The Richness of Banach Spaces within a Rigged Hilbert Space resp.: Banach Gelfand Triple

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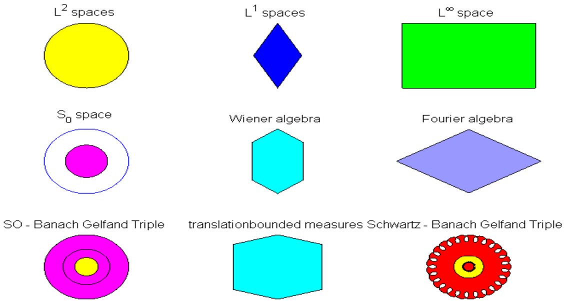
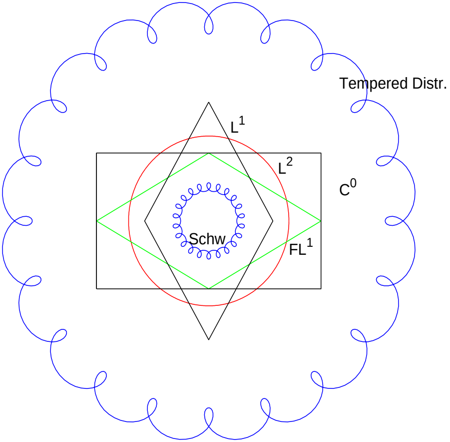
OVERVIEW over this lecture **28 MINUTES**

- The *classical view* on the **Fourier transform**, using $\mathbf{L}^1(\mathbb{R}^d), \mathbf{L}^2(\mathbb{R}^d), \mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d)$;
- Present some reflections concerning the (generalized) Fourier transform;
- Offer some hints concerning the **Banach Gelfand Triple** based on the Segal algebar $\mathbf{S}_0(\mathbb{R}^d)$;
- Define **Standard Spaces** and show their richness;
- Indicate a number of constructions within this family of space;
- Present a number of images of function spaces;

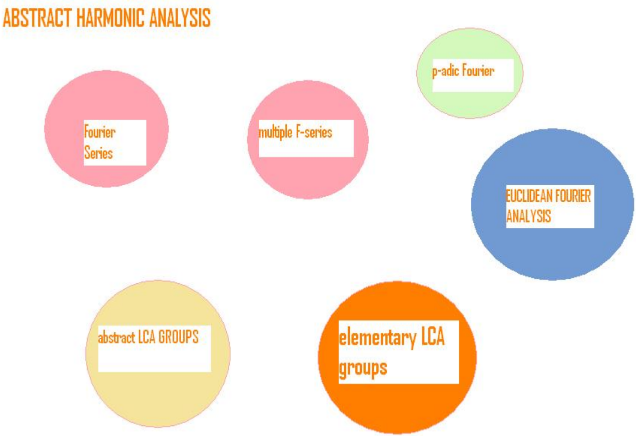
A variety of function spaces



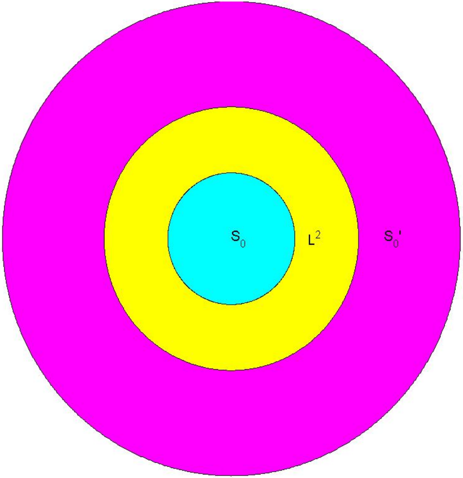
The classical view on the Fourier Transform



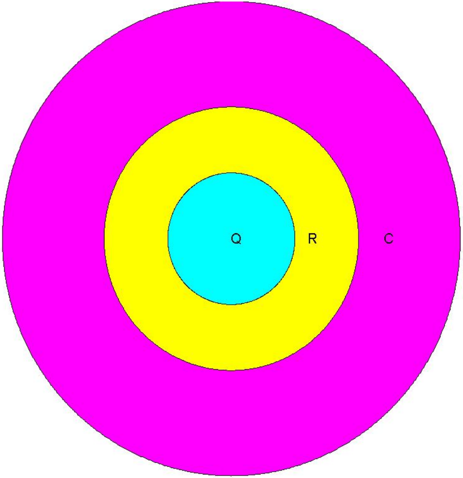
The classical view on the Fourier Transform



The classical view on the Fourier Transform



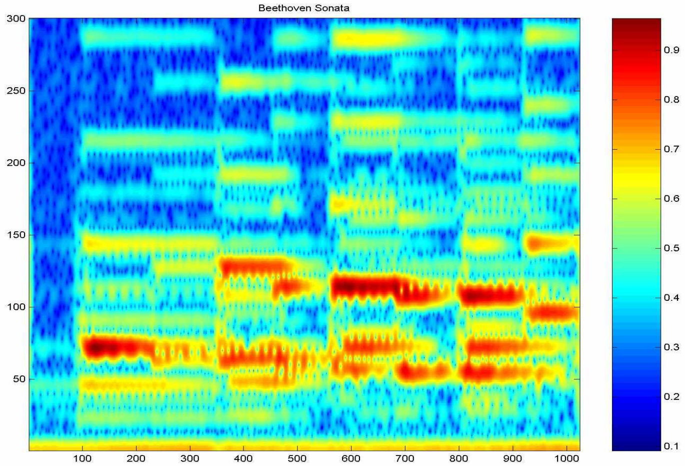
The classical view on the Fourier Transform



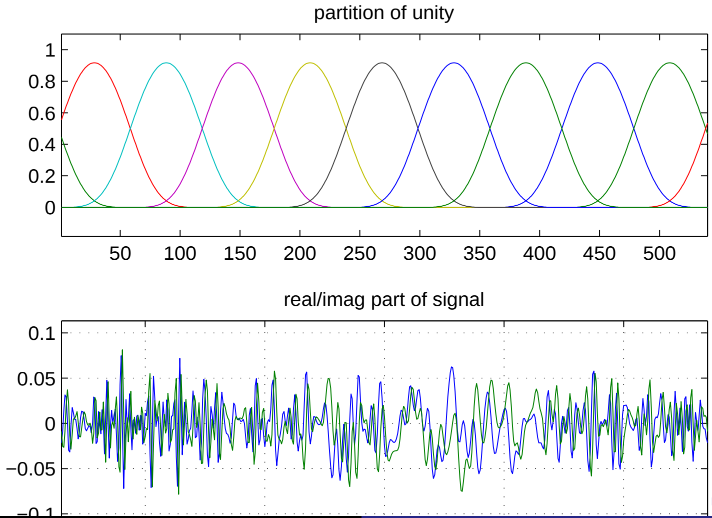
How and where do we our calculations

- All the actual computations are done in $\mathbb{Q}$, e.g. using finite decimal expressions, think of multiplications of fractions!
- The real number $\mathbb{R}$ have the advantage of being a *complete* metric spaces, this allows us to *define* numbers such as $\sqrt{2}$.
- Still we cannot solve quadratic equations, and *by the miracle* of adding the *imaginary unit* (i.e. introduce new objects: pairs of real numbers with a new multiplication!) one has an even more comprehensive field;
- IMPORTANT: each time one has a natural embedding of the smaller field within the larger object (e.g. convert fractions into periodic infinite decimal expressions);

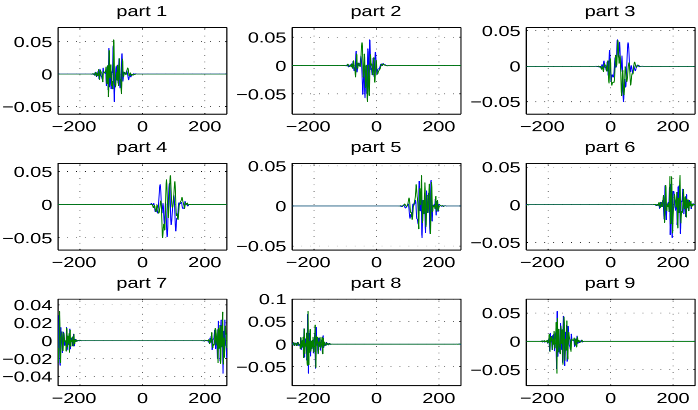
A Typical Musical STFT



The idea of a “localized Fourier Spectrum”



The localized Fourier transform (spectrogram)



Definition of the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$

A continuous, integrable function f on $\mathbb{R}^d$ belongs to Feichtinger's algebra $\mathbf{S}_0(\mathbb{R}^d)$, if its short-time Fourier transform

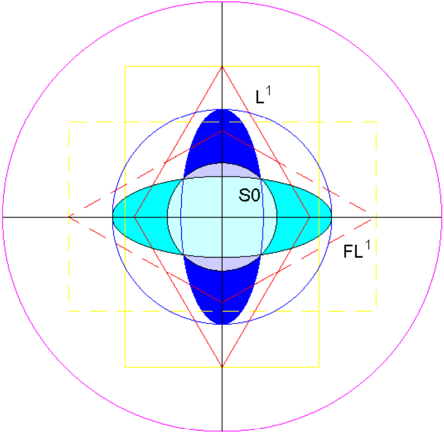
$$V_g f(x,\omega) := \int_{\mathbb{R}^{2d}} f(t)\,\overline{g(t-x)}\,e^{-2\pi i\,\omega\,t}dt, \qquad x,\omega \in \mathbb{R}^d,$$

is integrable, where $g(t) := e^{-\pi|t|^2}$ is the Gaussian window. Here, $|t|$ is the Euclidean norm. The $\mathbf{S}_0$-norm is given by

$$\|f\|_{\mathbf{S}_0} := \int_{\mathbb{R}^{2d}} |V_g f(x,\omega)|\,dx\,d\omega.$$

For various useful characterizations of $\mathbf{S}_0$ and its significance in time-frequency analysis,
The Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ is also described as the Wiener amalgam space $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$, where the local norm is the Fourier algebra norm. In particular, the compactly supported functions in $\mathcal{F}\mathbf{L}^1$ and in $\mathbf{S}_0(\mathbb{R}^d)$ are the same.

Intersection of Weighted L2-spaces with Sobolev



Describing the Fourier Transform

- At the level of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ the Fourier transform and! it's *inverse* are well defined integral transformation, even *Poisson's formula* is strictly valid;
- At the level of $\left(\mathbf{L}^2(\mathbb{R}^d),\ \|\cdot\|_2\right)$ one can express the fact that $\mathcal{F}$ is *unitary mapping*, preserving orthogonality and "energy";
- At the distributional level one can characterize the linear mapping which maps "pure frequencies" (in $\mathbf{L}^\infty \subset \mathbf{S}_0'(\mathbb{R}^d)$) to the corresponding Dirac measures;

The kernel theorem (despite lack of nuclearity!)

- $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is a ptw. and convolutive algebra, invariant under the Fourier transform;
- The so-called tensor product property of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ allows to prove a so-called kernel theorem: Every linear operator from $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ into *SORdPN* can be uniquely represented by a distributional kernel from $\mathbf{S}_0'(\mathbb{R}^{2d})$;
- The integral operator is *regularizing*, i.e. mapping w^*--convergent sequences in $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ into norm convergent sequences in $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ if and only if it is in $\mathbf{S}_0(\mathbb{R}^{2d})$.
- In between these two extremes one has: An operator is a Hilbert Schmidt operator on $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ if and only if its kernel is in $\mathbf{L}^2(\mathbb{R}^{2d})$.

Regularizing Operators

There are lots of regularizing operators:

- Any product-convolution operator (with a pointwise multiplier $h \in SORd$ and a convolutive kernel $g \in \mathbf{S}_0(\mathbb{R}^d)$ is also regularizing in the above sense.
- For $f \in SORdN$ the action of Dirac kernel (compression of some $g \in \left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ with $\hat{(}g) = 1$ is approaching the identity operator;
- Dilation of a function $h \in \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ (with $h(0) = 1$) is giving a pointwise-approximate identity by (ordinary) dilation.
- Finite partial sums of Gabor expansions are another very useful class of regularizing operators tending to the identity operator (on $\mathbf{S}_0'(\mathbb{R}^d)$ only in the w^*-topology).

Relative Completion and Minimal Space

- $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the *smallest* Banach space of functions which is isometrically invariant under time-frequency shifts (and containing at least on non-zero Schwartz function);
- $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is thus correspondingly the biggest space of (tempered) distributions which is isometrically invariant under time-frequency shifts
- We call *[restricted] standard spaces* Banach space $\left(\mathbf{B} \|\cdot\|_{\mathbf{B}}\right)$ betwenn $\mathbf{S}_0(\mathbb{R}^d)$ and $\mathbf{S}_0'(\mathbb{R}^d)$, which are also pointwise modules over $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$ and convolutive modules under $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.

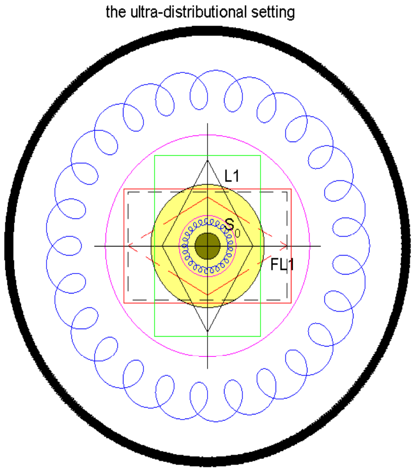
Relative Completion and Minimal Space

- For every standard space the closure of the test-function space $\mathbf{S}_0(\mathbb{R}^d)$ in $\left(\mathbf{B}\, \|\cdot\|_{\mathbf{B}}\right)$ is again a standard space; If this is the $\mathbf{B}$ itself it is called "minimal"; It's dual space is in the same class as well;
- For every standard space $\left(\mathbf{B}\, \|\cdot\|_{\mathbf{B}}\right)$ there is a *relative completion* of $\left(\mathbf{B}\, \|\cdot\|_{\mathbf{B}}\right)$ within $\mathbf{S}_0'(\mathbb{R}^d)$: on denotes by $\widetilde{\mathbf{B}}$ the Banach space of all limits of w^*-convergent, bounded sequences from $\left(\mathbf{B}\, \|\cdot\|_{\mathbf{B}}\right)$, with the infimum over all admissible norms of approximating sequences as natural norm. This is the largest space containing $\left(\mathbf{B}\, \|\cdot\|_{\mathbf{B}}\right)$ as subspace with the same norm
- every such "'maximal" is the dual space of some minimal space.

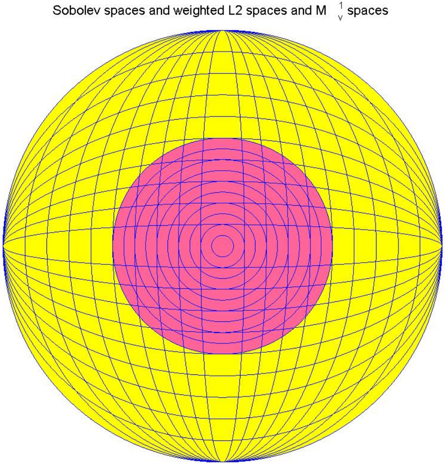
Relative Completion and Minimal Space

- A standard Banach space $(\mathbf{B} \| \cdot \|_{\mathbf{B}})$ is reflexive if and only it is minimal as well as maximal, and its dual has the same property!
- Given a standard space its Fourier image is a standard space as well;
- for every pair of standard spaces the set of all operator kernels mapping one into the other is again a standard space (on $\mathbb{R}^{2d}$);
- For any two standard spaces the set of pointwise multipliers from one into the other is either trivial or a standard space as well; same for convolution kernels;
- for any standard space the Wiener amalgam space $\mathbf{W}(\mathbf{B}, \ell^p)$ is again a standard space;

The setting of Ultradistributions



For pseudo-differential operators: Shubin Classes



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The END

All the BEST to Stevan!!

Thanks for going ahead.

and

Thank you for your attention

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