Constructivity versus Realizability

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Irregular Sampling

Let us take a systematic look at various questions related to irregular sampling.
We typically have a model assumption such as band-limitedness or membership in a spline-type space, implying the possibility of reconstruction of a function f in such a space from regular samplings $f(t_i)$.
Normally this is done by building some intermediate auxiliary function (e.g. a nearest neighborhood interpolation), followed by a projection operator.

Existence versus Reconstruction

There are many ways to describe the problem (realistically or superficially). E.g. one *assumes* that the set of kernels (K_i) realising the point evaluations

$$f \mapsto f(t_i) = \langle f, K_i \rangle$$

is a frame in the corresponding Hilbert space, and then one just applies the (inverse) frame operator.

Localization theory helps to predict good off-diagonal decay of the inverse Gramian, hence good concentration of the dual frame $\widetilde{K}_i$. Hence on can be sure that - once the dual frame as been calculated - the reconstruction is possible also for $f \in \mathbf{L}^p_w(\mathbb{R}^d)$, for suitable weighted $\mathbf{L}^p$-classes.

Solving equations numerically

A small algebraic homework:

$$x^2 - \frac{1}{\pi} = 0$$

$$(x + \sqrt{1/\pi}) \cdot (x - \sqrt{1/\pi}) = 0$$

Hence

$$x = \pm \frac{1}{\sqrt{\pi}}.$$

Existence versus Reconstruction

However, CAN we actually do what we are supposed to do according to the description of the various algorithms? We claim that we can - among others

- for correlation coefficients do perfect integrals
- in fact, infinitely many of them
- take the Fourier transform
- take the Fourier transform of a function
- divide by the periodized version
- do infinite linear combinations

- just to name to most crucial steps.

Existence versus Reconstruction

There is an addition problem, with most group theoretical problems (we have all kinds of groups within abstract Harmonic Analysis, we have FFT to imitate the Fourier transform, etc.), so we can do THE ANALOGUE of each problem on a finite group, or on the integers.
So unlike the situation of FEM one does not even notice how problematic the situations is, switching to the finite (dimensional) setting whenever we go to the computer.
In fact, engineers have to be brought to understand the problem, and awareness of the problem is one of our tasks.

Existence versus Reconstruction

Of course **approximation** and **continuity of operations** are the natural things to observe, but now we are dealing with functions, so we need the correct function space norms and approximation procedures in order to ensure good quality of our real world.

So the real challenge is to carefully split between discrete and finite sets of approximate data which are really what the computer can handle, and the connection/embedding of this object into the continuous model (where our functions and distributions live).

Typical Function Spaces to be used

The function spaces to be used are rather *NOT* just $\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^p$ etc., but rather

$$\left(\mathbf{W}(\mathbf{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\mathbf{W}}\right);$$

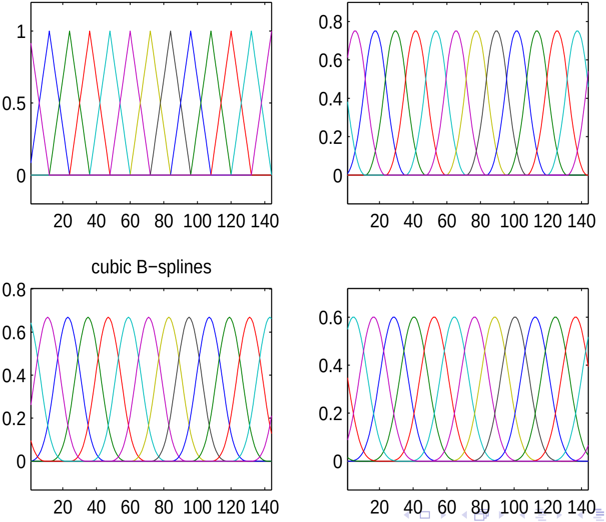
$$\mathbf{W}(\mathbf{L}^2, \ell^1)$$

$$(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1}) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d) = \left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right).$$

Partitions of Unity

One of the standard methods of recovery of smooth signals (resp. signals in spline-type spaces) is to first use the sampling values in order to build from them a step-function or a piecewise linear function, and then project that onto the space of functions under considerations)

Partitions of Unity



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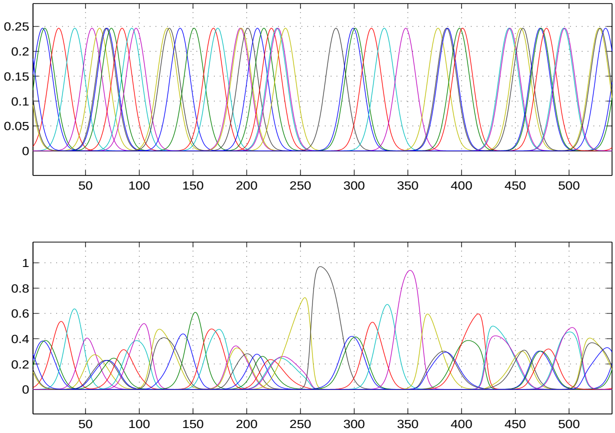

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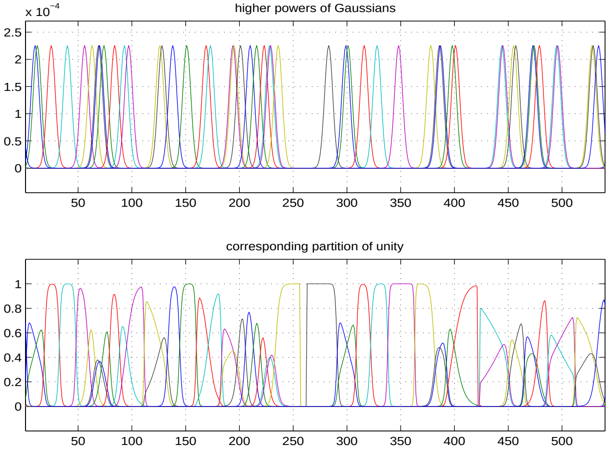

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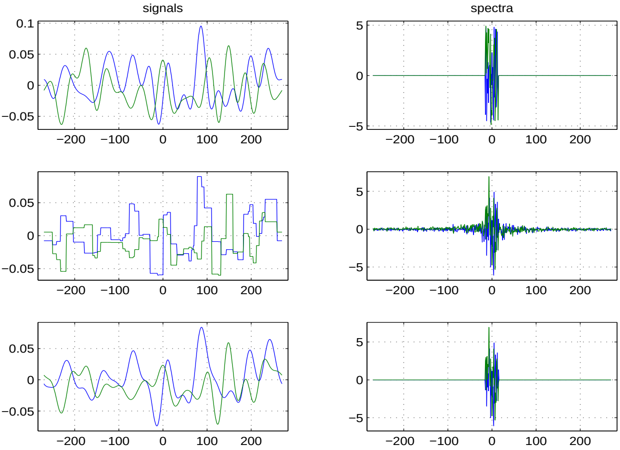

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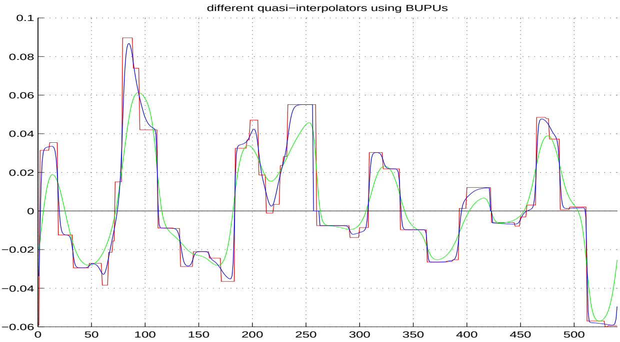

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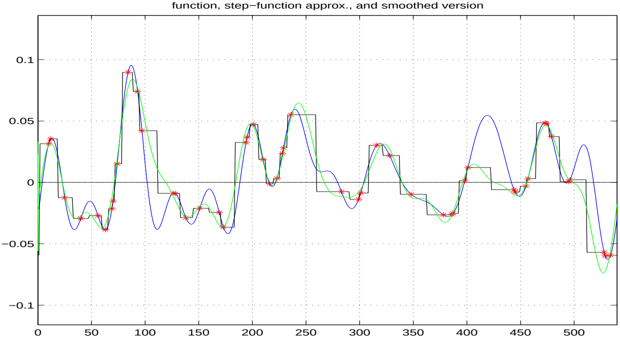

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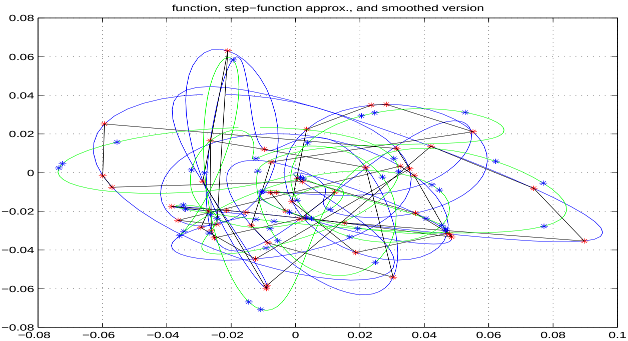

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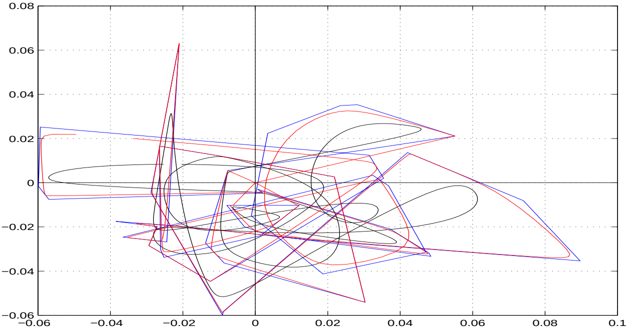


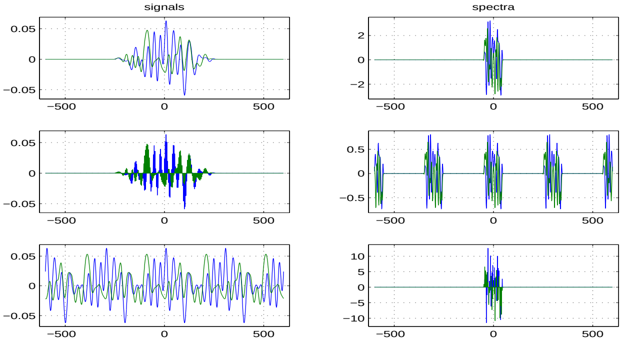
The Fourier transform on $\mathbf{S}_0(\mathbb{R}^d)$ via FFT

There are results by N. Kaiblinger on the approximation of the Fourier transform showing:

Theorem

Given $g \in \mathbf{S}_0(\mathbb{R}^d)$, one can take (in a regular fashion) any sufficiently wide collection of sampling values on any sufficiently fine grid, and use the information in a suitable adjusted $n-$-dim FFT algorithms, such that the FFT-sequence can be used to obtain a continuous function (gridding) which is close to $\hat{f}$ by any given degree required, measured in the $\mathbf{S}_0$-norm (hence accuracy in all the Lp-norms simultaneously).

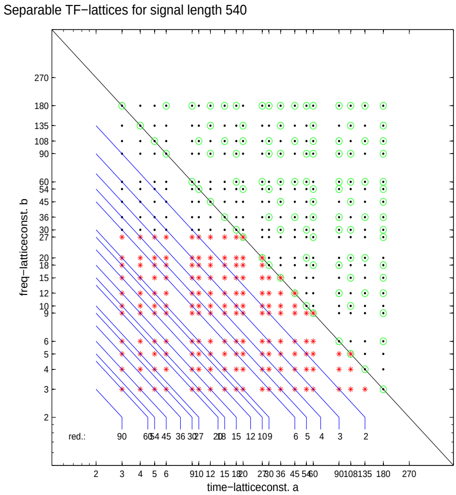
Sampling and Periodization



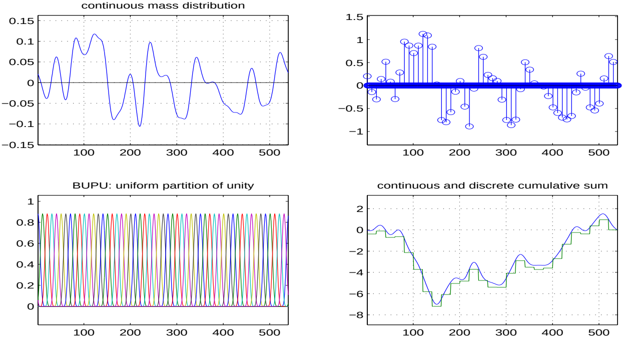
Dual Gabor atoms

Also by N. Kaiblinger we have a result about calculation dual Gabor atoms at a given precision.
In this case it is important to not only discretize appropriately and put the finite result back into $\mathbf{S}_0(\mathbb{R}^d)$ by appropriate quasi-interpolation, but also to use the fact that by choosing N large enough and rich enough of divisors one can find a discrete lattice of comparable redundancy and excentricity to do the job well.

Richness of Subgroups: Wexler Raz



Adjoint Action on Distributions: Discretization of Mass



Generalized Gabor Multipliers

In this subsection we want to indicate the relevance of the $2D$ version for the problem of approximating an operator by a so-called *generalized Gabor multiplier* with respect to the Hilbert Schmidt norm. Recall that an "ordinary" Gabor multiplier is constructed from a pair of "windows" (analysis window γ and synthesis window g), a lattice $\Lambda \lhd \mathbb{R}^{2d}$, and a multiplier sequence $(m_\lambda)_{\lambda\in\Lambda}$ (also called *upper symbol*), typically in $\ell^\infty(\Lambda)$, as follows

$$Tf = \sum_{\lambda\in\Lambda} m_\lambda \langle f, \pi(\lambda)\gamma\rangle \pi(\lambda) g. \qquad (1)$$

The connection between the problem of approximating HS-operators by Gabor multipliers using the KN-calculus is described in earlier papers.

Generalized Gabor Multipliers

A generalized Gabor multiplier is an operator which is a finite sum of operators of the above form. They are also studied by Dörfler and Toressani.

An alternative viewpoint on Gabor multipliers is to define the action of $\mathbb{R}^{2d}$ on operators by $\pi \otimes \pi^*(\lambda)\, T = \pi(\lambda) \circ T \circ \pi(\lambda)^{-1}$, and Q for the rank-one operator $f \mapsto \langle f, \gamma \rangle g$. Then a Gabor multiplier as defined above is an operator of the form

$$T = \sum_{\lambda \in \Lambda} m_\lambda \, \pi \otimes \pi^*(\lambda)\, Q = \sum_{\lambda \in \Lambda} m_\lambda Q_\lambda \qquad (2)$$

if we write $Q_\lambda = \pi \otimes \pi^*(\lambda)\, Q$.

Hence a generalized Gabor multipliers is obtained from a sequence $Q^1, \dots Q^k$ of rank one operators, with analysis windows $\gamma_1, \dots, \gamma_k$ and synthesis windows $g_1, \dots, g_k$, hence T is of the form

$$T = \sum_{j=1}^{k} \sum_{\lambda \in \Lambda} m_\lambda^j Q_\lambda^j. \tag{3}$$

Kohn-Nirenberg Calculus

The Kohn-Nirenberg mapping σ is a unitary mapping from the set of all Hilbert-Schmidt operators with the standard scalar product $\langle T, S\rangle_{HS} := trace(TS^*)$ onto $\mathbf{L}^2(\mathbb{R}^{2d})$. For us it is important that σ intertwines $\pi \otimes \pi^*$ with the ordinary translation operator, i.e. one has

$$\sigma[\pi \otimes \pi^*(\lambda)Q] = T_\lambda[\sigma(Q)]\,,\ \lambda \in \mathbb{R}^{2d}. \tag{4}$$

The Kohn-Nirenberg symbol of Q as above is given by

$$\sigma(Q)(x,\omega) = g(x)\overline{\hat{\gamma}(\omega)}exp(-2\pi ix\omega). \tag{5}$$

Hence the best approximation problem for generalized Gabor multipliers in the Hilbert-Schmidt norm is translated into a best-approximation problem for multi-windows spline-spaces in $\mathbf{L}^2(\mathbb{R}^{2d})$ over phase space. The kernel and the KNS-symbol $\sigma(Q)$ is in $\mathbf{S}_0(\mathbb{R}^{2d})$ if both $\gamma, g \in \mathbf{S}_0(\mathbb{R}^d)$.