A Banach Gelfand Triple Framework for Regularization and Approximation

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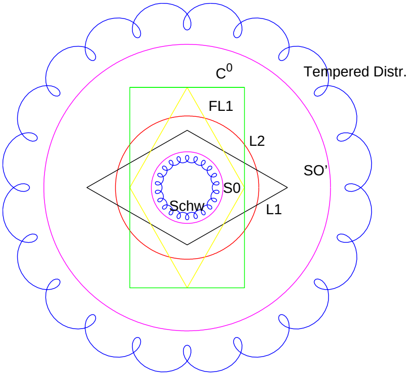
MOTIVATION

Classical View

- Fourier transforms are defined using (Lebesgue) integrals, but then inversion requires summability ideas;
- Fourier transforms are nice if one takes the Hilbert space point of view (unitary mappings);
- Fourier transforms are nice if one take the setting of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$;
- SUGGESTED here: $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$! With the ADVANTAGE over the Schwartz case: Just a Banach space (which is in fact isomorphic to ℓ^1), and furthermore naturally defined on any LCA group! (unike Schwartz-Bruhat)



Note that Fourier invariant spaces are depicted by using symbols which are symmetric under a rotation by 90 degrees.

Banach Gelfand Triples

Definition

A triple $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a **Banach Gelfand triple**.

The prototype of a Gelfand triple is $(\ell^1, \ell^2, \ell^\infty)$.
In most cases we could even start with an abstract Banach space B, as long as there is some (natural) embedding from B into its dual space. Then (by complex interpolation) a suitable Hilbert space can be found.
Similar constructions are known in the literature, under the name of Gelfand triples (e.g. $(\mathcal{S}, \mathbf{L}^2, \mathcal{S}')$), or *rigged Hilbert spaces* (with B e.g. some Sobolev space, i.e. three Hilbert spaces).

Banach Gelfand Triples Morphisms

Definition

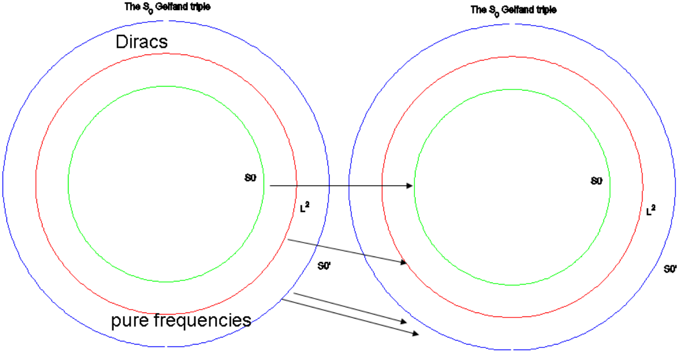
A linear mapping defines an (unitary) isomorphism between two Banach Gelfand Triples $(\mathbf{B}^1, \mathcal{H}_1, \mathbf{B}'_1)$ and $(\mathbf{B}^2, \mathcal{H}_2, \mathbf{B}'_2)$ are Gelfand triples if one has

1. A is an isomorphism between $\mathbf{B}^1$ and $\mathbf{B}^2$.
2. A is a [unitary operator resp.] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}'_1$ and $\mathbf{B}'_2$.

GTR-Examples

For the standard exampe $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ w^*-convergence corresponds to ordinary coordinate-wise convergence in ℓ^∞. This situation can be transferred to "abstract Hilbert spaces" $\mathcal{H}$. Given any orthonormal basis (h_n) one can relate $\boldsymbol{\ell}^1$ to the set $\mathbf{B}$ of all elements $f \in \mathcal{H}$ which have an *absolutely convergent* series expansions with respect to this basis. Clearly the ℓ^1-norm of the coefficients turns $\mathbf{B}$ into a Banach space. Moreover, the coefficient mapping which is unitary between $\mathcal{H}$ and ℓ^1 can also be extended in a most natural way to an isometry between $\mathbf{B}'$ and ℓ^∞. As a well known example of such a situation we may remind of the classical case of $\mathcal{H} = \mathbf{L}^2(\mathbb{T})$, with the usual Fourier basis the corresponding spaces are known as Wiener's $\mathbf{A}(\mathbb{T})$. The dual space is then $\mathcal{P}_M$, the space of pseudo-measures $= \mathcal{F}^{-1}[\boldsymbol{\ell}^\infty(\mathbb{Z})]$. Thus the FT extends naturally to a BGTr= Banach-Gefand-Triple-Isomorphism.

Gelfand triple mapping



Banach Gelfand Triples morphisms realized

Very often a Gelfand-Triple homomorphism T can be realized with the help of some kind of "summability methods". In the abstract setting this is a sequence of "regularizing" operators A_n with the following properties:

- each of the operators maps $\mathbf{B}_1'$ into $\mathbf{B}_1$;
- they are a uniformly bounded family of Gelfand-triple homomorphisms on $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$;
- $A_n f \to f$ in $\mathbf{B}_1$ for any $f \in \mathbf{B}_1$.

It then follows that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in $\mathbf{B}_2'$ (in the w^*-sense) for $f \in \mathcal{H}_1$ resp. $f \in \mathbf{B}_1'$ and thus describes concretely the prolongation to the full Gelfand triple. This continuation is unique due to the w^*-properties assumed for T (and the w^*-density of $\mathbf{B}_1$ in $\mathbf{B}_1'$).

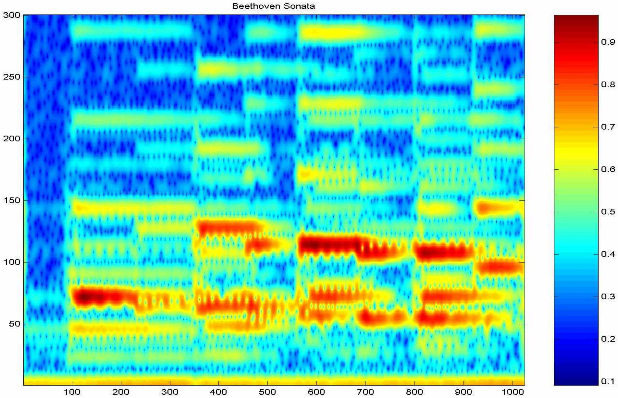
Definition of $S_0(\mathbb{R}^d)$

For the definition of $S_0(\mathbb{R}^d)$ we need the Short-Time Fourier Transform, defined as

$$V_g f(\lambda) = V_g f(t,\omega) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle,$$

where $\lambda = (t,\omega) \in \mathbb{R}^d$, while $T_x f(t) = f(t-x)$ is the time-shift and $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$ is the frequency shift operator. The shift operators' behavior under Fourier transform is as follows

$$(T_x f)\widehat{\ } = M_{-x}\hat{f}, \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}.$$


Beethoven Sonata
300
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200
150
100
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100
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400
500
600
700
800
900
1000
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

definition and basic properties of $\mathbf{S}_0(\mathbb{R}^d)$

A function $f \in \mathbf{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if, for some non-zero g (called the “window”) in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, it holds

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

The space $(S_0(\mathbb{R}^d), \| \cdot \|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

Minimality of $S_0(\mathbb{R}^d)$

Remark: Moreover, one can show that $S_0(\mathbb{R}^d)$ is the *smallest non-trivial Banach spaces with this property*, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda)\pi(\lambda) g d\lambda,$$

which implies

$$\|f\|_{\mathbf{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda) g\|_{\mathbf{B}} \, d\lambda = \|g\|_{\mathbf{B}} \|f\|_{S_0}.$$

This minimality property makes the following simple atomic characterization of $S_o(\mathbb{R}^d)$ plausible:

Lemma

For any non-zero $g \in S_o(\mathbb{R}^d)$ one has

$$S_o(\mathbb{R}^d) = \{f \in \mathbf{L}^2 \,|\, f = \sum_{n=1}^{\infty} a_n \, M_{\omega_n} T_{t_n} \, g, \textit{ with } \sum_{n=1}^{\infty} |a_n| < \infty\}$$

Moreover, the quotient norm $\|f\| := \inf\{\|a\|_1\}$, with the infimum taken over all admissible representation, defines a norm which is equivalent to the standard norm on $S_o(\mathbb{R}^d)$. In particular, the finite partial sums are norm dense in $\big(S_o(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\big)$ for every such g.

The dual space $\mathbf{S}_0'(\mathbb{R}^d)$

The dual space for $\left(S_o(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is of course denoted by $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ and can be characterized as the subspace of all tempered distributions $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ such that $V_{g_0}\sigma \in \mathbf{C}_b(\mathbb{R}^{2d})$, and the sup-norm $\|V_{g_0}\sigma\|_\infty$ is equivalent to $\|\sigma\|_{\mathbf{S}_0'}$ (for any fixed $g \in S_o(\mathbb{R}^d)$).
What is more interesting is the characterization of w^*-convergence in $\mathbf{S}_0'(\mathbb{R}^d)$:

Lemma

A sequence σ_n is convergent to a limit $\sigma_0 = w^* - lim_n \sigma_n$ if and only if one has uniform convergence of $V_{g_0}\sigma_n \to \; V_{g_0}\sigma_0$ for $n \to \infty$ over compact sets, or (equivalently) just pointwise convergence.

Proof.

The argument is quite easy, given the atomic characterization of $S_o(\mathbb{R}^d)$: It is obvious that w^* convergent sequence σ_n satisfies $\sigma_n(\pi(\lambda)g)$ for any $\lambda = (t, \omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, hence pointwise convergence, and by the usual approximation argument uniform convergence over compact sets [a]. Conversely, pointwise convergence implies that one has $\sigma_n(f) \to \sigma_0(f)$ for all functions f having a finite atomic representation. Since these functions are dense in $\left(S_o(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ one has the same property for arbitrary $f \in S_o(\mathbb{R}^d)$, hence w^*-convergence □

[a] For the convergence of nets one has to assume boundedness for this argument!

From the practical point of view this means, that one has to "look at the spectrograms" of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution σ over compact sets.
Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^*-dense subsets of $\mathbf{S}_0'(\mathbb{R}^d)$.
Let us look at some concrete examples:

spline-type approximation of test functions

There are various important families which can be used to do a w^*-approximation of other distributions. In order to show how one can approximate a given $\sigma_0 \in \mathbf{S}_0'(\mathbb{R}^d)$ by discrete measures let us remind about another interesting property of $S_o(\mathbb{R}^d)$:

Lemma

Let $\Psi = (\psi_n)_{n\in\mathbb{N}}$ be a partition of unity with compact support, and write $D_\rho(h)(z) := h(\rho z)$ for $z \in \mathbb{R}^d, \rho > 0$, then $D_\rho\Psi = (D_\rho\psi_n)_{n\in\mathbb{N}} = (D_\rho T_n\psi)$ has the property that the corresponding spline-type quasi-interpolation operators

$$\mathrm{Sp}_\Psi f := \sum_{n\in\mathbb{Z}} f(\rho n) D_\rho T_n \psi(z)$$

converges to $f \in S_o(\mathbb{R}^d)$ for every $f \in S_o(\mathbb{R}^d)$, for $\rho \to \infty$.

w^*-approximation by discrete measures

Given this strong convergence of the sequence of quasi-interpolation operators, which in the most simple case are simply piecewise linear interpolations (or their tensor products) on the line, one gets of course automatically the w^*-convergence for the corresponding sequence of adjoint operators, acting on $\left(S_o(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ (which of course includes the space of bounded Radon-measures as a [small] subspace). Checking the details it is easy to find that $\mathrm{Sp}_\Psi^* = D_\Psi$, defined:

$$D_\Psi(\sigma) = \sum_{n \in \mathbb{Z}} \sigma(\psi_n)\delta_n \qquad (1)$$

Noting that the adjoint operator of D_ρ is the mass-preserving (resp. $\mathbf{L}^1-$normalized) dilation St_ρ we obtain that

$$\sum_{n\in\mathbb{Z}} \sigma(D_\rho\psi_n)\delta_{n/\rho} \to \sigma \text{ for } \rho \to \infty$$

in the w^*-sense, for every $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$. Clear enough, one can replace the (potentially) infinite sums for each fixed $\rho > 0$ by a finite sum (extending the summation over larger and larger domains), without changing the claim, hence the Dirac measures form a w^*-total subset of $\mathbf{S}_0'(\mathbb{R}^d)$.
Due to the Fourier invariance one has of course that the "pure frequencies", $t \to exp(2\pi is \cdot t)$ for another w^*-total subset of $\mathbf{S}_0'(\mathbb{R}^d)$. It is this fact which one uses in the "soft" interpretation of the Fourier inversion formula.

Regularizing operators

As we have seen in a Banach Gelfand triples a standard way of extending a morphism, given on the small (inner) space is to make use of Cauchy-sequences in the corresponding Hilbert space [2] and finally by taking suitable w^*-limits to extend the mapping to the full range of $\mathbf{S}_0'(\mathbb{R}^d)$.
There are different types of regularization operators that we can consider. In each case we require that each individual operator is a bounded and continuous Banach Gelfand triple morphism, however with the extra property of mapping the "big space" into the small space, while in the individual action on eachg of the small layers the action is bounded and in fact an approximation of the identity.

[2] this is how one usually proves the Plancherel theorem or other unitary mappings, by testing them on $\mathcal{S}(\mathbb{R}^d)$ or $S_o(\mathbb{R}^d)$

Doing the Fourier transform via FFTs

The w^* to w^* continuity can also be used do relate the ordinary Fourier transform (defined by the integral, e.g. the Riemannian integral on $S_o(\mathbb{R}^d)$) to the FFT algorithm.
The philisophy is the following:

1. Take a sampled and periodized (by a multiple of the sampling step) version of $f\ inS_o(\mathbb{R}^d)$;
2. Apply a DFT (resp. FFT) to the resulting finite vector (one full period);
3. insert the resulting sequence back into the Fourier domain, generating a discrete and periodic sequence of Diracs;
4. Via quasi-interpolation and localization to the "basic period" we get a good $\mathbf{S}_0$-approximation of $\hat{f}$.

A collection of regularizing operators

Theorem

(Regularization via Product-Convolution operators)
Assume that $g \in S_o(\mathbb{R}^d)$ *satisfies* $\int_{\mathbb{R}^d} g(x)dx = 1$, *and that* $h \in S_o(\mathbb{R}^d)$ *satisfies* $h(0) = 1$. *Then the family of product convolution operators, where convolution takes place with respect to the Dirac sequence* $(St_\rho g)_{\rho \to 0}$ *and with the plateau-type functions* $(D_\rho h)_{\rho \to 0}$, *i.e.*

$$T_\rho(f) = D_\rho h \cdot (St_\rho g * f) \quad \text{or} \quad St_\rho g * (D_\rho h \cdot f) \qquad (2)$$

are a convergent net of regularizing operators, for $\rho \to 0$.

Proof.

(PC-CP-Operators): We have to show various things:

- Each of these operators maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $S_o(\mathbb{R}^d)$. Since $S_o(\mathbb{R}^d)$ is invariant with respect to each automorphism of $\mathbb{R}^d$, hence in particular with respect to dilations, it is enough to show that e.g. $h \cdot (g * \sigma)$ belongs to $S_o(\mathbb{R}^d)$ for $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$. this is verified easily using the *convolution relations for Wiener amalgam spaces*:

$$\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) \cdot [\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) * \mathbf{W}(\mathcal{F}\mathbf{L}^\infty, \ell^\infty)] \subset \quad (3)$$

$$\subset \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) \cdot \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^\infty) \subset \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1). \quad (4)$$

- $\|St_\rho g * f - f\|_{\mathbf{S}_0} \to 0 \quad \forall f \in S_o(\mathbb{R}^d)$;
- $\|D_\rho h \cdot f - f\|_{\mathbf{S}_0} \to 0 \quad \forall f \in S_o(\mathbb{R}^d)$.

Characterization via Gabor expansions

It is a well-established (and important) result meanwhile (due to Gröchenig and Leinert in its most general form) that a Gabor frame generated from some Gabor atom $g \in S_o(\mathbb{R}^d)$ and any lattice $\Lambda \lhd \mathbb{R}^{2d} = \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ also has a *dual atom* belonging (automatically) to $S_o(\mathbb{R}^d)$. As a consequence on has the following fact:

Lemma

The canonical Gabor coefficients of for a Gabor system generated from (g, Λ) as just described, or alternatively the samples of $(V_\gamma \sigma(\lambda))_{\lambda \in \Lambda}$ belong to $(\ell^1, \ell^2, \ell^\infty)$ if and only if $\sigma \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

regularization via Gabor Multipliers

We start from the canonical representation of σ via

$$\sigma = \sum_{\lambda \in \Lambda} V_{\tilde{g}}\sigma(\lambda)\pi(\lambda)g; \tag{5}$$

and has boundedness of the coefficient guaranteed, for any $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$, but on the other hand also w^*-convergence of the sum in the general case. Multiplying with any sequence of $\ell^1(\Lambda)$-functions $m^\rho(\lambda)$, with $m^\rho(\lambda) \to 1$ for $\rho \to 0$ will then do the job, i.e.

$$A_\rho f = \sum_{\lambda \in \Lambda} m^\rho(\lambda) V_{\tilde{g}}\sigma(\lambda)\pi(\lambda)g;$$

clearly maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $S_o(\mathbb{R}^d)$, but on the other hand acts like an approximation of the identity operator on $S_o(\mathbb{R}^d)$ and $\mathbf{L}^2(\mathbb{R}^d)$, because one has convergence properties in $\ell^1(\Lambda)$ or $\ell^2(\Lambda)$ respectively.

Approximation by finite groups

Due to the w^*-density of $S_o(\mathbb{R}^d)$ within $\mathbf{S}_0'(\mathbb{R}^d)$ it is enough (from the w^*-point-of-view) to approximate now $f \in$ *SORd* appropriately by an object for some finite Abelian group. Since we are starting from a discrete but non-periodic function after sampling we only have to periodize it appropriately. Hence the best way to approximate a function $f \in S_o(\mathbb{R}^d)$ is to sample it appropriately first, and then periodize the samples, or equivalenty (if sampling rate and periodization are multiples of each other) periodize and sample afterwards! Although the resulting periodic and discrete distributions (sitting on some fine lattice) are "living" (as distributions) on $\mathbb{R}^d$ they can be identified with their values over a fundamental domain (with respect to lattice describing the periodicity, typically $N \cdot \mathbb{Z}^d$).

How to do FT via FFT?

A well known problem in classical Fourier analysis is to find out, how to use the FFT algorithm in order to show that the continuous Fourier transform of a given function $f \in S_o(\mathbb{R}^d)$. By taking pictures one can imagine that the sequence of complex numbers which arises when one takes a sufficiently fine and sufficiently wide (over all relevant portions of f) *finite* sampling sequence, it should "resemble" the shape of $\mathcal{F}f$. What is valid is the fact, that we can obtain *exactly* a sampled and periodized version of $\hat{f}$. Although it is clear that the inherent decay of $\hat{f} \in S_o(\mathbb{R}^d)$ implies that one can recover the samples of its non-periodized version with only a small ℓ^1-error, it is not so obvious (but true) that the doing a quasi-interpolation allows to approximately recover (now) $\hat{f}$, with an error which tends to zero in the $S_o(\mathbb{R}^d)-$norm.