Modulation Spaces

A quantitative view on the Riemann-Lebesgue Theorem

It is the motivation of this to describe the family of so-called *modulation spaces*, introduced by the speaker around 1983. As results from the last 10 years show very clearly these spaces are very well suited for a description of many problems in the area of time-frequency analysis. By TF-analysis we understand the analysis of functions and distributions by means of their local Fourier expansions (Short-Time Fourier Transform with respect to a given "window"-function, typically a Gaussian kernel).

The original the original definition was modeled in analogy to the characterization of Besov spaces, by replacing the dyadic partitions of unity by uniform partitions (e.g. integer translates of a given B-spline). The by now *classical* family of modulation spaces $M^s_{p,q}(R^d)$ show very similar behaviour compared to the family of Besov spaces $B^s_{p,q}(R^d)$, with respect to duality, interpolation, trace theorem, and the like. Later on an enlarged family of modulation spaces (sometimes called ultra-modulation spaces) was defined (in the spirit of *coorbit spaces*, jointly developed with Gröchenig), simply defined by the membership of the short-time FT in some translation invariant function space over the so-called time-frequency plane (or phase space).

It will be shown, that modulation spaces are exactly those Banach spaces of distributions which can be characterized by the Gabor coefficients of their elements (e.g. belonging to some weighted mixed-norm L^p-spaces). Among the modulation spaces the Segal algebra $S_0(R^d)$ (corresponding to Gabor coefficients in ℓ^1) and its dual (exactly the space of tempered distibutions with bounded coefficients) appear to be particularly useful, e.g. as a replacement for the Schwartz space and the space of tempered distributions respectively. Both spaces are invariant with respect to the Fourier transform, and all the L^p-spaces contain $S_0(R^d)$ and are embedded into $S_0'(R^d)$. Moreover there is a kernel theorem, i.e. bounded linear mappings from $S_0(R^d)$ to $S_0'(R^d)$ can be characterized by a distributional kernel in $S_0'(R^{2d})$.

As time permits also a few consequences for classical questions, such as the characterization of Fourier multipliers, or the use of summability kernels will be given. The basic reference is: Feichtinger, H.G.; (M. Krishna, R. Radha; S. Thangavelu; ed.) Modulation spaces of locally compact Abelian groups, in Proc. Internat. Conf. on Wavelets and Applications, Allied Publishers, New Delhi (2003) [Chennai, January 2002] p.1-56.

`LINK: http://www.univie.ac.at/NuHAG/NuHAGread/modspa03.pdf`

Riemann-Lebesgue Lemma for functions on the torus:

The Fourier coefficients of an L^1-function decay at infinity.

Hausdorff-Young is connecting this result with the "ideal" case, concerning L^2-functions: Pareseval's relationsship tells us, that the FT is isometric between $\boldsymbol{L}^2(\mathbb{T})$ and $\ell^2(\mathbb{Z})$. Problems with the inversion theorem suggest to distinguish the elements which have an absolutely convergent Fourier series $\boldsymbol{A}(\mathbb{T})$. Of course one can talk about the dual space of $\boldsymbol{PM}(\mathbb{T}) := \boldsymbol{A}(\mathbb{T})'$ and call it the space of pseudo-measures, in order to come up with the following observation: The preimage of the triple of Banach spaces $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$ is the triple $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$.

The Fourier transform is also well compatible with duality of spaces so that we can also transfer the fact that $\ell^1(\mathbb{Z})$ is the dual space of $c_0(\mathbb{Z})$ (the null-sequences over $\mathbb{Z}$) into this picture: by calling $\mathcal{F}^{-1}(c_0)$ the space of pseudo-functions, we can claim, that the pseudo-functions, which are the closure of $\boldsymbol{A}(\mathbb{T})$ within $\boldsymbol{PM}(\mathbb{T})$, whose dual space is just $\boldsymbol{A}(\mathbb{T})$.

So the Riemann-Lebesgue Lemma is a statement that $\boldsymbol{L}^1(\mathbb{T})$ is contained in the pseudo-functions (as a proper subspace). Since $\boldsymbol{L}^p(\mathbb{T}) \subseteq \boldsymbol{L}^1(\mathbb{T})$ for any $p \geq 1$ this is true for general L^p-functions.

When it comes to general LCA groups, including the Euclidean case of $G = \mathbb{R}^d$ the situation becomes more involved. In particular, there are *no inclusion relations* between the L^p-spaces, and the standard definition of the Fourier algebra $\mathcal{F}L^1 = \mathcal{F}(\boldsymbol{L}^1(\mathbb{R}^d))$ does not help very much in order to prove the inversion theorem (e.g. in a pointwise sense), leave alone the validity of Poisson's formula, for which a number of additional assumptions are required (i.e. a good combination of decay/smoothness).

The only theorem which is valid *is Plancherel's Theorem*, describing the Fourier transform as a unitary mapping from $\boldsymbol{L}^2(\mathbb{R}^d)$ onto itself. The *Riemann-Lebesgue Theorem* can be described as Banach algebra homomorphism from $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$ into $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, where the first is of course understood as Banach algbra with respect to convolution.

Aside from $\boldsymbol{L}^2(\mathbb{R}^d)$ there is only one other "nice" space of functions which is invariant under the Fourier transform, namely $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions, which gives rise to an extended Fourier transform for the space $\boldsymbol{\mathcal{S}'}(\mathbb{R}^d)$ of tempered distributions. Obviously this is a highly useful tool in the theory of partial differential equations. Unfortunately is generalization to LCA groups, to Schwartz-Bruhat space $\boldsymbol{\mathcal{S}}(G)$ is not at all easy to handle or even define.

I want to explain next that time-frequency analysis methods, in particular the theory of Gabor expansions, allows to establish of Banach spaces of functions resp. tempered distributions which exhibit properties similar to those for the triple $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$.

Since the global Fourier transform is not able to describe well the local properties of a function one it is nowadays ("post wavelet area") natural to make use of a localized version of the Fourier transform. Indeed, engineers have used the *Short-time Fourier Transform* , also called the *Sliding Window Fourier Transform* long before wavelet theory had been discovered, i.e. by 1977.

Although all the results described below can be properly formulated in the context of locally compact Abelian groups, especially for $\mathbb{R}^d$, we restrict our attention to $\mathbb{R}$.

$$\begin{aligned} T_x f(z) \quad &:= \quad f(z-x) \\ M_s f(z) \quad &:= \quad e^{2\pi i s \cdot z} \cdot f(z) \\ \pi(\lambda) = \pi(t,s) \quad &:= \quad M_s T_z \end{aligned}$$

The short-time Fourier transform of $f \in \boldsymbol{L}^2(\mathbb{R})$ with respect to some window $g \in \boldsymbol{L}^2(\mathbb{R})$ is defined as the following correlation function of $\lambda \in \mathbb{R} \times \hat{\mathbb{R}}$:

$$V_g f(\lambda) = \langle f, \pi(\lambda g) \rangle$$

It would be natural to make use - similar to the basis or "pure frequencies" in the case of $\mathbb{T}$ which has tow parameters, one for the time, the other for the frequency. Although there is a simple way of creating such an orthonormal basis for $\boldsymbol{L}^2(\mathbb{R})$ by restricting $f \in \boldsymbol{L}^2(\mathbb{R})$ to the sequence of intervals. This corresponds to an orthonormal system of the form $\pi(n,m)\mathbf{b}$, where $\mathbf{b}$ is the box function or indicator function for $[0,1]$, and (n,m) runs through the Neumann lattice $\mathbb{Z} \times \mathbb{Z}$.

Unfortunately not even a test function will have - in general - well decaying coefficients with respect to such an orthonormal system, because jumps between the left and the right end of the interval prohibit that the restriction of the test function belongs to $\boldsymbol{A}(\mathbb{T})$.

According to the Balian-Low theorem it is *not possible at all* to find a function which would at the same time allow to characterize the usual test functions by well decaying coefficients and at the same time forming a complete ONB for $\boldsymbol{L}^2(\mathbb{R})$. The wish (following D. Gabor, 1946) of having an ONB of the form $\pi(n,m)g$ and the request of completeness in $\boldsymbol{L}^2(\mathbb{R})$ are not compatible!

It has turned out in the last 20 years that one can have a very good understanding of so-called Gabor (Banach) frames, which are redundant, i.e. do not lead to unique coefficients for functions to be represented.

Let us discuss these Banach frames by way of a particular example: We fix a pair of lattice constants (a, b) with $ab < 1$, and consider Gabor systems arising from the specific Gabor atom $g = g_0$, the Gaussian function, given by $g_0(t) = e^{-\pi t^2}$. It is invariant with respect to the Fourier transform and has the extra advantage of being optimally concentrated in a TF-sense, expressed by achieving equality in the Heisenberg uncertainty relation.

The **Gaussian Gabor family generated by the triple** (g_0, a, b) is then given as

$$g_{n,m} := \pi(an, bm)g_0, \quad n, m \in \mathbb{Z}.$$

It is well known by know that for any such family (independent of the particular choice of (a, b)), is a frame for the Hilbert space $\boldsymbol{L}^2(\mathbb{R})$, in particular there exists $C = C(a, b,)$ and a linear mapping from $\boldsymbol{L}^2(\mathbb{R})$ into $\big(\boldsymbol{\ell}^2(\mathbb{Z}^2), \|\cdot\|_2\big)$, $f \mapsto \mathbf{c} = \mathbf{c}(f)$ such $\|\mathbf{c}\|_2 \leq C\|f\|_2$ and

$$f = \sum_{\mathbb{Z}\times\mathbb{Z}} c_{n,m} g_{n,m}.$$

$\mathbf{c}$ is the minimal norm solution to the problem. Given this situation it is natural to ask what the functions are for which this series is absolutely convergent. Obviously these functions belong to any Banach space containing g_0 and for which the opertors $\pi(\lambda)$ are all isometric, such as all the $\boldsymbol{L}^p$-spaces.

Let us give the following definition:

$$\boldsymbol{S}_0(\mathbb{R}) = \{f \in \boldsymbol{L}^2(\mathbb{R}),\ \mathbf{c} \in \boldsymbol{\ell}^1(\mathbb{Z} \times \mathbb{Z})\},$$

endowed with the norm $\|f\|_{\boldsymbol{S}_0(\mathbb{R})} := \|\mathbf{c}\|_{\boldsymbol{\ell}^1(\mathbb{Z}\times\mathbb{Z})}$.

Then it is already clear that $\boldsymbol{S}_0(\mathbb{R})$ is a Banach space, continuously embedded into $\boldsymbol{L}^p(\mathbb{R})$ for any $p \geq 1$. Moreover, it is easy to verify that $\mathcal{F}(\boldsymbol{S}_0(\mathbb{R})) = \boldsymbol{S}_0(\mathbb{R})$, if we have $a = b$.

From *this* definition it is not clear why the space should not depend on the lattice constants (a, b), and why the space should have an equivalent norm for which it is isometrically invariant under the operators $\pi(\lambda)$, $\lambda \in \mathbb{R} \times \mathbb{R}^d$.

This follows from an early result in the area which expresses the norm on $\boldsymbol{S}_0(\mathbb{R})$ using the STFT:

THEOREM: $f \in \boldsymbol{S}_0(\mathbb{R})$ if and only if $V_{g_0}f \in \boldsymbol{L}^1(\mathbb{R} \times \mathbb{R}^d)$. Moreover, for any (a, b) with $ab < 1$ the discrete norm as defined above is equivalent to the "continuous" norm $\|f\|_{\boldsymbol{S}_0(\mathbb{R})} := \|V_{g_0}f\|_{\boldsymbol{L}^1(\mathbb{R}\times\mathbb{R}^d)}$.

The minimality and the various invariance properties of this space (described as the "minimal TF-invariant Segal algebra" by the author in 1979) makes this Banach space an ideal substitute for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R})$.

Since the coefficient mapping extends to $\boldsymbol{\mathcal{S}}'(\mathbb{R})$ the following characterization is of interest: $\sigma \in \boldsymbol{S}_0'(\mathbb{R})$ if and only if it has a representation with $\mathbf{c} \in \ell^\infty(\mathbb{Z} \times \mathbb{Z})$.

It is helpful to introduce the notion of a Gelfand Triple:

A Banach space $(B, \| \cdot \|_B)$ which is continuously and densely embedded into some Hilbert space $\mathcal{H}$ gives raise to the *Gelfand triple* $(B, \mathcal{H}, B')$. Note that one has w^*-density of $\mathcal{H}$ in B'.

A *homomorphism of Banach Gelfand triples* is a bounded linear mapping, which maps the dual spaces into each other, but also the intermediate Hilbert space and the underlying Banach spaces, and if it is continuous with respect to the corresponding norm topologies as well as $w^* - w^*$-continuous on the dual spaces. It is called a unitary homomorphism if it is unitary at the Hilbert space level.

In this sense there is a larger picture to the situation described above: The mapping $f \mapsto \mathbf{c}$ is a right inverse to the synthesis mapping $\mathbf{c} \mapsto \sum_{\mathbb{Z}\times\mathbb{Z}} c_{n,m} g_{n,m}$, but it extends from the Hilbert space setting to a *retract* between the Gelfand triples $(\boldsymbol{S}_0(\mathbb{R}), \boldsymbol{L}^2(\mathbb{R}), \boldsymbol{S}_0'(\mathbb{R}))$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z} \times \mathbb{Z})$.

Another statement describes the Fourier transform:

THEOREM: $\mathcal{F}$ defines a unitary Gelfand triple automorphism on the Gelfand triple $(\boldsymbol{S}_0(\mathbb{R}), \boldsymbol{L}^2(\mathbb{R}), \boldsymbol{S}_0'(\mathbb{R}))$. It is the unique GT-isomorphism mapping the "pure frequecies" $x \mapsto exp(2\pi isx)$ into the Dirac measures δ_s.

The triple $(\boldsymbol{S}_0(\mathbb{R}), \boldsymbol{L}^2(\mathbb{R}), \boldsymbol{S}_0'(\mathbb{R}))$ is also a suitable tool to establish the so-called kernel theorem. Before doing so let us recall that the standard Hilbert space theory allows to identify $\boldsymbol{L}^2(\mathbb{R}^{2d})$ with the Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ in a standard way. In the S_0-setting this unitary mapping extends to a $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$. Hence we have:
THEOREM: The kernel theorem establishes a unitary Gelfand triple isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R})$ and $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$.

For the case of a kernel in $\boldsymbol{S}_0$ one can find the "kernel" exactly as one finds the entries of a matrix (by applying the matrix to a unit vector): $K(x, y) = T(\delta_y)(x)$ (which is well defined because $\delta_y \in \boldsymbol{S}_0'(\mathbb{R})$ and its image under T belongs to $\boldsymbol{S}_0(\mathbb{R})$ (which is a space of continuous functions).

Another principe is the Kohn-Nirenberg representation of (pseudo-) differntial operators, or the spreading representation: in the finite case one can write every $n \times n$ matrix as a linear combination of matrices representing (cyclic) TF-shifts.
THEOREM: The spreading mapping is a unitary Gelfand triple isomorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R})$ into itself which is uniquely determined by the fact that it maps the kernel of the TF-shift operators $\pi(\lambda)$ onto δ_λ for any $\lambda \in \mathbb{R} \times \mathbb{R}$.

There are various types of more general modulation spaces, which can all be characterized by the behavior of the STFT (with respect to some Schwartz window g) of their elements (so they provide an alternative description for the standard Sobolev spaces in terms of Gabor expansions).

The classes $\boldsymbol{M}^s_{p,q}(\mathbb{R})$ (introduced around 1983) are modelled after the corresponding family of Besov (and Triebel-Lizorkin) spaces. They show similar properties, but they coincide with their Besov counterparts only for the case $p = q = 2$.

One can show duality and interpolation results for this family of spaces, and even a trace theorem is true, with the same loss of smoothness as in the case of Besov spaces.

There is an interesting alternative approach using radial symmetric weights of polynomial type. Especially interesting are the spaces characterized b a weighted $\boldsymbol{L}^p$-condition on their STFTs. These spaces. These spaces are invariant with respect to the Fourier transform but also under the whole metaplectic group (e.g. fractional Fourier transforms as well). For the case $p = 2$ these spaces can be completely characterized via weighted ℓ^2-conditions on their coefficients with respect to the classical Hermite orthonormal basis.