Banach Gelfand Triples motivated by Time-Frequency Analysis

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OVERVIEW over this lecture **40 MINUTES**

- This is a talk about Banach Gelfand Triples
- explaining some background in time-frequency analysis
- showing some applications in Fourier Analysis
- indicating its relevance for numerical applications
- and for teaching purposes
- OVERALL:
- perhaps changing your view on Fourier Analysis

Calculating with all kind of numbers

The most beautiful equation

$$e^{2\pi i} = 1.$$

It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

$$cos(2\pi) + i * sin(2\pi) = 0 \quad \text{in} \quad \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency (FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \mathbf{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{2}$$

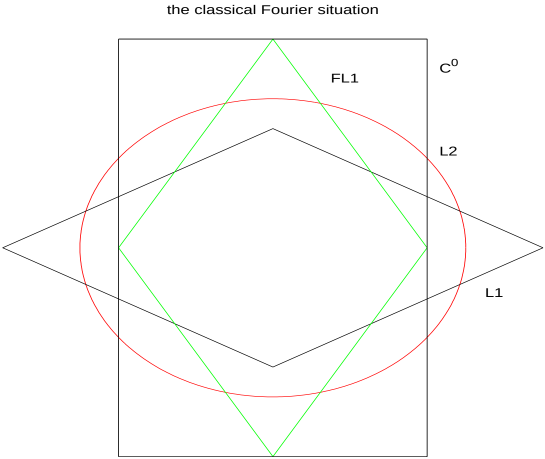
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$.
One often speaks of Fourier analysis being the first step, and the Fourier inversion as a method to build f from the pure frequencies (we talk of Fourier synthesis).

The classical situation with Fourier

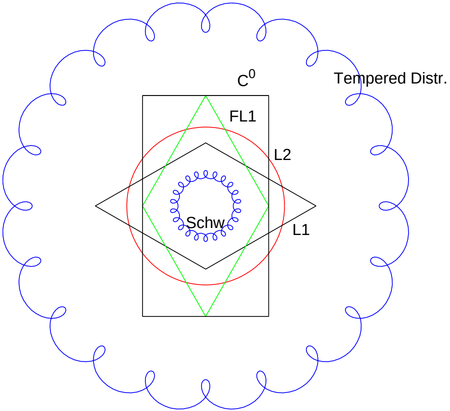
Unfortunately the Fourier transform does not behave well with respect to $\mathbf{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \mathbf{L}^1(\mathbb{R}^d)$ we have $\hat{f} \in \mathbf{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\mathbf{L}^1(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d)$ is isometric in the $\mathbf{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\mathbf{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\hat{f}$.

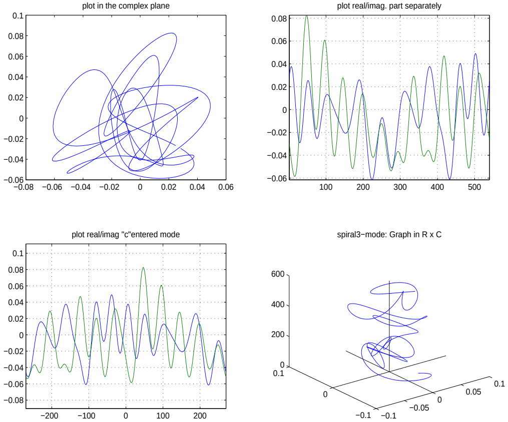
A schematic description of the situation



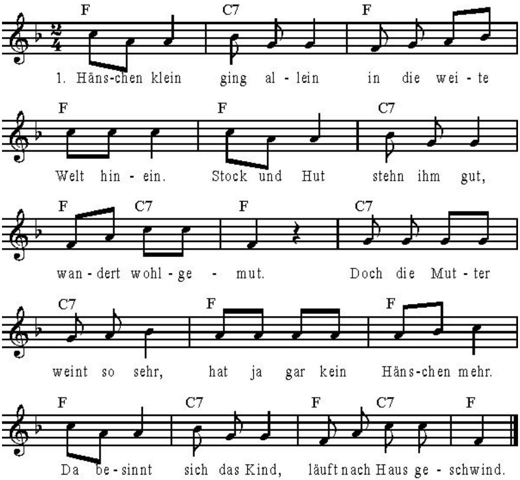
The situation using Schwartz spaces



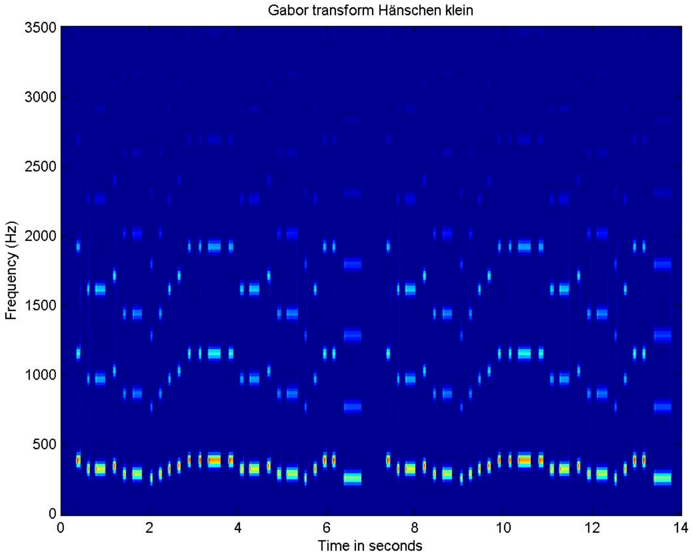
Complex-valued Functions on the Torus



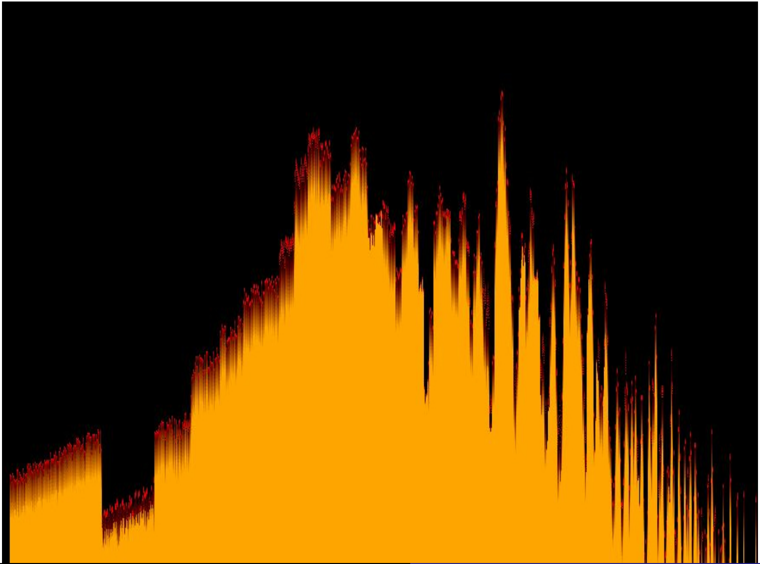
Time-Frequency Analysis and Music



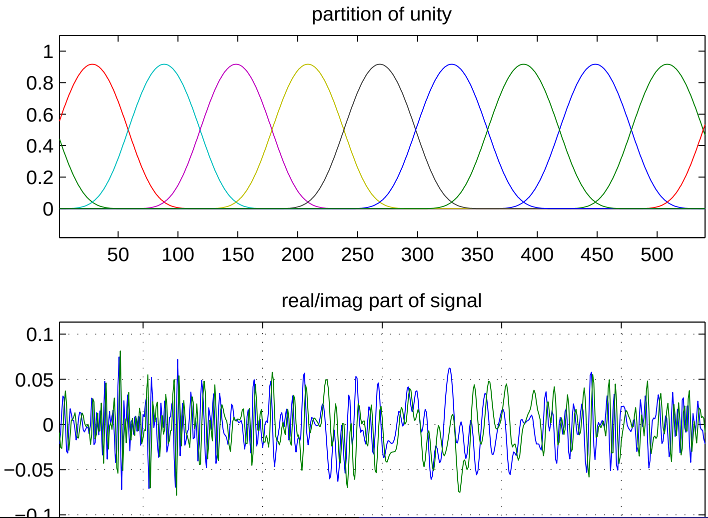
The Short-Time Fourier Transform of this Song



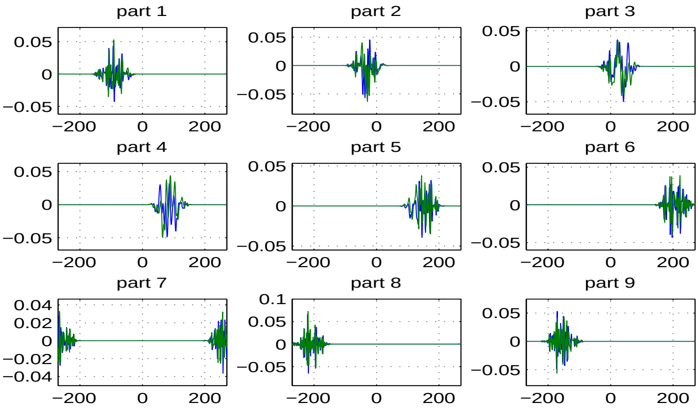
Another (Standard) representation of a Musical STFT



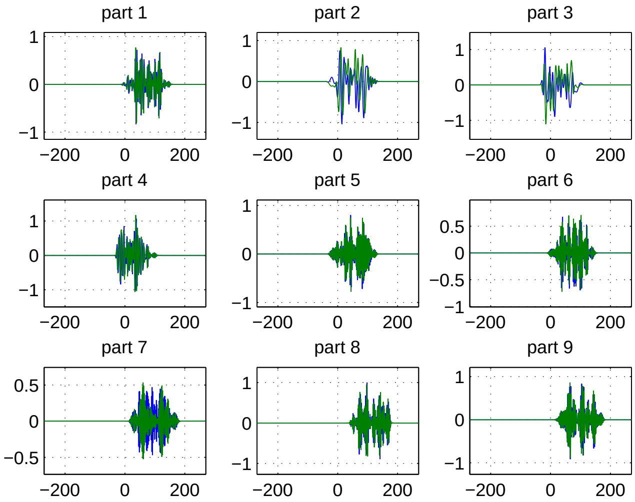
The idea of a “localized Fourier Spectrum”



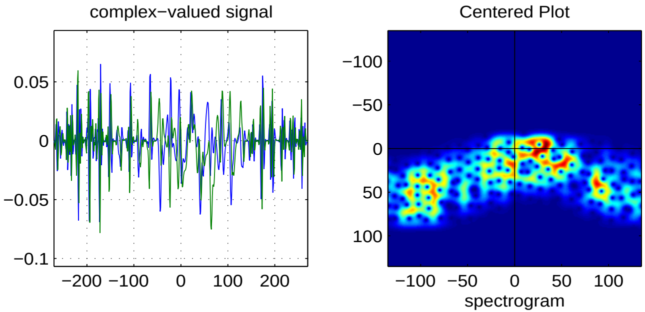
The localized Fourier transform (spectrogram)



Spectral decomposition: variable bandwidth



STFT of a function of “variable band-width”



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

$$(T_x f)\widehat{} = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$*, then the following holds:*

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$*, and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

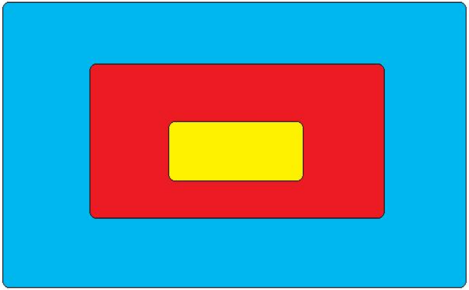
Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

The SO-Banach Gelfand Triple (Rigged Hilbert Space)

TEST FUNCTIONS - Hilbert space L2 - generalized functions = DISTRIBUTIONS



Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\mathbf{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{T}) = \mathbf{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{3}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

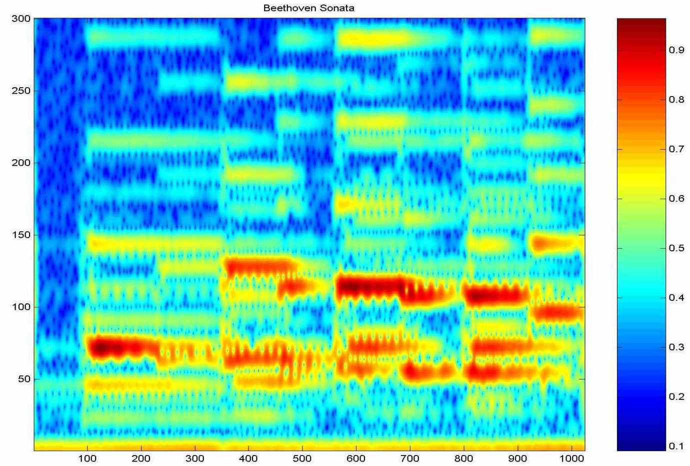
It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
Therefore it is interesting to check what the w^*-convergence looks like:

Lemma

For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ a sequence σ_n is w^*-convergent to σ_0 if and only the spectrograms $V_g(\sigma_n)$ converge uniformly over compact sets to the spectrogram $V_g(\sigma_0)$.

The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

A Typical Musical STFT



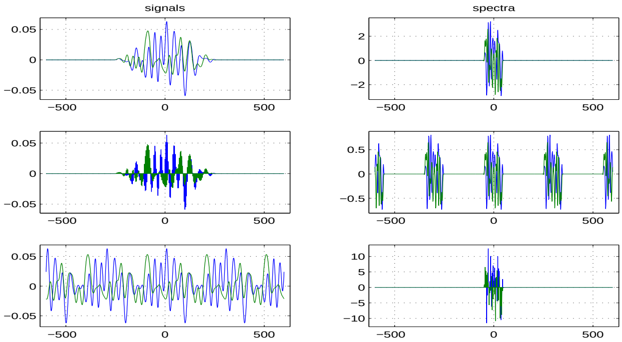
The w^*- topology: dense subfamilies

From the practical point of view this means, that one has to look at the spectrograms of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution $V_g(\sigma_0)$ over compact sets.
The approximation of elements from $\mathbf{S}_0'(\mathbb{R}^d)$ takes place by a bounded sequence.
Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^*-dense subsets of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.
Let us look at some concrete examples: Test-functions, finite discrete measures $\mu = \sum_i c_i \delta_{t_i}$, trigonometric polynomials $q(t) = \sum_i a_i e^{2\pi i \omega_i t}$, or discrete AND periodic measures (this class is invariant under the generalized Fourier transform and can be realized computationally using the FFT).

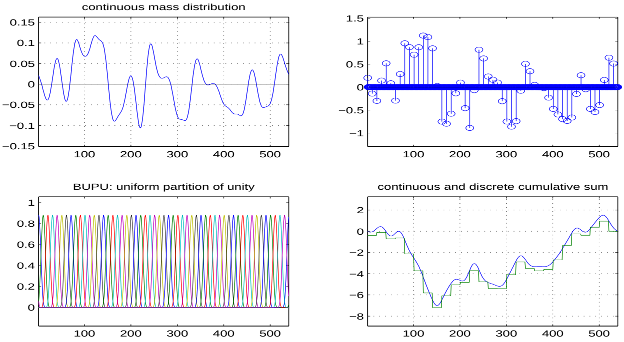
The w^*- topology: approximation strategies

- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \to \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an $\mathbf{L}^1$-Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite disrete sequence using quasi-interpolation (N. Kaiblinger): $Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x)$.

Sampling and Periodization



Adjoint Action on Distributions: Discretization of Mass



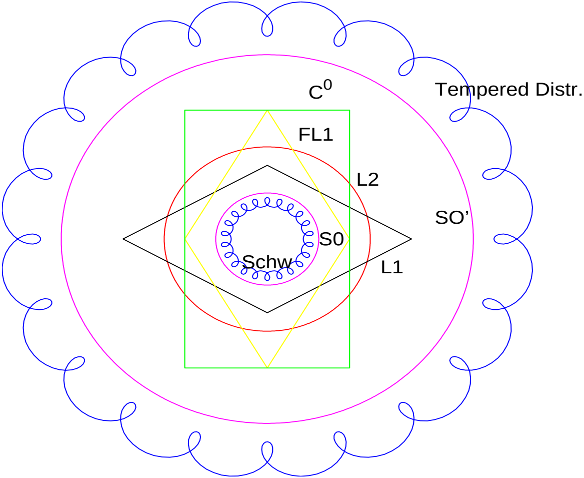
Interesting Consequence for Operators: Kernel Theorem

The Banach Gelfand Triple also appears to be appropriate for a natural generalization of things we are used to have in the context of finite dimensional vector spaces.

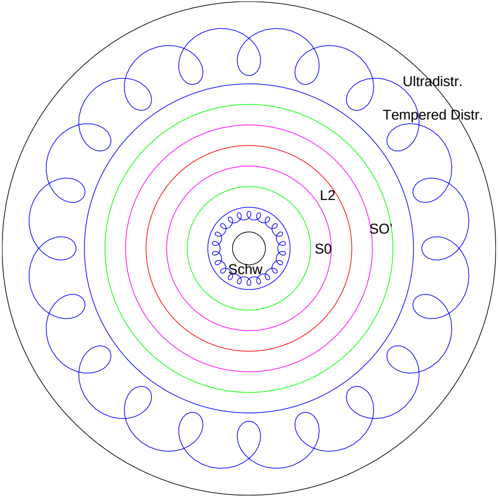
For example: the analogue of the matrix representation of a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ using a uniquely determined (once the bases are fixed) $m \times n$-matrix A: Every linear operator from $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ into $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ has a (distributional) kernel $\sigma \in \mathbf{S}_0'(\mathbb{R}^{2d})$.

This is very much in the spirit of Dirac's calculus!

The new view on the Fourier Transform



Ultradistributions and the Fourier Transform



OUTLOOK on further application areas

- teaching of Fourier Analysis to engineers;
- treatment of generalized stochastic processes: they are interpreted as bounded linear mappings from $\mathbf{S}_0(\mathbb{R}^d)$ to some abstract Hilbert space (of random variables, with expectation zero);
- replacing $\mathcal{S}(G)$ over LCA groups (Schwartz-Bruhat), in a convenient way;
- description of pseudo-differential operators (Kohn-Nirenberg, Weyl calculus, spreading representation);
- numerical approximation of all that, ...

more information are found ...

at the NuHAG Web-Page

www.nuhag.eu

see DB+tools >> Talks (for example)

Afterthoughts

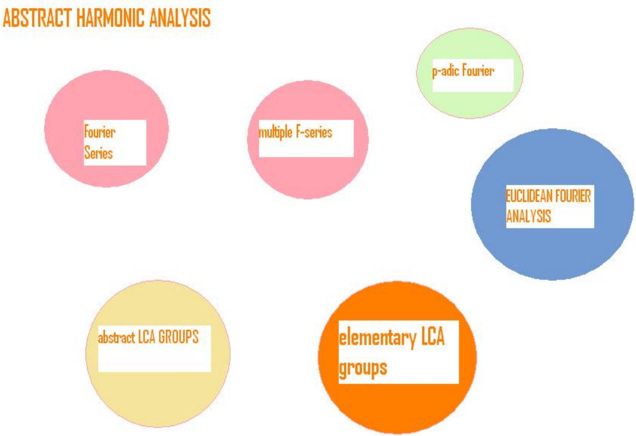
What is the future of Harmonic Analysis?

Is there a place for Abstract Harmonic Analysis?

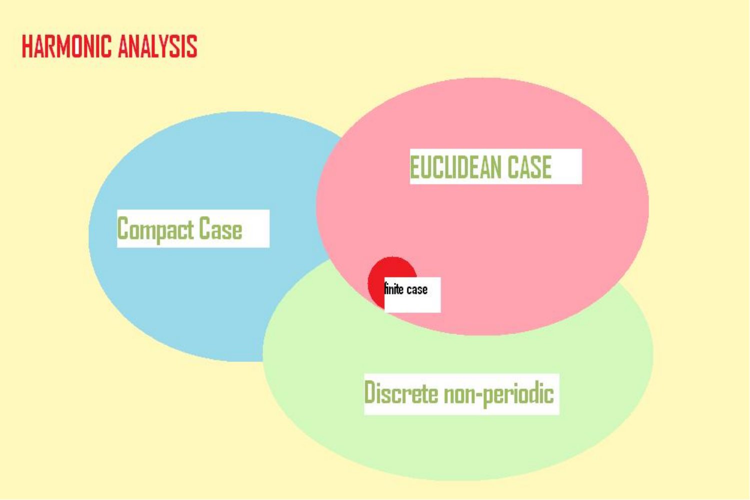
What is the role of Computational Harmonic Analysis?

Constructive versus *realizable* methods!

The classical view of Abstract Harmonic Analysis

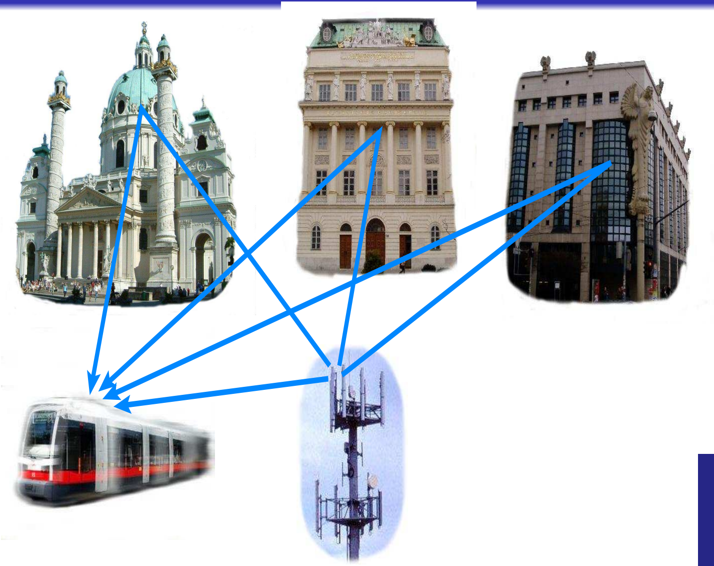


A more INTEGRATED viewpoint



Application Areas: Mobile Communication

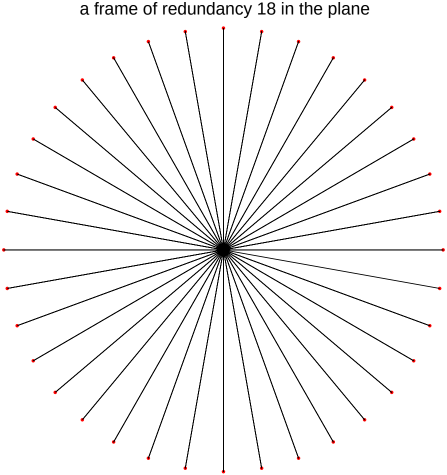
Mobile Communication 2



Audio Processing and Gabor Multipliers 1

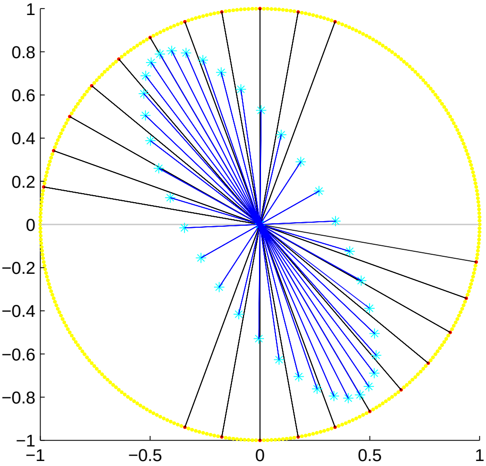
Audio Processing and Gabor Multipliers 2

A generic, high redundancy frame in the plane

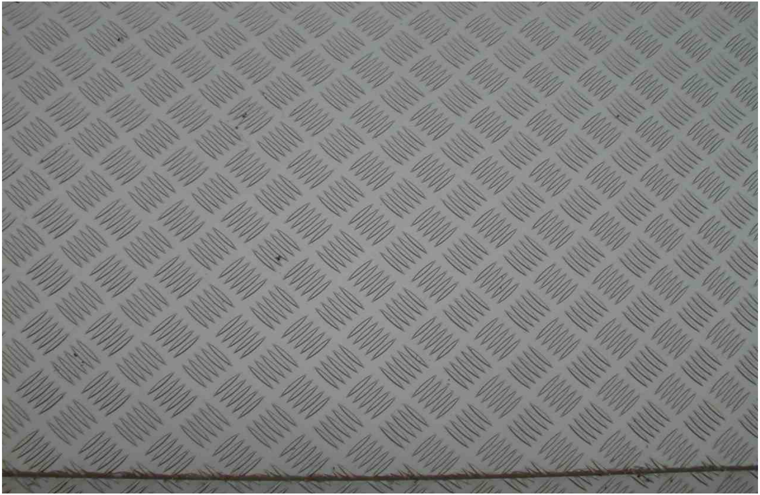


The action of a corresponding frame multiplier

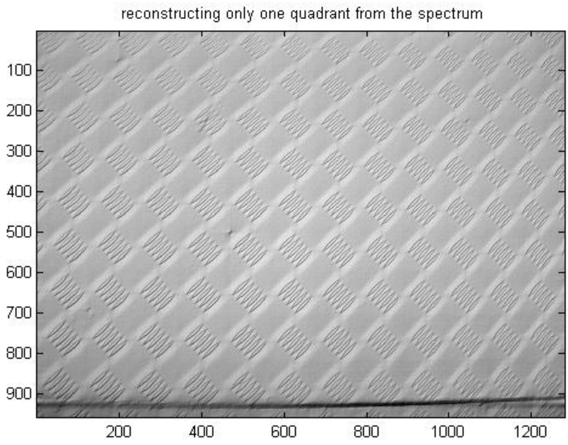
The effect of a frame multiplier in the plane:



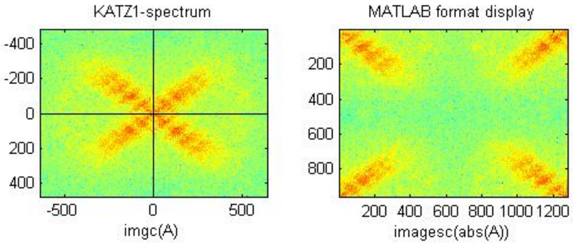
2D-Gabor Transform



2D-Gabor Analysis: Test Images



2D-Gabor Transform: Test-Images 2



2D-Gabor Transform: Plane Waves

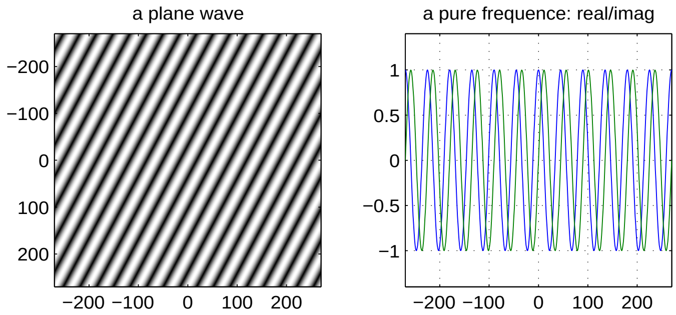


Image Compression: a Test Image