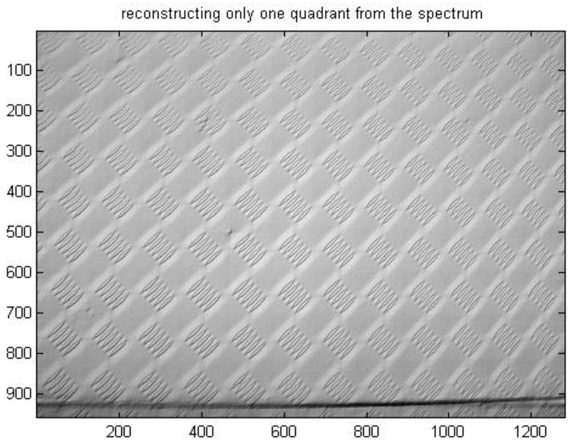
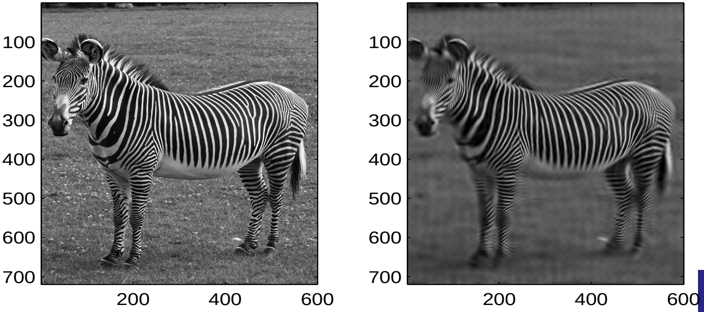


Image Compression



Various ongoing projects at NuHAG

For details see www.nuhag.eu >> research ;
Together with electrical engineers:

- MOHAWI: Modern Harmonic Analysis and Wireless Communication (slowly varying channels);
- SISE: Signal and Information Processing in Science and Engineering, a national priority program (2008-2011);
- SPORTS: Sparse Signals and Operators: Theory, Methods, and Applications (related to compressed sensing questions)

But also other projects related to medical signal processing (defibirillation), acoustic signal processing (frame multipliers), scattered data approximation, and those of more theoretical nature (e.g. symplectic geometry, metaplectic group).

Modulation spaces and Pseudo-Differential Operators

Another outcome of the research in time-frequency analysis is a more precise understanding of what "good windows" are (they should have at least an integrable STFT, i.e. they should be long to the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ (Feichtinger's algebra), which certainly captures in a precise way what engineers (but also classical Fourier analysts working on summability theory would have used) would accept (certainly not the SINC function or the Dirichlet kernel, but very well something like a Fejer kernel or some De-la-Vallee Poussin kernel (see [2]).

Time-frequency analysis resp. Gabor analysis also opens up a new approach to the realization and inversion of "slowly time-variant" operators, which are "diagonally concentrated" with respect to good Gabor frames (or Wilson bases) just in the same way as Calderon-Zygmund operators have a good (and sparse) matrix representation with respect to orthonormal wavelet bases.

Comparison with Wavelet Theory I

There are both similarities and differences compared to wavelet theory. Let us start with the similarities : In both cases ...

- ... there is some continuous transform, the continuous wavelet transform resp. the sliding window or Short-Time Fourier transform, allowing a representation of general elements from $\mathbf{L}^2(\mathbb{R}^d)$ as continuous and bounded function on an appropriate domain (upper half-space and phase-space resp.);
- ... for *admissible* wavelets this mapping is isometric from $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ into a larger (reproducing kernel) Hilbert space;
- ... there is some locally compact group in the background, such that the transform equals $V_g(f)(x) := \langle f, \pi(x)g \rangle$.

The unifying theory behind this is the coorbit theory by F/Gröchenig, established around 1987 ([3]).

Comparison with Wavelet Theory II

On the other hand there are substantial differences :

- The groups $\mathcal{G}$ acting via some (integrable and irreducible [projective]) representation, i.e. a mapping $x \mapsto \pi(x)$, a unitary operator on $\mathcal{H}$ for each $x \in \mathcal{G}$ are quite different. The "$ax + b$" group of affine transformation on the one hand, and the Heisenberg group $\mathbb{H}^d$ on the other hand;
- The first group is non-unimodular, accounting e.g. for a non-trivial admissibility condition, while $\mathbb{H}^d$ is nilpotent;
- For us the existence of "nice" wavelets g allowing to have good orthonormal systems of the form $(\pi(x_n)g)_{n\geq 1}$, in contrast to the Balian-Low phenomenon;
- While the wavelet transform is well suited for a description of Besov-Triebel-Lizorkin spaces the same is true for the less well-known class of modulation spaces for the STFT.

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