Hans G. Feichtinger
NuHAG - University of Vienna
http://www.nuhag.eu

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↬Banach Gelfand Triples and their applications in Harmonic and Functional Analysis↫

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My Personal Background (from AHA to CHA)

- Trained as an **abstract harmonic analyst** (Advisor Hans Reiter) at the Univ. of Vienna
- working on function spaces on locally compact groups, distribution theory
- turning to applications (signal processing, image processing), wavelets
- doing **numerical work** on scattered data approximation, **Gabor analysis**
- having a keen interest in connecting theory with **applications** through efficient numerical algorithms underpinned by theoretical foundations

OUTLINE of the TALK:

1. Recall the USE and the CONCEPTS of Fourier Analysis (locally compact Abelian groups OR tempered distributions);
2. Propose the use of a particular BANACH GELFAND TRIPLE;
3. Concepts from time-frequency analysis and in particular from GABOR ANALYSIS;
4. The numerical challenges of Gabor analysis;
5. {The transition from continuous to the finite setting}.

Motivation: Where and How do we need Fourier Analysis?

- At which level of generality (using Riemannian integrals, Lebesgue integrals, generalized functions)? Which tools?
- We may need it in order to define Sobolev or Bessel potential spaces;
- In which setting should it be explained and in which order? (classical books start from Fourier series, go then to FT on $\boldsymbol{L}^1(\mathbb{R}^d)$, FFT, maybe tempered distributions);
- What is the natural setting: of course LCA groups? (according to A. Weil); but or practical purposes often "elementary LCA groups";
- What kind of Fourier Analysis is needed to teach engineers and our students (impulse response, transfer function, filter, . . .);

What do we have (?) to teach our students?

The typical VIEW that well trained mathematicians working in the field may have, is that ideally a STUDENT have to

- learn about Lebesgue integration (to understand Fourier integrals);
- learn about Hilbert spaces and unitary operators;
- learn perhaps about L^p-spaces as Banach spaces;
- learn about topological (nuclear Frechet) spaces like $\mathcal{S}(\mathbb{R}^d)$;
- learn about tempered distributions;
- learn quasi-measures, to identify TLIS as convolution operators;

Classical Approach to Fourier Analysis

- Fourier Series (periodic functions), summability methods;
- Fourier Transform on $\mathbb{R}^d$, using Lebesgue integration;
- sometimes: Theory of Almost Periodic Functions;
- Generalized functions, tempered distributions;
- Discrete Fourier transform, FFT (Fast Fourier Transform), e.g. FFTW;
- Abstract ($>>$ Conceptional !)Harmonic Analysis over LCA groups;
- . . . but what are the connections?? What is needed for computations?

What are our goals when doing Fourier analysis?

- find relevant "harmonic components" in [almost] periodic functions;
- define the Fourier transform (first $\boldsymbol{L}^1(\mathbb{R}^d)$, then $\boldsymbol{L}^2(\mathbb{R}^d)$, etc.);
- describe time-invariant linear systems as convolution operators;
- describe such system as Fourier multipliers (via transfer functions);
- deal with (slowly) time-variant channels (communications) ;
- describe changing frequency content ("musical transcription");
- define operators acting on the spectrogram (e.g. for denoising) or perhaps pseudo-differential operators using the Wigner distribution;

CLAIM: What is really needed!

In contrast to all this the CLAIM is that just a bare-bone version of functional analytic terminology is needed (including basic concepts from Banach space theory, up to w^*-convergence of sequences and basic operator theory), and that the concept of Banach Gelfand triples is maybe quite useful for this purpose. So STUDENTS SHOULD LEARN ABOUT:

- refresh their linear algebra knowledge (ONB, **SVD**!!!, linear independence, generating set of vectors), and matrix representations of linear mappings between finite dimensional vector spaces;
- Banach spaces, bd. operators, dual spaces norm and w^*-convergence;
- about Hilbert spaces, orthonormal bases and unitary operators;
- about frames and Riesz basis (resp. matrices of maximal rank);

BRANCH 1: A. Weil: LCA groups G is the natural setting!

First of all one has plenty of continuous functions k in $\boldsymbol{C}_c(G)$, i.e. with compact support on such a "locally compact" group G, and the space $\boldsymbol{C}_0(G)$, the closure of those test functions in $\left(\boldsymbol{C}_b(G), \|\cdot\|_\infty\right)$ is non-trivial. It is also clear what $\widehat{G}$, namely the group of all continuous (group) homomorphism from G into the standard group $\mathbb{T}$ (dual group).

Consequently we have (keeping the Riesz-representation theorem in mind) $\boldsymbol{M}_b(G)$ is well defined as the space of bounded linear functionals on $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$. As a dual Banach space it carries two topologies, the norm topology (usually called the total variation norm on measures) and the w^*-topology (vague convergence, e.g. used in the central limit theorem).

Among the most simply functionals in $\boldsymbol{M}_b(G)$ the Dirac measures $\delta_x, x \in G$ which send $f \in \boldsymbol{C}_0(G)$ into $f(x)$. Note that $\delta_x \to \delta_0$ for $x \to 0$ only in the w^*- topology. They are w^*-total in all of $\boldsymbol{M}_b(G)$.

Now one could start talking about the existence of the (invariant) Haar measure, $\boldsymbol{L}^1(G)$ and $\boldsymbol{L}^2(G)$ and the Fourier transform on those spaces. However I prefer to introduce first convolution and the Fourier-Stieltjes transform. Obviously we can define translation already now on $\boldsymbol{C}_0(G)$ as well as on $\boldsymbol{M}_b(G)$ (in the usual way, by adjoint action), let us call them T_z. First of all one has to show (which is not difficult) that there is a $natural$ identification between the bounded linear operators on $\boldsymbol{C}_0(G)$ which commute with translations, the so-called translation invariant linear systems and the elements of $\boldsymbol{M}_b(G)$, where we have exact correspondence between $\delta_z \in \boldsymbol{M}_b(G)$ and the operator T_z.
Since these operators clearly form a (closed) subalgebra of the operator algebra on $\boldsymbol{C}_0(G)$ it is clear that we can transfer the multiplication of operators to some natural "multiplication of bounded measures", which we call convolution and write $*$. Obviously we have

$$\delta_x * \delta_y = \delta_{x+y}.$$

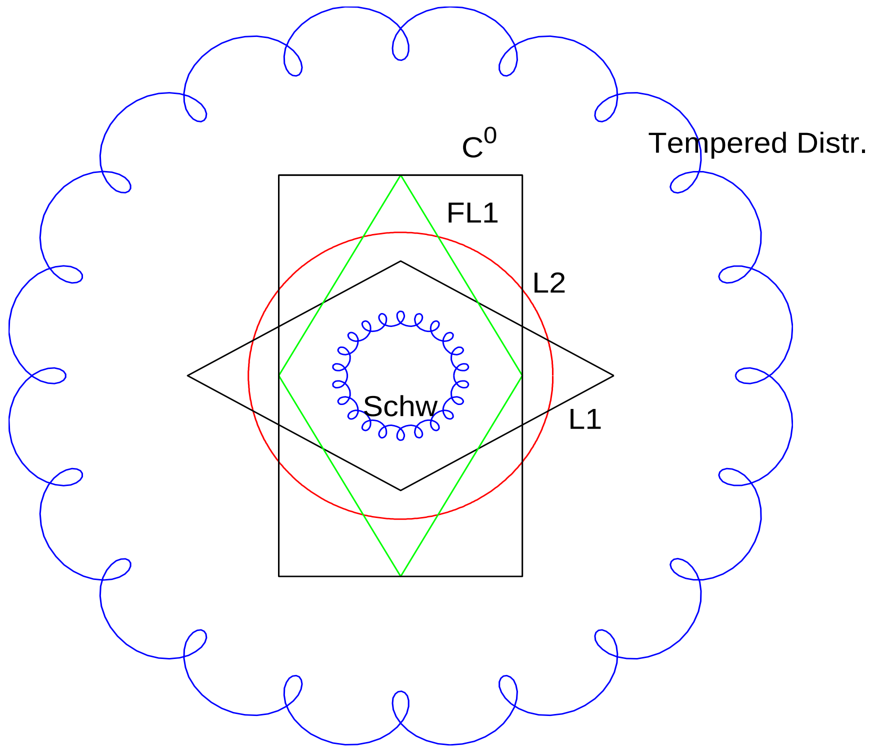
Unifying aspect of this view-point

- Basic concepts, like the Haar measure (Lebesgue measure on $\mathbb{R}^d$, the counting measure on a discrete group) find a common interpretation;
- For any LCA groups there are *characters* (for mathematicians) or *pure frequencies*. They may look differently (e.g. plane waves vs. a pure sinusoidal tone), but share equal properties;
- Translation invariant operators can be written as convolution operators (although formulas have a different appearance);
- There is exactly one "Fourier transform", which is just an orthogonal change of bases for finite Abelian groups (nicely realized using the FFT!), but appears to be much more complicated otherwise (by Plancherel's theorem it is at least unitary on $\boldsymbol{L}^2(G)$!).

BRANCH 2: The "usual program": FT on $L^1(G), L^2(G)$ etc.

We define $\left(L^1(G), \|\cdot\|_1\right)$, the Fourier transform on it (using Lebesgue integrals), and show - e.g. using classical summability arguments - that the inversion can be done "somehow".
Then one goes on to Plancherel's theorem (by showing the the Fourier transform is isometric on $L^1 \cap L^2$ and applying an approximation argument for general elements $f \in L^2(G)$).
Still the picture is quite a bit "imcomplete" an distorted if you look at it from the modern time-frequency point of view (the roles of $L^1(\mathbb{R}^d)$ on the one side, naturally associated with convolution, has nothing comparable on the "other" side).
Of course the theory by L. Schwartz, using the space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions and its dual, $\mathcal{S}(\mathbb{R}^d)$ (the *tempered distributions*) give a more beautiful picture. But if you have seen the complications of the Schwartz-Bruhat space for LCA groups you will not propagate this approach.

The classical view on the Fourier Transform



From tempered distributions to Banach Gelfand Triples

- Typical questions of (classical and modern) Fourier analysis
- Fourier transforms, convolution, impulse response, transfer function
- The Gelfand triple $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')(\mathbb{R}^d)$, of Schwartz functions and tempered distributions; maybe *rigged Hilbert spaces*;

WHAT WE WANT TO DO TODAY:

- The Banach Gelfand Triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ and its use;
- various (unitary) Gelfand triple isomorphisms involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

LET US START WITH SOME FORMAL DEFINITIONS:

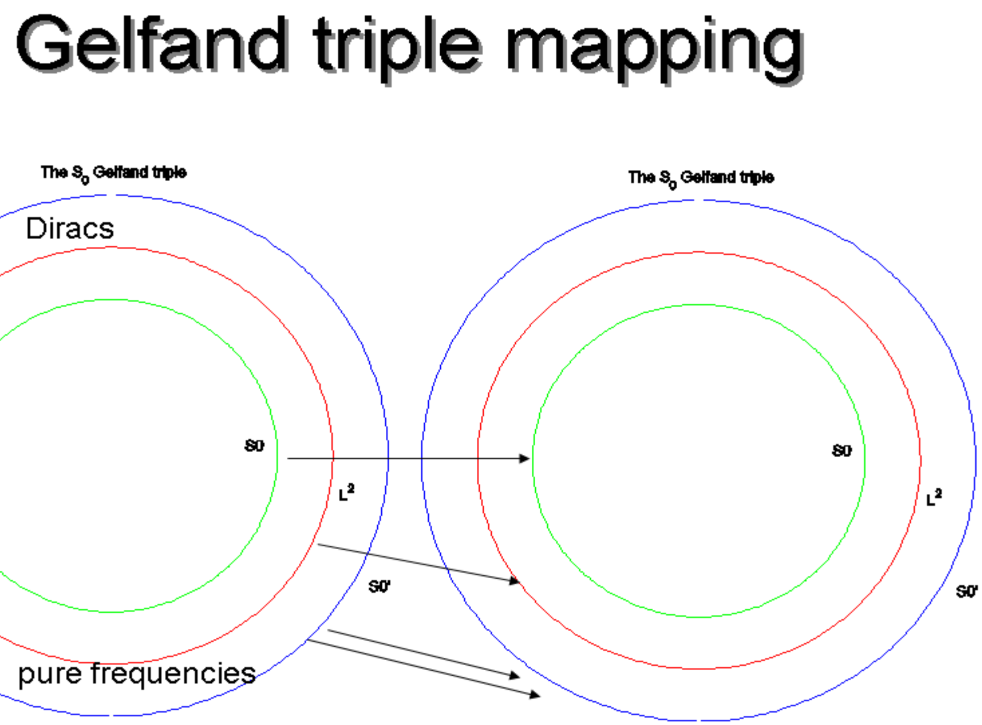
Definition 1. *A triple $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.*

Definition 2. *If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if*

1. *A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.*
2. *A is a [unitary operator resp.] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.*
3. *A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.*

The prototype is $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$. w^*-convergence corresponds to coordinate convergence in ℓ^∞. It can be transferred to "abstract Hilbert spaces" $\mathcal{H}$. Given any orthonormal basis (h_n) one can relate $\boldsymbol{\ell}^1$ to the set of all elements $f \in \mathcal{H}$ which have an *absolutely convergent* series expansions with respect to this basis. In fact, in the classical case of $\mathcal{H} = \boldsymbol{L}^2(\mathbb{T})$, with the usual

Fourier basis the corresponding spaces are known as Wiener's $\boldsymbol{A}(\mathbb{T})$. The dual space is then $\mathcal{P}_M$, the space of pseudo-measures $= \mathcal{F}^{-1}[\boldsymbol{\ell}^\infty(\mathbb{Z})]$.



Realization of a GT-homomorphism

Very often a Gelfand-Triple homomorphism T can be *realized with the help of some kind of "summability methods"*. In the abstract setting this is a sequence (or more generally a net) A_n, having the following property:

- each of the operators maps $\boldsymbol{B}_1'$ into $\boldsymbol{B}_1$;
- they are a uniformly bounded family of Gelfand-triple homomorphism on $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$;
- $A_n f \to f$ in $\boldsymbol{B}_1$ for any $f \in \boldsymbol{B}_1$;

It then *follows* that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in $\boldsymbol{B}_2'$ (in the w^*-sense) for $f \in \mathcal{H}_1$ resp. $f \in \boldsymbol{B}_1'$ and thus describes concretely the prolongation to the full Gelfand triple. This continuation is unique due to the w^*-properties assumed for T (and the w^*-density of $\boldsymbol{B}_1$ in $\boldsymbol{B}_1'$).

Typical Philosophy

One may think of $\boldsymbol{B}_1$ as a (Banach) space of test functions, consisting of "decent functions" (continuous and integrable), hence $\boldsymbol{B}_1{}'$ is a space of "generalized functions, containing at least all the $\boldsymbol{L}^p$-spaces as well as all the bounded measures, hence in particular finite discrete measures (linear combinations of Dirac measures).

At the INNER $=$ test function level every "transformation" can be carried out very much as if one was in the situation of a finite Abelian group, where sums are convergent, integration order can be interchanged, etc.. At the INTERMEDIATE level of the Hilbert space one has very often a unitary mapping, while only the OUTER LAYER allows to really describe what is going on in the *ideal limit case*, because instead of unit vectors for the finite case one has to deal with Dirac measures, which are only found in the big dual spaces (but not in the Hilbert space!).

Using the BGTR-approach one can achieve . . .

- a relative simple minded approach to Fourier analysis (can be motivated by linear algebra);
- results based on standard functional analysis only;
- provide clear rules, based on basic Banach space theory;
- comparison with extensions $\mathbb{Q} >> \mathbb{R}$ resp. $\mathbb{R} >> \mathbb{C}$;
- provide confidence that "generalized functions" really exist;
- provide simple descriptions to the above list of questions!

Key Players for Time-Frequency Analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

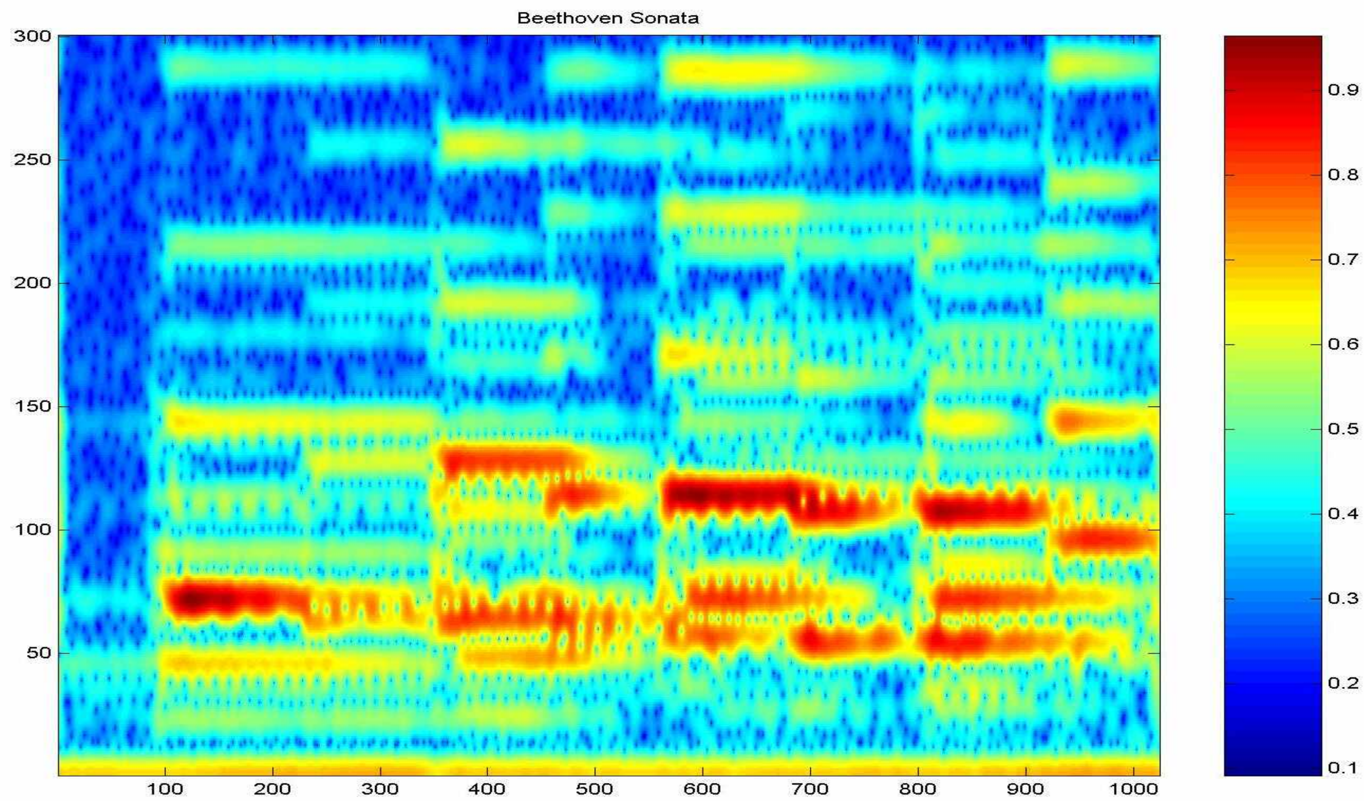
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = V_g f(t, \omega) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \quad \lambda = (t, \omega);$$

A Typical Musical STFT



Some algebra in the background: The Heisenberg group

Weyl commutation relation

$$T_x M_\omega = e^{-2\pi i x\cdot\omega} M_\omega T_x, \quad (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

$\{M_\omega T_x : (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$ is a **projective representation** of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $L^2(\mathbb{R}^d)$. **Heisenberg group** $\mathbb{H} := \{\tau M_\omega T_x : \tau \in \mathbb{T}, (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d\}$

Schrödinger representation $\{\tau M_\omega T_x : (x,\omega,\tau) \in \mathbb{H}\}$ is a square-integrable (irreducible) group representation of $\mathbb{H}$ on the Hilbert space $L^2(\mathbb{R}^d)$. Then the STFT $V_g f$ is a representation coefficient.

Moyal's formula or orthogonality relations for STFTs:
Let f_1, f_2, g_1, g_2 be in $L^2(\mathbb{R}^d)$. Then

$$\left\langle V_{g_1} f_1, V_{g_2} f_2 \right\rangle_{L^2(\mathbb{R}^{2d})} = \langle f_1, f_2 \rangle_{L^2(\mathbb{R}^d)} \langle g_2, g_1 \rangle_{L^2(\mathbb{R}^d)}.$$

Reconstruction formula
Let $g, \gamma \in L^2(\mathbb{R}^d)$ with $\langle g, \gamma \rangle \neq 0$. Then for $f \in L^2(\mathbb{R}^d)$ we have

$$f = \frac{1}{\langle g, \gamma \rangle} \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(x, \omega) \pi(x, \omega) \gamma \, dx d\omega.$$

So typically one chooses $\gamma = g$ with $\|g\|_2 = 1$.

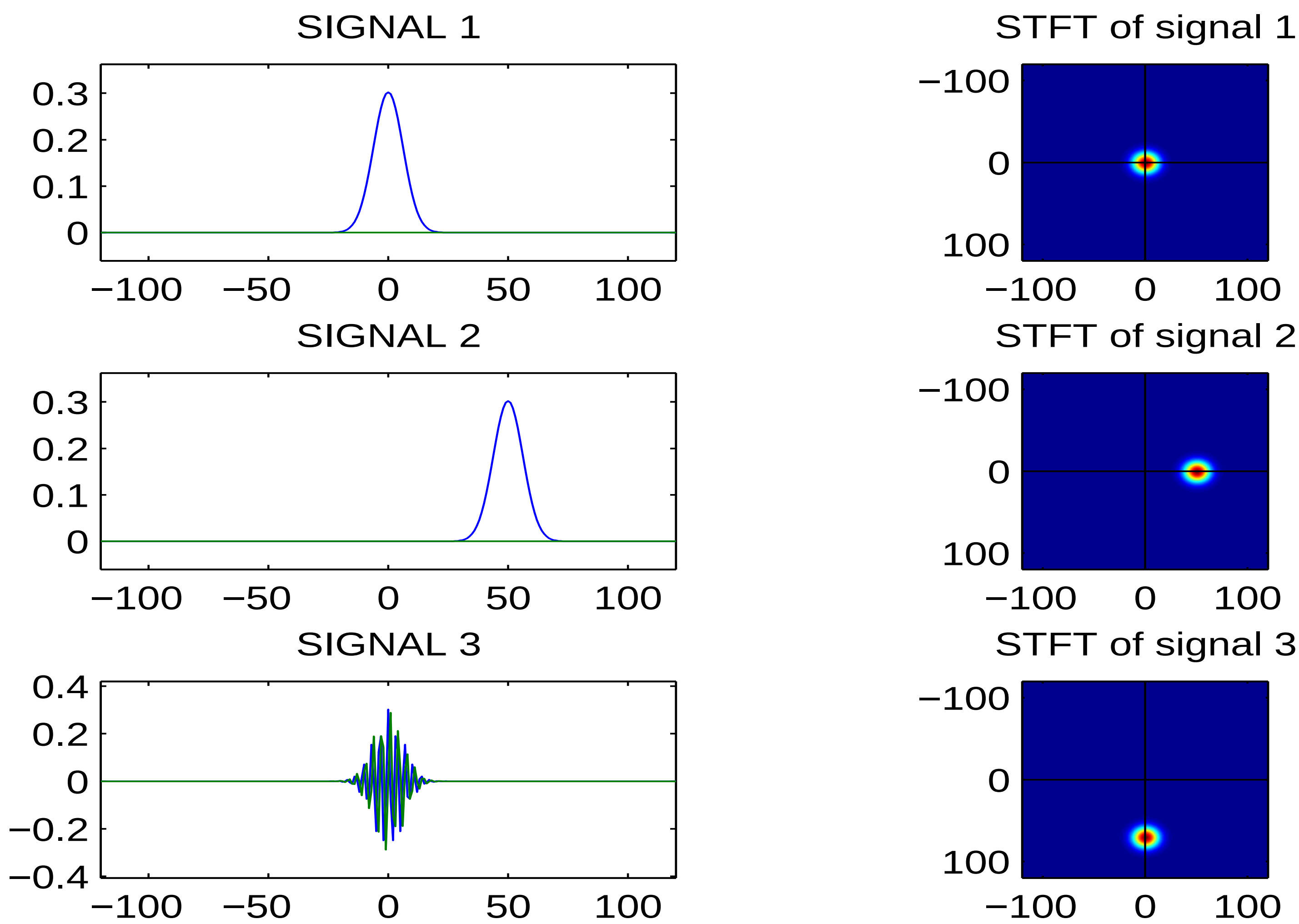
Primer on Gabor analysis: Atomic Viewpoint

D.GABOR's suggested to replace the continuous integral representation by a discrete series and still claim that one should have a representation of arbitrary elements of $\boldsymbol{L}^2(\mathbb{R})$!
Let $g \in L^2(\mathbb{R}^d)$ and Λ a lattice in time-frequency plane $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

$$\mathbf{f} = \sum_{\lambda \in \boldsymbol{\Lambda}} \mathbf{a}(\lambda) \pi(\lambda) \mathbf{g}, \quad \text{for some} \quad \mathbf{a} = (\mathbf{a}(\lambda))_{\lambda \in \boldsymbol{\Lambda}}$$

is a so-called **Gabor expansion** of $f \in L^2(\mathbb{R}^d)$ for the **Gabor atom** g.

1946 - D. Gabor: $\Lambda = \mathbb{Z}^2$ and Gabor atom $g(t) = e^{-\pi t^2}$.


SIGNAL 1
0.3
0.2
0.1
0
−100
−50
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STFT of signal 1
−100
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100
SIGNAL 2
0.3
0.2
0.1
0
−100
−50
0
50
100
STFT of signal 2
−100
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100
SIGNAL 3
0.4
0.2
0
−0.2
−0.4
−100
−50
0
50
100
STFT of signal 3
−100
0
100
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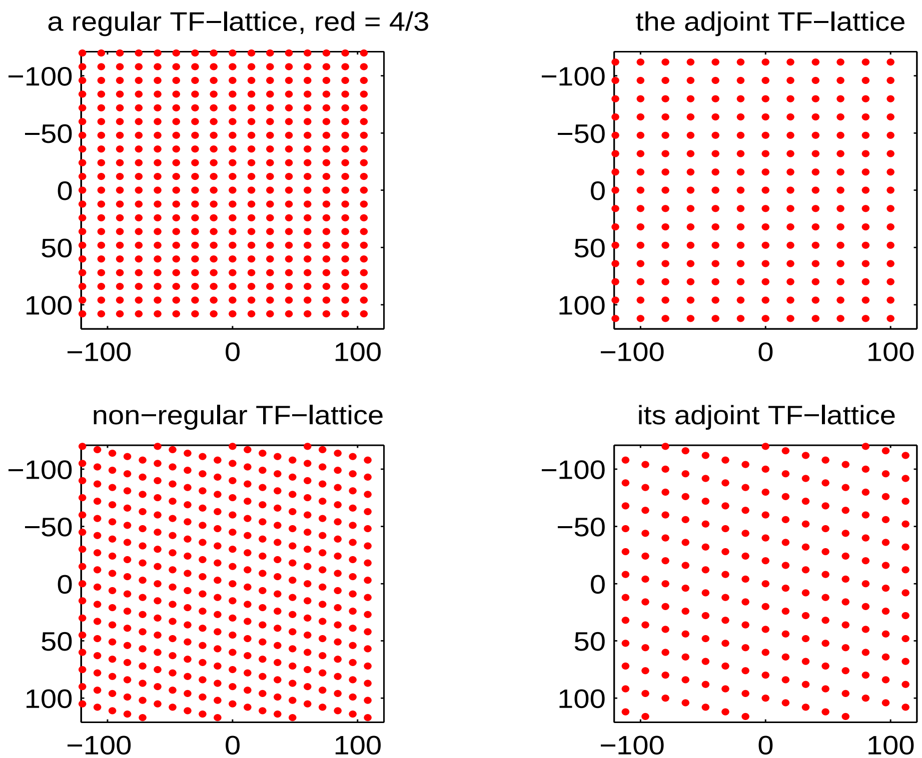
100

Examples of finite Gabor families

Signal length $n = 240$, lattice Λ with $320 = 4/3 * n$ [$180 = 3/4 * n$] points.



Gabor analysis over finite Abelian groups

In order to find out, if a given family $(\pi(\lambda)g) = (g_\lambda)_{\lambda \in \Lambda}$ generates the space of all signals over the group G, which is either $\ell^2(G)$ or $\mathbb{C}^n$, with $n = \#(G)$. For a long time (for large n, and due to the non-orthogonality of the family (g_λ) this was considered a computationally intensive task. However again group theoretical considerations can help out. One better considers the so called frame operator $S_{g,\Lambda} : \mathbf{x} \to \sum_\Lambda \langle \mathbf{x}, g_\lambda \rangle g_\lambda,$ which is (in the case of a spanning family (g_λ) an invertible and positive definite matrix, but beyond that the commuation relations imply that

$$\pi(\lambda) \circ S = S \circ \pi(\lambda), \quad \lambda \in \Lambda.$$

This implies that the whole problem of inverting S is reduced to the (much easier) task of finding the dual atom $\tilde{g} := S^{-1}(g)$, or solve the positive definite system $S(h) = g$ for h, e.g. by the use of conjugate gradients.

The benefit of having a dual Gabor atom (and duality is a symmetric relationship because the frame operator induced by t $\tilde{g}$ is just the inverse of the frame operator!) is that one can use one for analysis and the other for synthesis as follows:

Seen as a sampling problem, one reconstructs the signal f from the samples of $V_g(f)$ over Λ by the formula $f = S^{-1}S(f) = \sum_\lambda V_g f(\lambda)\pi(\lambda)\tilde{g}$.

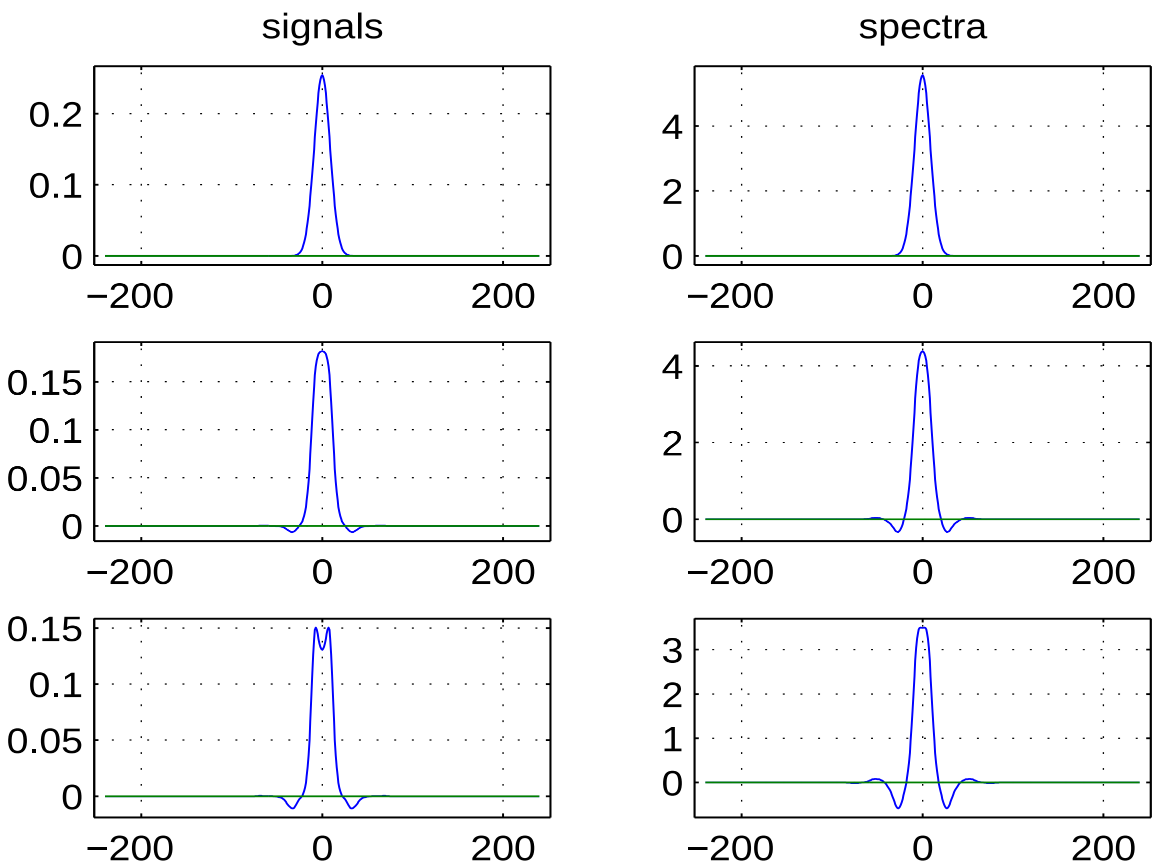
On the other hand, if one takes the atomic point of view, i.e. if one want to fulfill Gabor's wishes by providing in a most efficient ways coefficients for a given function f in order to write it as an (unconditionally convergent) Gabor sum, then one will prefer the formula $f = S^{-1}S(f) = \sum_\lambda V_{\tilde{g}} f(\lambda)\pi(\lambda)g$.

There is also a symmetric way, of modifying both the analysis and synthesis operator in order to (by choosing $h = S^{-1/2}g$)

$$f = \sum_\lambda V_h f(\lambda)\pi(\lambda)h = \sum_\lambda \langle f, h_\lambda \rangle h_\lambda.$$

This looks very much like an orthonormal expansion (although it is not), and h is called a tight Gabor atom.

Gabor atom, with canonical tight and dual Gabor atoms



Introducing $S_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$ $\quad (Fei, 1979)$

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is (by definition) in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often the Gaussian is used as a window. Note that

$$V_g f(x,\omega) = (\widehat{f \cdot T_x g})(\omega), \quad \text{i.e., } g \text{ localizes } f \text{ near } x.$$

Lemma 1. *Let* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$*, then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$*, and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Remark 2. *Moreover one can show that* $\boldsymbol{S}_0(\mathbb{R}^d)$ *is the smallest non-trivial Banach spaces with this property, i.e., it is continuously embedded into any such Banach space. As a formal argument one can use the continuous inversion formula for the STFT:*

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g f(\lambda) \pi(\lambda) g d\lambda$$

which implies

$$\|f\|_{\boldsymbol{B}} \leq \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(\lambda)| \|\pi(\lambda) g\|_{\boldsymbol{B}} \, d\lambda = \|g\|_{\boldsymbol{B}} \|f\|_{\boldsymbol{S}_0}.$$

Basic properties of $S_0(\mathbb{R}^d)$ resp. $S_0(G)$

THEOREM:

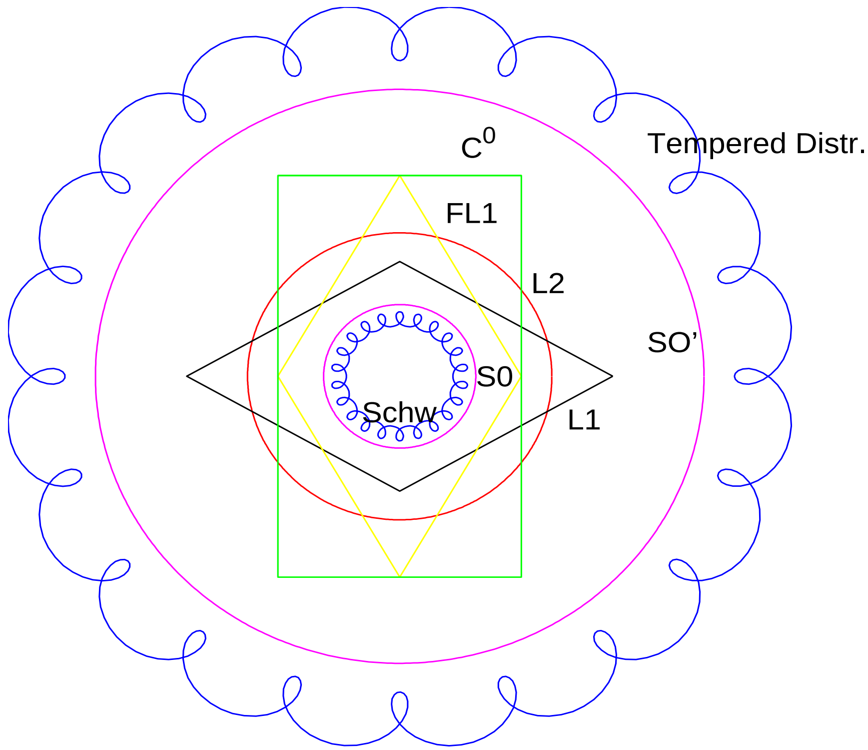
- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $\boldsymbol{S}_0(G)$; $[with\,(\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}\boldsymbol{S}_0(G) = \boldsymbol{S}_0(\hat{G})$; (Invariance under the Fourier Transform);
- $T_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(G/H)$; (Integration along subgroups);
- $R_H \boldsymbol{S}_0(G) = \boldsymbol{S}_0(H)$; (Restriction to subgroups);
- $\boldsymbol{S}_0(G_1)\hat{\otimes}\boldsymbol{S}_0(G_2) = \boldsymbol{S}_0(G_1 \times G_2)$; (tensor product stability).

Basic properties of $S_0'(\mathbb{R}^d)$ resp. $S_0'(G)$

THEOREM: (Consequences for the dual space $\boldsymbol{S_0}'(\mathbb{R}^d)$)

- $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ is in $\boldsymbol{S_0}'(\mathbb{R}^d)$ if and only if $V_g\sigma$ is bounded;
- w^*-convergence in $\boldsymbol{S_0}'(\mathbb{R}^d) \approx$ pointwise convergence of $V_g\sigma(\lambda)$;
- $\left(\boldsymbol{S_0}'(G), \|\cdot\|_{\boldsymbol{S_0}'}\right)$ is a Banach space with a translation invariant norm;
- $\boldsymbol{S_0}'(G) \subseteq \mathcal{S}'(G)$, i.e. $\boldsymbol{S_0}'(G)$ consists of tempered distributions;
- $\boldsymbol{P}(G) \subseteq \boldsymbol{S_0}'(G) \subseteq \boldsymbol{Q}(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq \boldsymbol{S_0}'(G)$; (contains translation bounded measures).

Schwartz space, S_0, L^2, S_0', tempered distributions



Basic properties of $S_0'(\mathbb{R}^d)$ continued

THEOREM: $\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$ for $f \in \boldsymbol{S}_0(\hat{G}), \sigma \in \boldsymbol{S}_0{}'(G)$

- defines a Generalized Fourier Transforms, with $\mathcal{F}(\boldsymbol{S}_0'(G)) = \boldsymbol{S}_0'(\hat{G})$.

- $\sigma \in \boldsymbol{S}_0'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in \boldsymbol{S}_0'(G/H)$ such that $\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle$.

- $\boldsymbol{S}_0'(H)$ can be identified with a subspace of $\boldsymbol{S}_0'(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in \boldsymbol{S}_0'(G)$ one has $\sigma \in i_H(\boldsymbol{S}_0'(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

The usefulness of $S_0(\mathbb{R}^d)$: maximal domain for Poisson

Theorem 1. *(***Poisson's formula***) For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and any discrete subgroup H of $\mathbb{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \qquad (1)$$

with absolute convergence of the series on both sides.

By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $\boldsymbol{S}_0'(\mathbb{R}^d)$ is invariant under the (generalized) Fourier transform. Sampling corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$, while it corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side = periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ all this makes perfect sense.

Regularizing sequences for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$

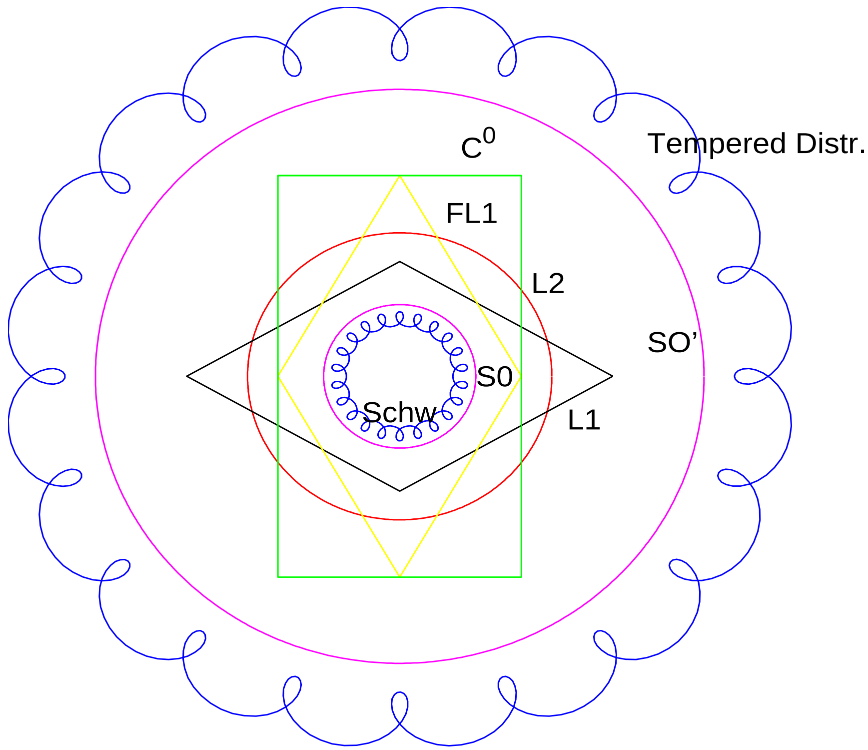
Wiener amalgam convolution and pointwise multiplier results imply that

$$\boldsymbol{S}_0(\mathbb{R}^d) \cdot (\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d), \quad \boldsymbol{S}_0(\mathbb{R}^d) * (\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) \subseteq \boldsymbol{S}_0(\mathbb{R}^d)$$

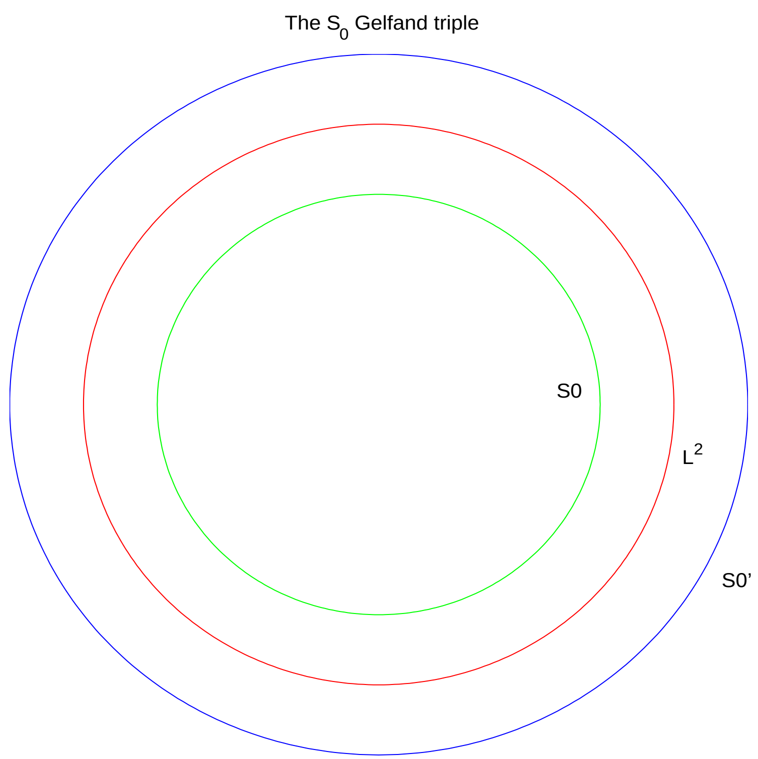
e.g. $\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) * \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^\infty)$.

Let now $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ be given with $h(0) = 1$. Then the dilated version $h_n(t) = h(t/n)$ are a uniformly bounded family of multipliers on $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$, tending to the identity operator in a suitable way. Similarly, the usual Dirac sequences, obtained by compressing a function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ are showing a similar behavior: $g_n(t) = n \cdot g(nt)$ Following the above rules the combination of the two procedures, i.e. product-convolution or convolution-product operators of the form provide suitable regularizers: $A_n f = g_n * (h_n \cdot f)$ or $B_n f = h_n \cdot (g_n * f)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



The Fourier transform is a prototype of a Gelfand triple isomorphism.

EX1: The Fourier transform as Gelfand Triple Automorphism

Theorem 2. *Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}^d})$,*

(2) *$\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}^d})$,*

(3) *$\mathcal{F}$ is a weak*-weak* (and norm-to-norm) continuous isomorphism between $\boldsymbol{S}_0'(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{2}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, or $(f, g) \in \boldsymbol{L}^2(\mathbb{R}^d) \times \boldsymbol{L}^2(\mathbb{R}^d)$ or other pairings from the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The properties of Fourier transform can be expressed by a Gelfand bracket

$$\langle f, g \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} = \langle \hat{f}, \hat{g} \rangle_{(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')} \tag{3}$$

which combines the functional brackets of dual pairs of Banach spaces and of the inner-product for the Hilbert space.

One can characterize the Fourier transform as the *uniquely determined* unitary Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ which maps pure frequencies into the corresponding Dirac measures (and vice versa). [1]

One could equally require that TF-shifted Gaussians are mapped into FT-shifted Gaussians, relying on $\mathcal{F}(M_\omega T_x f) = T_{-\omega} M_x (\mathcal{F} f)$ and the fact that $\mathcal{F} g_0 = g_0$, with $g_0(t) = e^{-\pi |t|^2}$.

[1] as one would expect in the case of a finite Abelian group.

EX.2: The Kernel Theorem for general operators in $\mathcal{L}(S_0, S_0')$

Theorem 3. *If* K *is a bounded operator from* $\boldsymbol{S}_0(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0'(\mathbb{R}^d)$*, then there exists a unique kernel* $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ *such that* $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ *for* $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$*, where* $g \otimes f(x, y) = g(x)f(y)$*.*

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Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dyg(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

This result is the "outer shell of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

Again the complete picture can again be best expressed by a unitary Gelfand triple isomorphism. We first describe the innermost shell:

Theorem 4. *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with* $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ *if and only if the corresponding operator* K *maps* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* w^*-*topology into the norm topology of* $\boldsymbol{S}_0(\mathbb{R}^d)$.

Remark: Note that for "regularizing" kernels in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Note that $\delta_y \in \boldsymbol{S}_0'(\mathbb{R}^d)$ implies that $K(\delta_y) \in \boldsymbol{S}_0(\mathbb{R}^d)$ by the regularizing properties of K, hence the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')).$$

The Kohn Nirenberg Symbol and Spreading Function

In the setting of a finite group (such as $G = \mathbb{Z}_n$) it is easy to show that the collection of all matrices which are composed of time-frequency shifts (there are $n = \#(G)$ of each sort, so altogether n^2 such operators, span the whole n^2-dimensional space $\mathcal{M}_n$ of *all* $n \times n$-matrices. In fact, it is easy to show that they form an orthonormal basis with respect to the scalar product introduced by transferring the Euclidean structure of $\mathbb{R}^{n^2}$ back to these matrices (where it becomes the Frobenius or Hilbert Schmidt scalar product).

If $Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$ then $\sigma(K) = \int_{\mathbb{R}^d} k(x, x-y)e^{-2\pi i y \cdot \omega} dy$. In signal analysis $\sigma(K)$ was introduced by Zadeh and is called the *time-varying transfer function* of a system modelled by K.
The nice invariance properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ and hence of $\boldsymbol{S}_0'(\mathbb{R}^d)$ allow for simple arguments within the context of Banach Gelfand Triples over $\mathbb{R}^d$.

The spreading symbol as Gelfand Triple mapping

The Kohn-Nirenberg symbol $\sigma(T)$ of an operator T (respectively its *symplectic* Fourier transform, the *spreading distribution* $\eta(T)$ of T) can be obtained from the kernel using some automorphism and a partial Fourier transform, which again provide unitary Gelfand isomorphisms. In fact, the symplectic Fourier transform is another unitary Gelfand Triple (involutive) automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Theorem 5. *The correspondence between an operator T with kernel K from the Banach Gelfand triple $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and the corresponding spreading distribution $\eta(T) = \eta(K)$ in $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ is the uniquely defined Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ mapping the time-frequency shift $M_y \circ T_x$ to $\delta_{(x,y)}$, the Dirac at (x, y).*

Kohn-Nirenberg and Spreading Symbols of Operators

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x + \frac{y}{2}, x - \frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y - x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

Kohn-Nirenberg correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator with *symbol* σ

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

is the *pseudodifferential operator* with Kohn-Nirenberg symbol σ.

$$\begin{aligned} K_\sigma f(x) &= \int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}\sigma(x,\omega)e^{-2\pi i(y-x)\cdot\omega}d\omega\Big)f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. Formulas for the (integral) kernel k: $k = \mathcal{T}_a\mathcal{F}_2\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x) \\ &= \widehat{\sigma}(\eta, y-x)e^{2\pi i\eta\cdot x}d\eta. \end{aligned}$$

3. The *spreading representation* of the same operator arises from the identity

$$K_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$

$\widehat{\sigma}$ is called the spreading function of the operator K_σ.

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *Rihaczek distribution* is defined by

$$R(f,g)(x,\omega) = e^{-2\pi i x \cdot \omega} \overline{\widehat{f}(\omega)} g(x).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, R(f,g) \rangle = \langle K_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Weyl correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator

$$L_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-\pi i \xi \cdot u} f(x) du d\xi$$

is called the *pseudodifferential operator* with *symbol* σ. The map $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and σ the Weyl symbol of the operator L_σ.

$$\begin{aligned} L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \widehat{\sigma} e^{-\pi i u \cdot \xi} T_{-u} M_\xi f(x) du d\xi \\ &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \widehat{\sigma}(\xi, y - x) e^{-2\pi i \xi \frac{x+y}{2}} \Big) f(y) dy. \end{aligned}$$

2. Formulas for the kernel k from the KN-symbol: $k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_1^{-1}\widehat{\sigma}\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2^{-1}\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma. \end{aligned}$$

3. $\langle L_\sigma f, g\rangle = \langle k, g \otimes \overline{f}\rangle$. (Weyl operator vs. kernel)

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the *cross Wigner distribution* of f, g is defined by

$$W(f,g)(x,y) = \int_{\mathbb{R}^d} f(x+t/2)\overline{g}(x-t/2)e^{-2\pi i\omega\cdot t}dt = \mathcal{F}_2\mathcal{T}_s(f \otimes \overline{g})(x,\omega).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, W(f,g) \rangle = \langle L_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator L_σ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

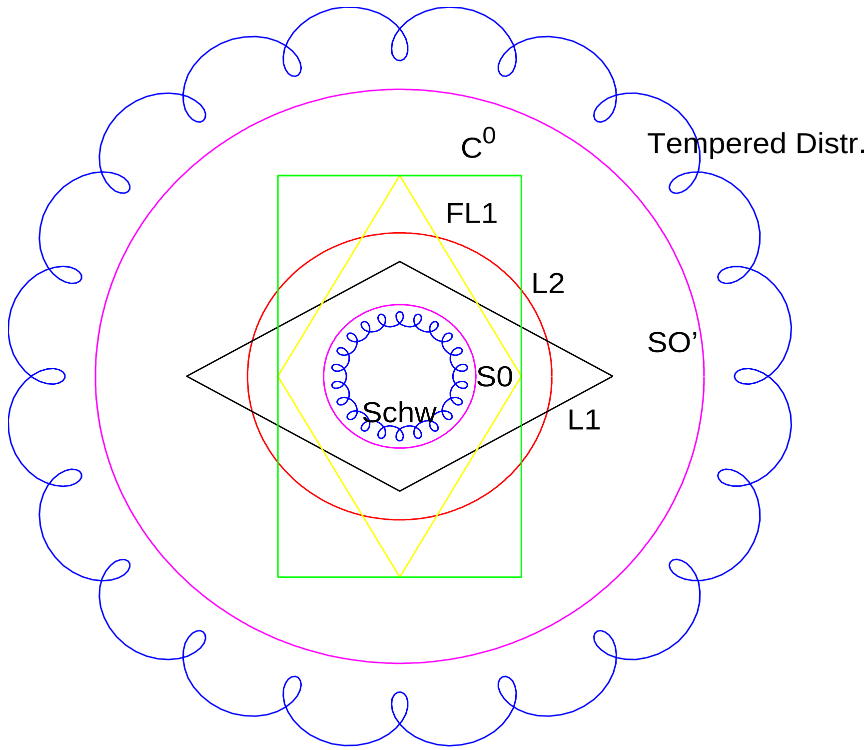
$$(\mathcal{U}\sigma)(\xi, u) = \mathcal{F}^{-1}(e^{\pi i u \cdot \xi} \widehat{\sigma}(\xi, u)).$$

$$K_{\mathcal{U}\sigma} = L_\sigma$$

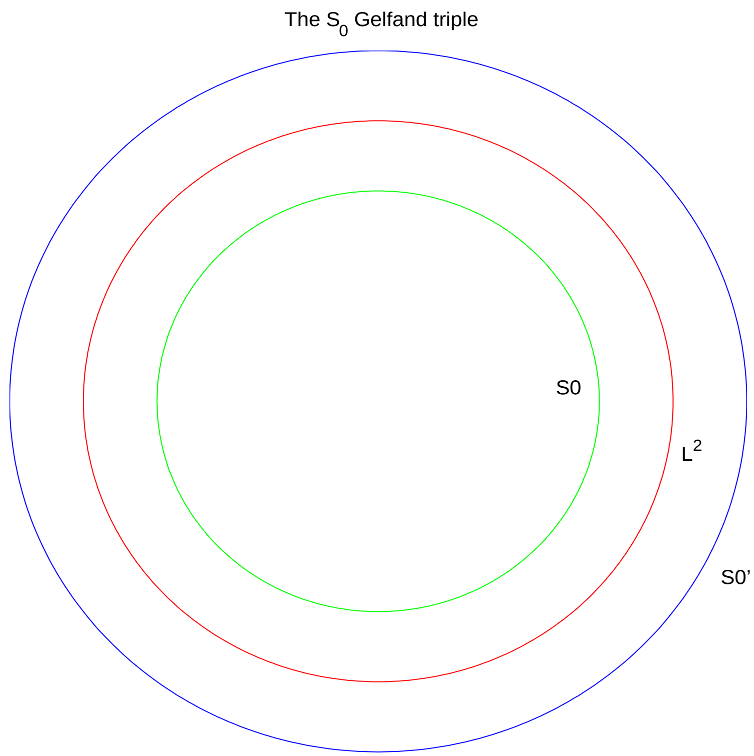
describes the connection between the Weyl symbol and the operator kernel.

In all these considerations the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be correctly replaced by $\boldsymbol{S}_0(\mathbb{R}^d)$ and the tempered distributions by $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple (S_0, L^2, S_0')



Fourier transform is a prototype of a unitary Gelfand triple isomorphism.

Examples of Gelfand Triple Isomorphisms

1. The standard Gelfand triple $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$.

2. The family of orthonormal Wilson bases (obtained from Gabor families by suitable pairwise linear-combinations of terms with the same absolute frequency) extends the natural unitary identification of $\boldsymbol{L}^2(\mathbb{R}^d)$ with $\ell^1(I)$ to a unitary Banach Gelfand Triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)(I)$.

   This isomorphism leeds to the observation that essentially the identification of $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ boils down to the identification of the bounded linear mappings from $\ell^1(I)$ to $\ell^\infty(I)$, which are of course easily recognized as $\ell^\infty(I \times I)$ (the bounded matrices). The fact that tensor products of $1D$-Wilson bases gives a characterization of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ over $\mathbb{R}^{2d}$ then gives the kernel theorem.

Automatic Gelfand-triple invertibility

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):
Theorem 6. *Assume that for* $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *the Gabor frame operator*

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

is invertible as an operator on $\boldsymbol{L}^2(\mathbb{R}^d)$*, then it is also invertible on* $\boldsymbol{S}_0(\mathbb{R}^d)$ *and in fact on* $\boldsymbol{S}_0'(\mathbb{R}^d)$*.*
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S *is (resp. extends to) an isomorphism of the Gelfand triple automorphism for* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$*.*
In a recent paper K. Gröchenig shows among others, that invertibility of S follows already from a dense range of $S(\boldsymbol{S}_0(\mathbb{R}^d))$ in $\boldsymbol{S}_0(\mathbb{R}^d)$.

Robustness resulting from those three layers:

In the present situation one has also (in contrast to the "pure Hilbert space case") various robustness effects:

1) One has robustness against jitter error. Depending (only) on Λ and $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ one can find some $\delta_0 > 0$ such that the frame property is preserved (with uniform bounds on the new families) if any point $\lambda \in \Lambda$ is not moved more than by a distance of δ_0.

2) One even can replace the lattice generated by some non-invertible matrix $\mathring{A}$ (applied to $\mathbb{Z}^{2d}$) by some "sufficiently similar matrix $\mathcal{B}$ and also preserve the Gabor frame property (with continuous dependence of the dual Gabor atom $\tilde{g}$ on the matrix $\mathbf{B}$) (joint work with N. Kaiblinger, Trans. Amer. Math. Soc.).

Stability of Gabor Frames with respect to Dilation (F/Kaibl.)

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

$$F_g = \big\{(g, L) \in X \times \mathrm{GL}(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k \in \mathbb{Z}^{2d}}\big\}. \tag{4}$$

The set $F_{\boldsymbol{L}^2}$ need not be open (even for good ONBs!). But we have:

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The set $F_{\boldsymbol{L}^2}$ need not be open (even for good ONBs!). But we have:

Theorem 7. (i) *The set* $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ *is open in* $\boldsymbol{S}_0(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.
(ii) $(g, L) \mapsto \widetilde{g}$ *is continuous mapping from* $F_{\boldsymbol{S}_0(\mathbb{R}^d)}$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$.

Stability of Gabor Frames with respect to Dilation (F/Kaibl.)

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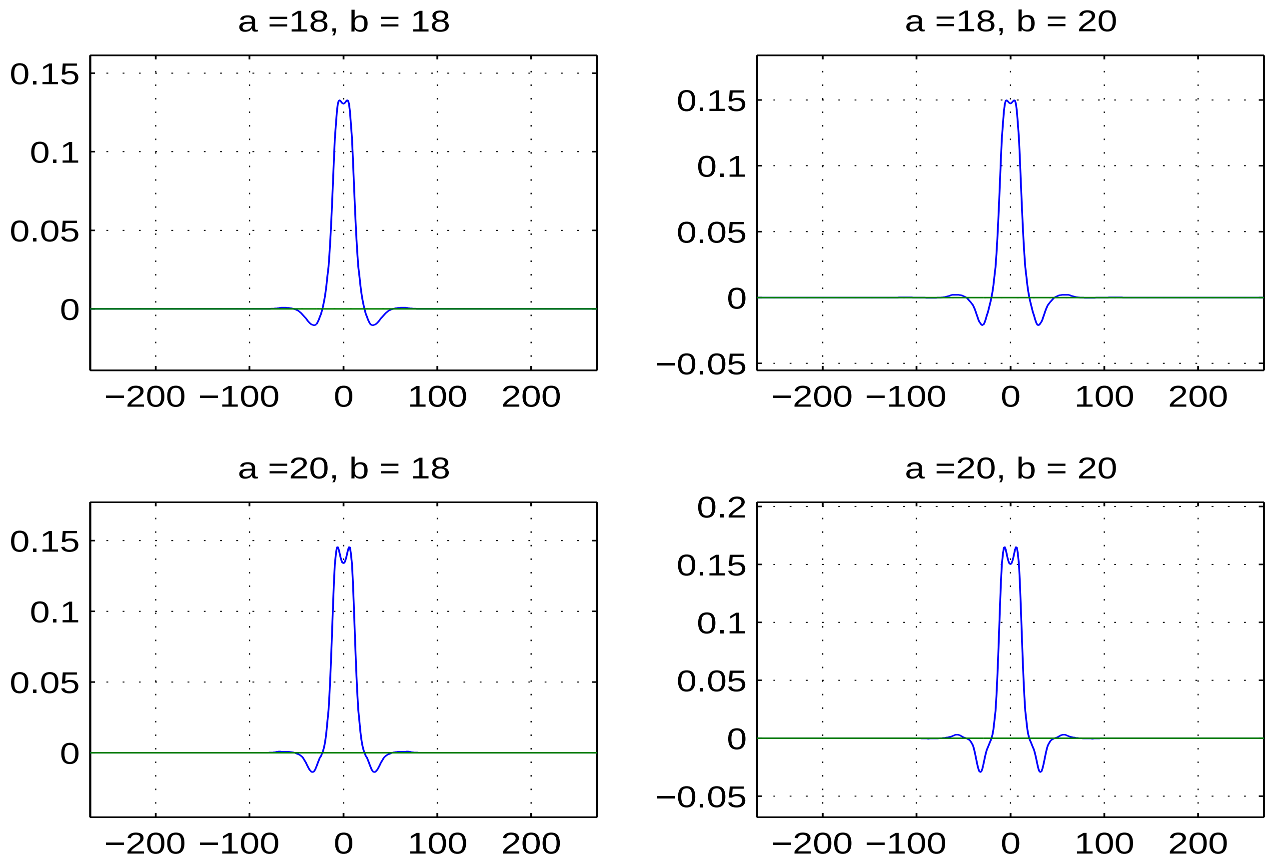
There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Corollary 3. (i) *The set* $F_{\mathcal{S}}$ *is open in* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times \mathrm{GL}(\mathbb{R}^{2d})$.

(ii) *The mapping* $(g, L) \mapsto \widetilde{g}$ *is continuous from* $F_{\mathcal{S}}$ *into* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

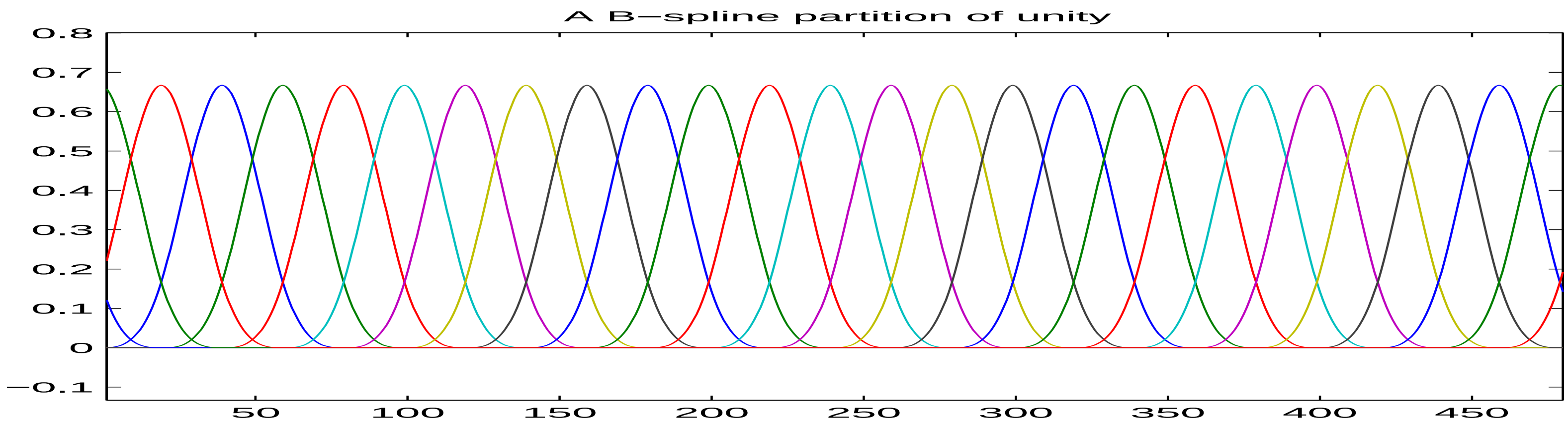
On the continuous dependence of dual atoms on the TF-lattice



Bounded Uniform Partitions of Unity

Definition 3. *A bounded family* $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ *in a Banach algebra* $(\mathbf{A}, \|\cdot\|_A)$ *is a regular* *A-Bounded Uniform Partition of Unity if*

$$\sum_{n\in\mathbb{Z}^d} \psi(x-n) = 1 \quad \textit{for all} \quad x \in \mathbb{R}^d$$



BUPUs and Quasi-interpolation

Having such a partition of unity $\Psi = (\psi_n)_{n \in \mathbb{Z}^d}$ one can of course apply dilations (by some factor $h > 0$) in order to obtain arbitrary fine partitions of unit, upon replacing each of the functions ψ_n by $\psi_{n,h}(x) = \psi_n(x/h)$, let us call this system Ψ_h. It is easy to check that the so-called spline operators $f \mapsto Sp_\Psi$, defined by

$$Sp_\Psi(f)(x) = \sum_n f(n)\psi_n(x)$$

is bounded on $\boldsymbol{S}_0(\mathbb{R}^d)$, and that $Sp_h := Sp_{\Psi_h}$ converges uniformly to f for any $f \in \boldsymbol{C}_0(\mathbb{R}^d)$. But in fact we have

Lemma 4. *For any* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *one has:* $\|Sp_h f - f\|_{\boldsymbol{S}_0} \to 0 \quad$ *as* $h \to 0$.

BUPUs and approximation of $\sigma \in S_0'(\mathbb{R}^d)$ by discrete measures

We have just seen that the (uniformly bounded) family of operators $f \to Sp_h$ converges in the strong operator topology to $Id_{\boldsymbol{S}_0}$, therefore its adjoint will provide a weak-* approximation to the identity in $\boldsymbol{S}_0'(\mathbb{R}^d)$:
We call these operators D_Ψ resp. D_h, because they are *discretization* operators. It is easy to check that $D_\Psi\sigma(f) := \sigma(Sp_\Psi f)$ is of the form

$$D_\Psi(\sigma) = \sum_n \sigma(\psi_n)\delta_n.$$

In this sense we will find that for each $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$: $\sigma = lim_{w^*} D_h\sigma$. Since it is also possible to approximate test functions $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ by compactly supported test functions one finds that the set of finite linear combinations of Dirac measures $\mu = \sum_{j\in F} c_j\delta_{x_j}$, where F is an arbitrary finite sense, are w^*-dense in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Quasi-interpolation and discretization

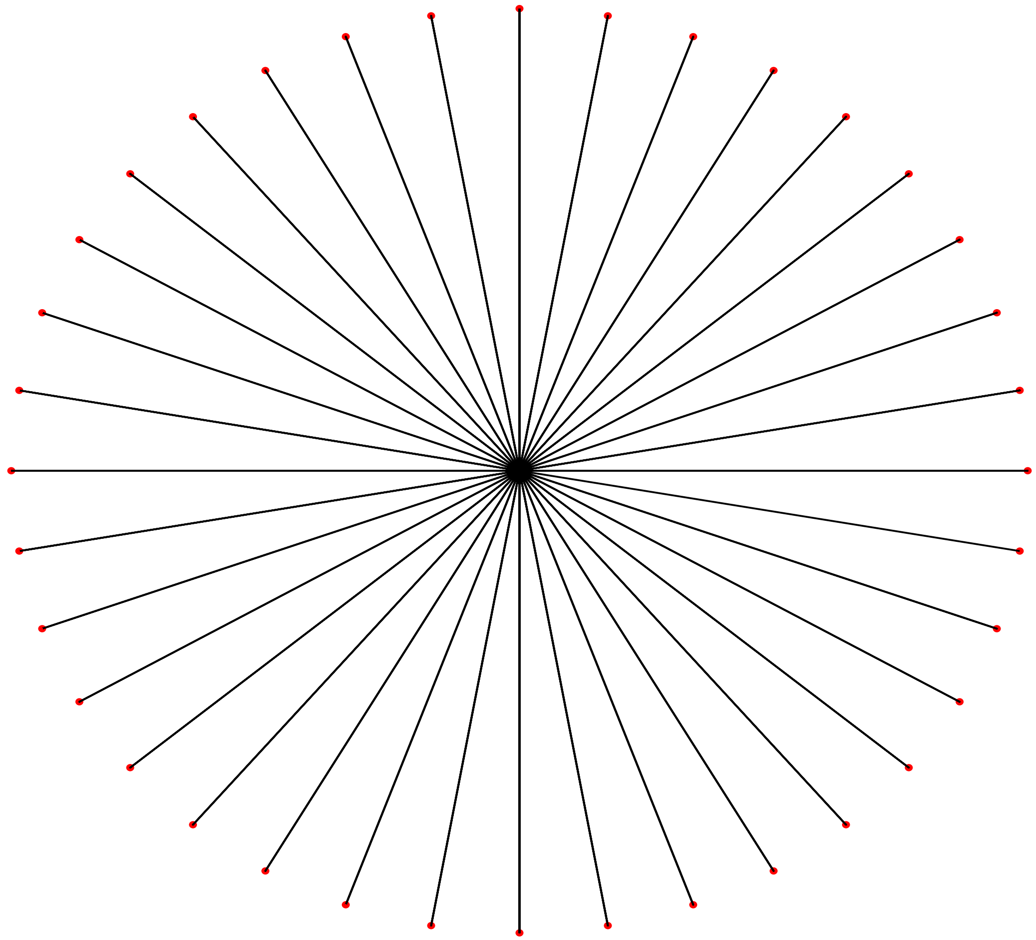
HGFei/Kaiblinger have shown (J. Approx. Th.) that piecewise linear interpolation resp. quasi-interpolation (using for example cubic splines), i.e. operators of the form satisfy

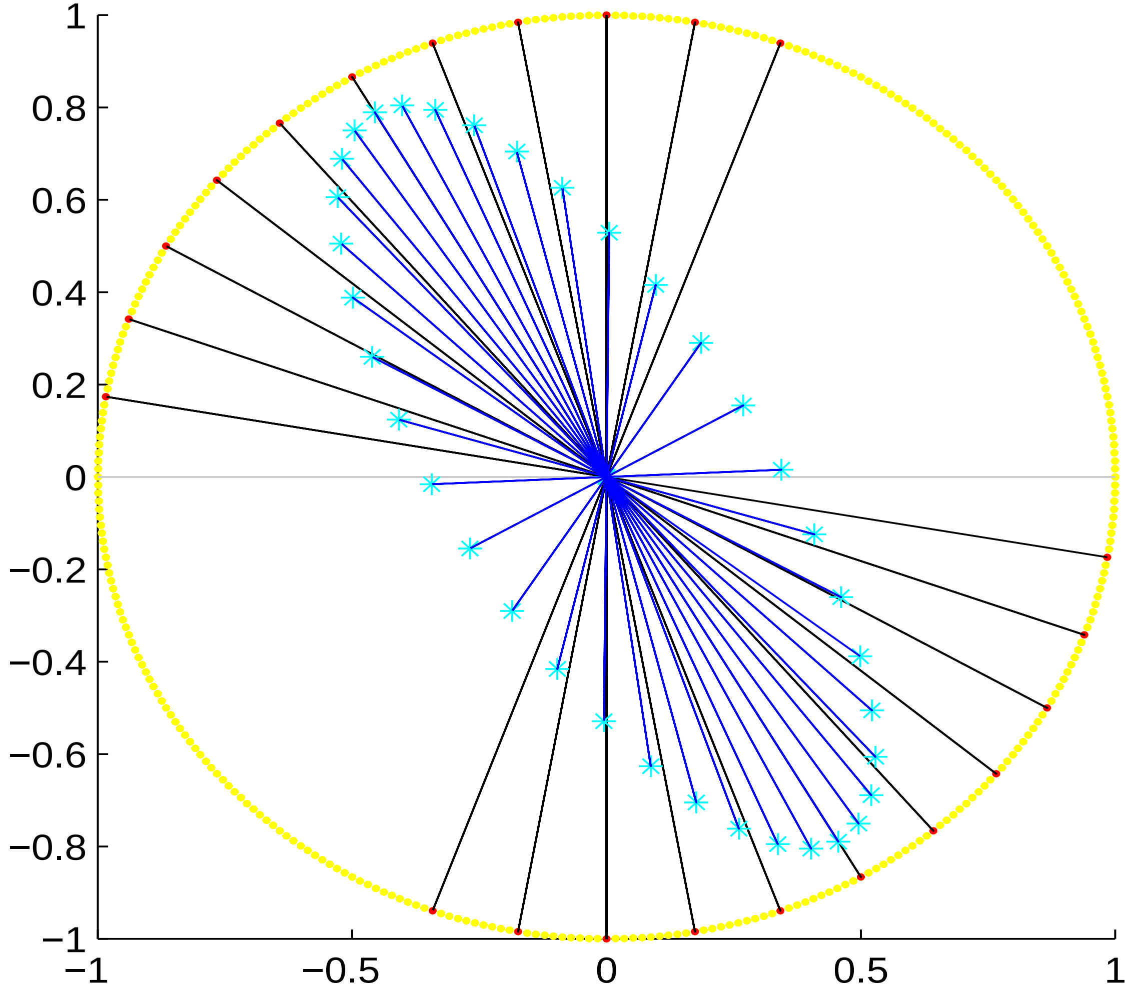
$$Q_h f = \sum_{k \in \mathbb{Z}^d} f(hk) T_{hk} \psi_h$$

are norm convergent to $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ in the $\boldsymbol{S}_0$-norm.
This is an important step for his work on the approximation of "continuous Gabor problems by finite ones" (handled computationally using MATLAB, for example), a subject which has been driven further to the context of Gabor Analysis (using code for the determination of dual Gabor atoms over finite Abelian groups in order to determine approximately solutions to the continuous question).

Some idea about frames and frame multipliers

a frame of redundancy 18 in the plane




1
0.8
0.6
0.4
0.2
0
−0.2
−0.4
−0.6
−0.8
−1
−1
−0.5
0
0.5
1

THE END!

THANK you for your attention! HGFei

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