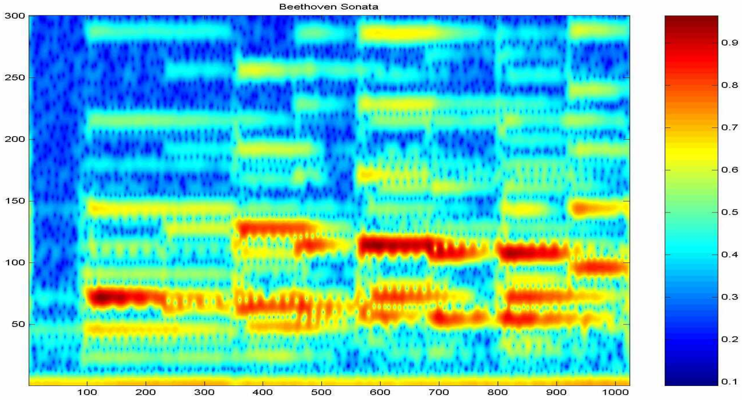
# Mathematics of Time-Frequency Analysis
## *motivational presentation*

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
**www.nuhag.eu**

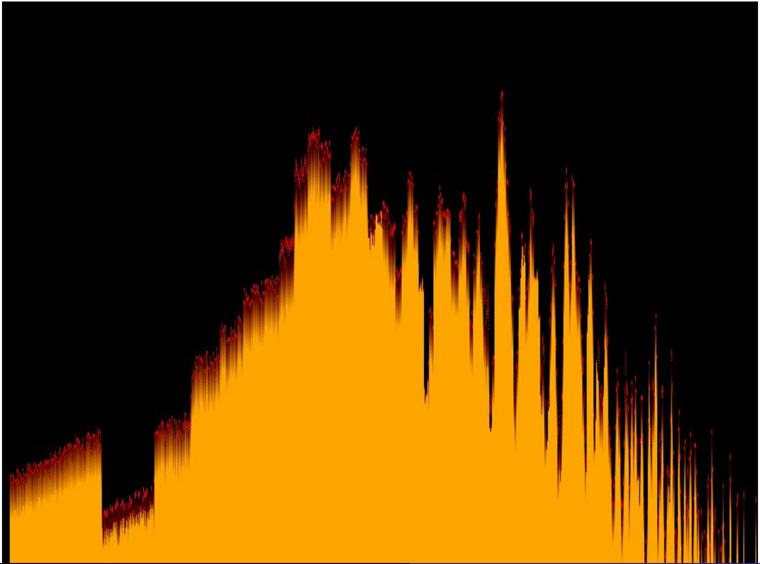
ERSKINE Fellow at University of Canterbury, 2010
July/August 2010, talk delivered July 29th, 2010

# Beethoven's piano sonata

Let us also **listen** to some (other) music and start STX$^T$*M* (ARI, Vienna) or simple the Wavplayer! (Visualization via fire or water!).



# Gabor Analysis in our kid's daily live (MP3)



# It all began with the unit circle

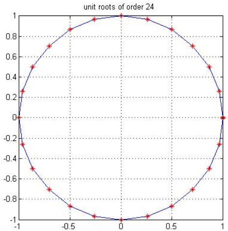
We all know about the unit circle

$$\mathbb{T} := \{z \quad | \quad z \in \mathbb{C}, |z| = 1\}$$

which can be parameterized by the complex *exponential function*

$$t \rightarrow exp(2\pi it) = e^{2\pi jt}.$$

It is a (compact) group under pointwise multiplication, with the *unit-roots of order N* being its (closed) subgroups ($N = 2^k!$)



# ... and the trigonometric functions

As we know due to **Euler** we have

$$e^i x = cos(x) + i \cdot sin(x), \quad x \in \mathbb{R},$$

$$cos(x) = \frac{1}{2}(e^{ix} + e^{-ix}), sin(x) = \frac{1}{2i}(e^{ix} - e^{-ix})$$

*classical Fourier series* describing a periodic function

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_k cos(2\pi kt) + b_k sin(2\pi kt)$$

is thus fully equivalent to the orthonormal series expansions

$$f(t) = \sum_{k \in \mathbb{Z}} c_k e^{2\pi ikt} = \sum_{k \in \mathbb{Z}} \hat{f}(k)\chi_k(t) = \sum_{k \in \mathbb{Z}} \langle f, \chi_k \rangle \chi_k(t),$$

where engineers call $\chi_k$ a *pure frequency*.

# The continuous Fourier transform

$$\hat{f}(s) = \int_{\mathbb{R}} f(t)\overline{\chi_s(t)}dt = \int_{\mathbb{R}} f(t)e^{-2\pi ist}dt$$

and an inversion of the form

$$f(t) = \int_{\mathbb{R}} \hat{f}(s)\chi_s(t)ds = \int_{\mathbb{R}} \hat{f}(s)e^{2\pi ist}ds.$$

Like the FFT it is isometric with respect to the *energy*-norm $\|f\|_2 := \sqrt{\int_{\mathbb{R}} |f(t)|^2 dt}$, and satisfies $\langle f, g\rangle = \langle \mathcal{F}f, \mathcal{F}g\rangle$.

**Theorem (Theorem of Plancherel)**

*The Fourier transform is unitary on $\mathbf{L}^2(\mathbb{R})$ onto itself. It is unitary, i.e. the kernel of the inverse mapping is just the conjugate of the forward Fourier transform, or equivalently, $\mathcal{F}\mathcal{F}f(t) = f(-t)$.*

# Why is it so important and useful?

1. Probability: the Fourier transform turns *convolution*

$$f * g(x) := \int_{\mathbb{R}^d} g(x-y) f(y) dy$$

into pointwise multiplication $\widehat{f * g} = \hat{f} \cdot \hat{g}$, the so-called **convolution theorem**.

2. The pure frequencies (or *characters*) are the only **bounded eigenvectors to shift operators** due to

$$\chi_s(t-z) = e^{-2\pi i s z} \chi_s(t).$$

3. **Translation invariant linear systems**, i.e. mappings $T$ commuting with shift-operators pop up everywhere ($>$ signal and image processing, PDE).

# The continuous Fourier transform

Looking at the formulas once more:

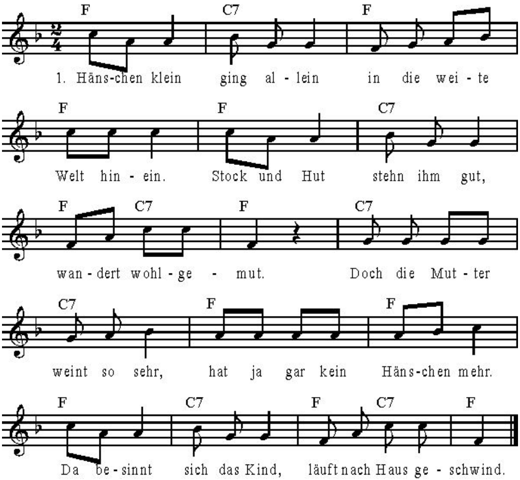
$$\hat{f}(s) = \int_{\mathbb{R}} f(t)\overline{\chi_s(t)}dt = \int_{\mathbb{R}} f(t)e^{-2\pi ist}dt$$

and an inversion of the form (IF $\hat{f}$ is **integrable**!)

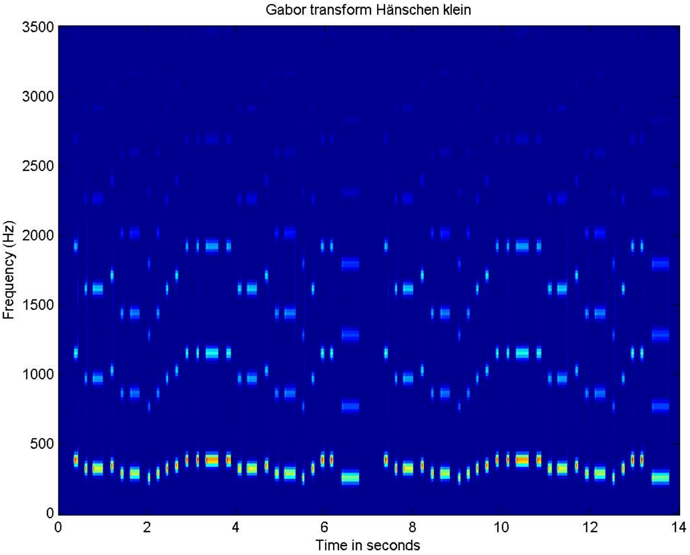
$$f(t) = \int_{\mathbb{R}} \hat{f}(s)\chi_s(t)ds = \int_{\mathbb{R}} \hat{f}(s)e^{2\pi ist}ds.$$

Do we have to learn Lebesgue integration theory to understand the fine details of the Fourier transform? (answer later: better learn **distribution theory!**)
But what are we achieving in this way: To approximate (? in which sense) a harmless and smooth bump-function $f$ by certain linear combinations of pure frequencies (trigonometric polynomials).
Well, we have now an uncountable continuous family of them!

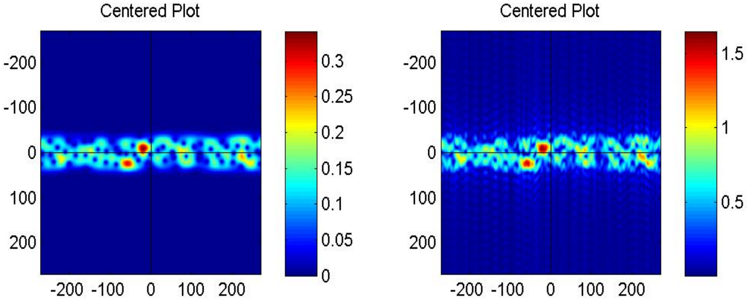
## Time-Frequency Analysis and Music



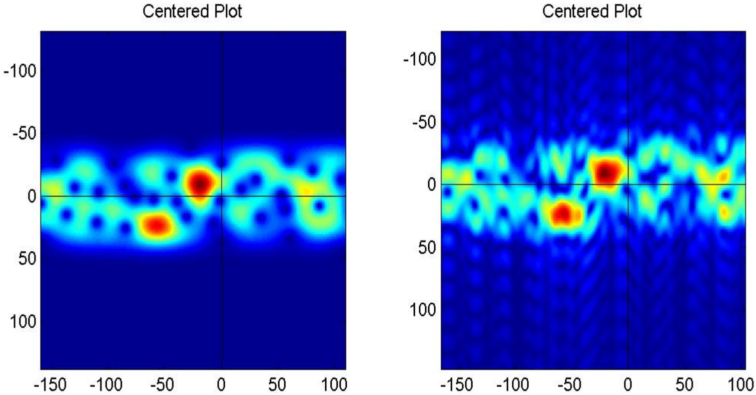
# The Short-Time Fourier Transform of this Song



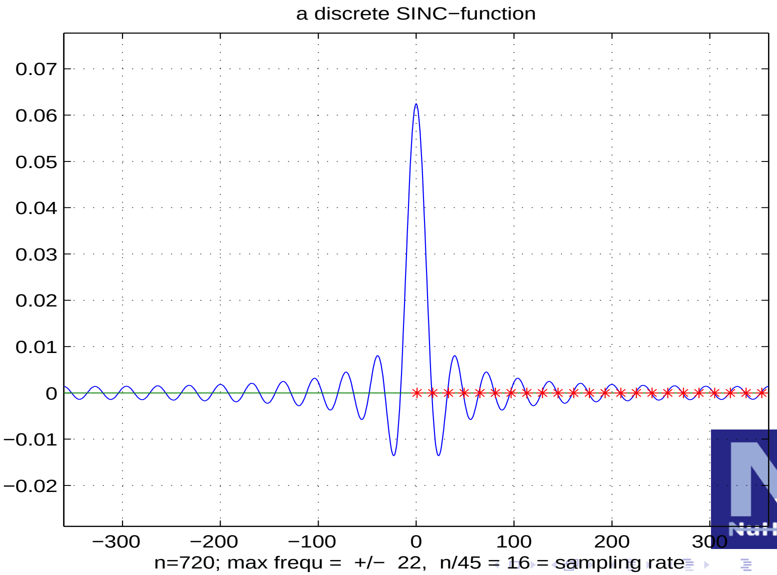
# Choice of the window. Why not BOX-function?



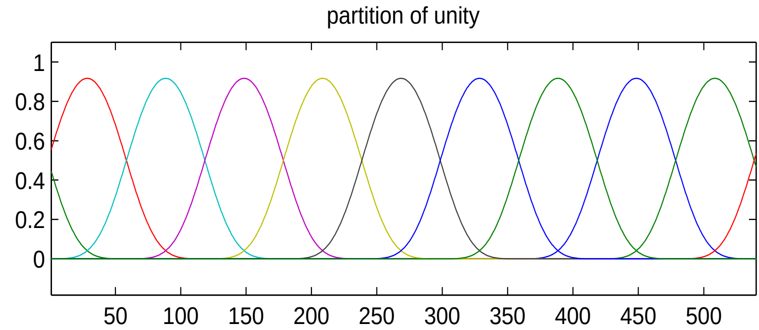
# Choice of the window. Why not BOX-function?

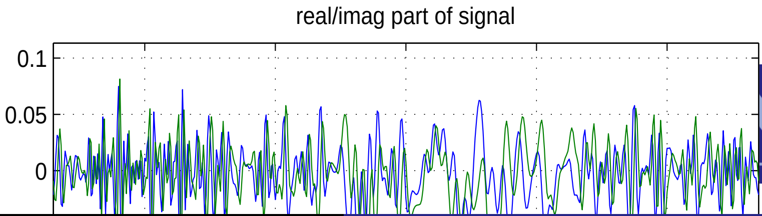


# .. due to the bad decay of SINC

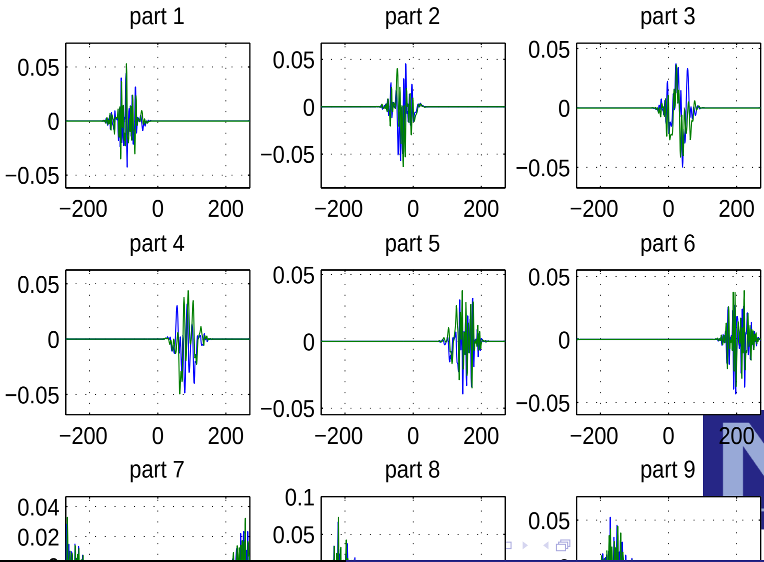


# Motivated by MUSICAL SCORE one could do ?

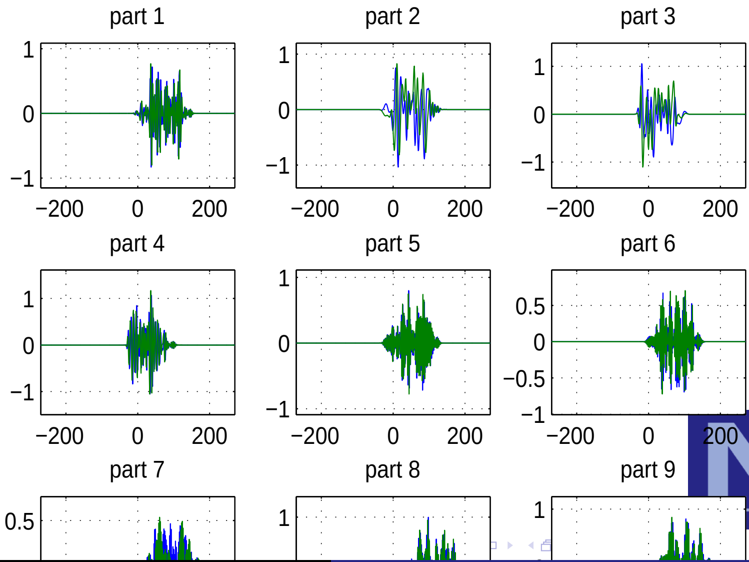


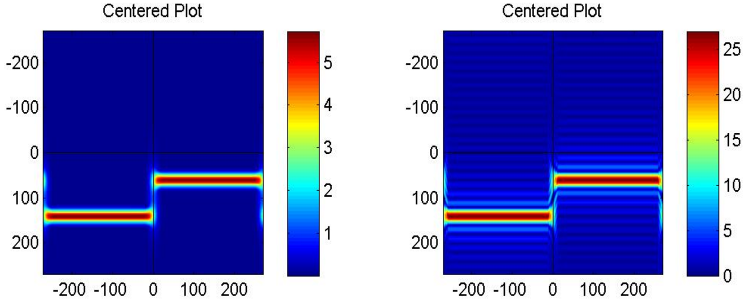
# … and cut the signal into pieces



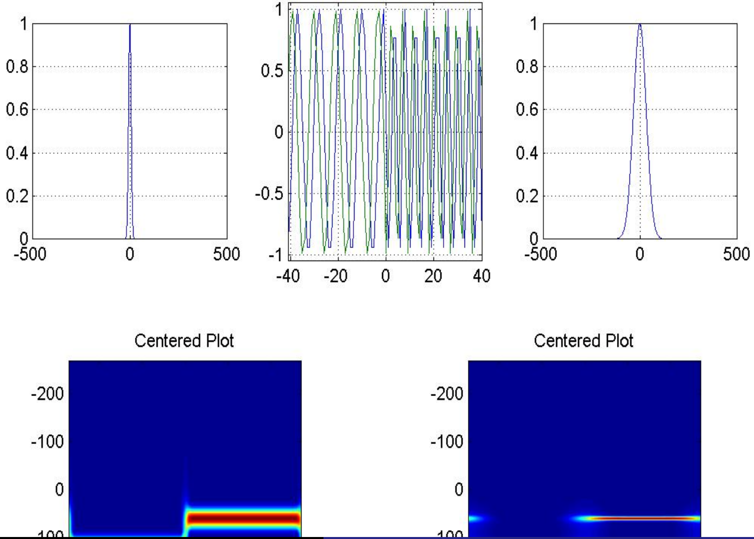
# ... and do localized spectra



# Choice of the window. Why not BOX-function?

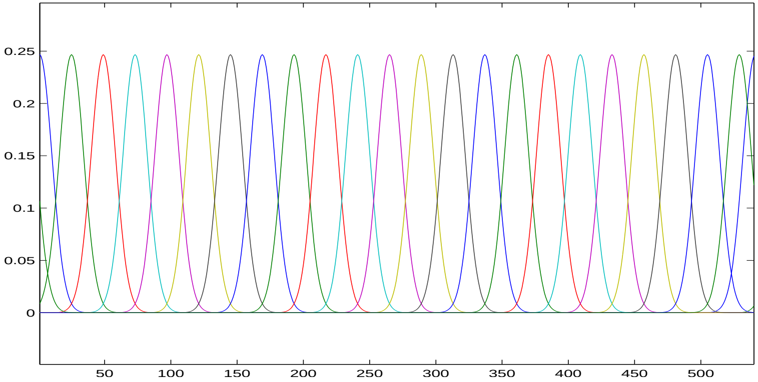


# Choice of the window. Why not BOX-function?

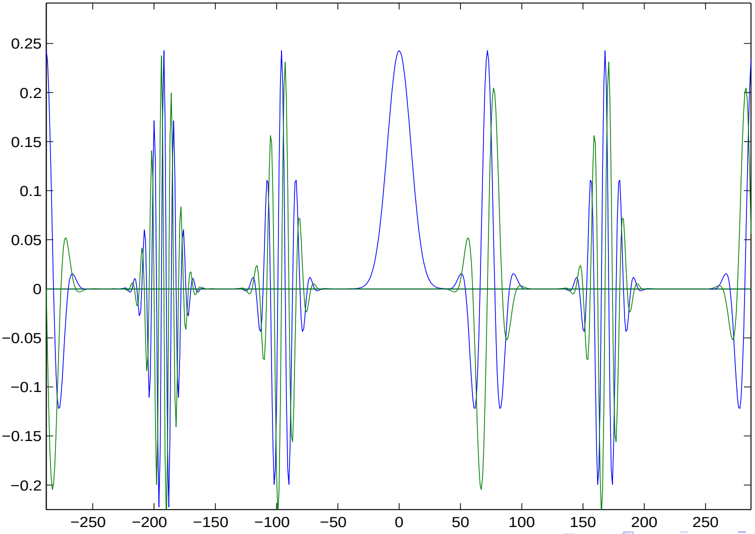


# D.Gabor's suggestion of 1946

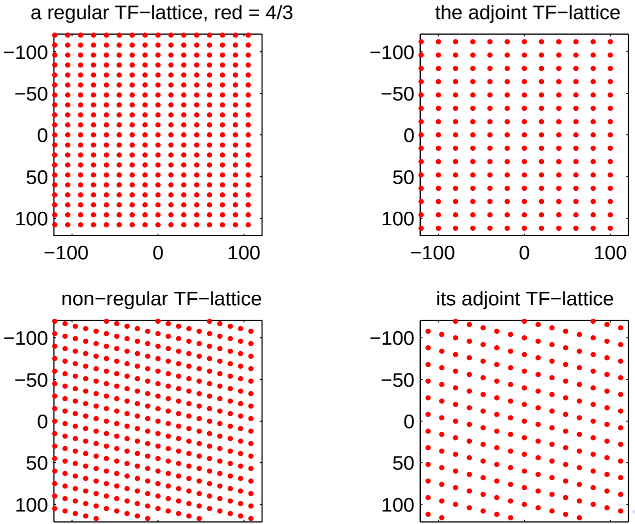
Choose the Gauss-function, because it is the unique minimizer to the *Heisenberg Uncertainty Relation* and choose the critical, so-called von-Neumann lattice, which is simply $\mathbb{Z}^2$.



# The Gaborian Building blocks



# Building blocks are associated with points in phase space or the TIME-Frequency Plane



# The Key Players (why is it called TF-analysis)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^{d^d}$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(t, \omega) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega);$$

# Modern Viewpoint

Todays Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \mathcal{S}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal $f$ resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda) g \rangle$.
We speak of *tight Gabor frames* $(g_\lambda)$ if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \mathbf{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!! (this gives an idea how one might produce tight frames in general).

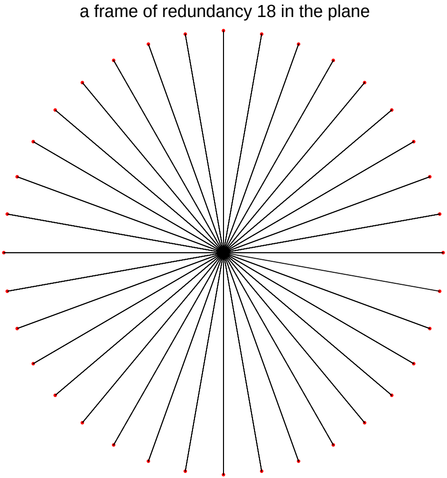
# Modern Viewpoint II

Another basic fact is that for each $g \in \mathcal{S}(\mathbb{R}^d)$ one can find, if $\Lambda$ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\tilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \tilde{g}_\lambda \qquad (1)$$

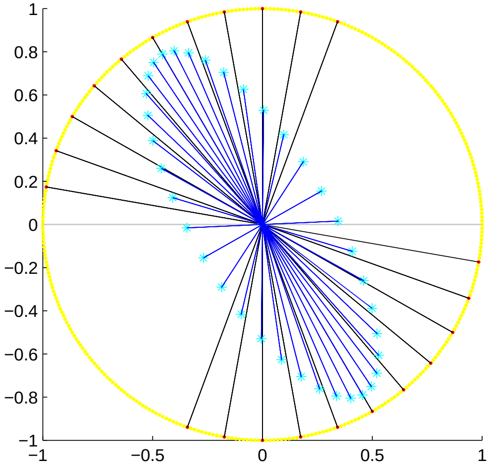
$\tilde{g}$ can be found as the solution of the (positive definite) linear system $S\tilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\tilde{g}$ instead of $g$ for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.
Thus (1) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).

# A generic, high redundancy frame in the plane

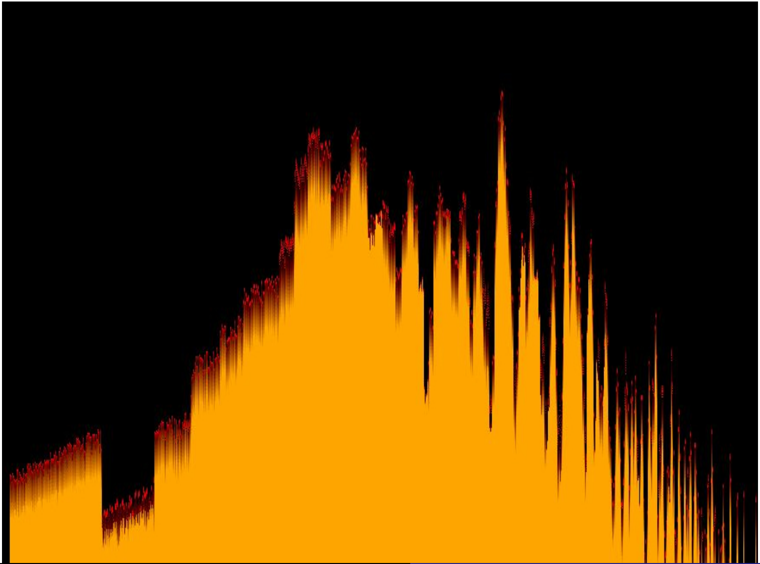


# The action of a corresponding frame multiplier

The effect of a frame multiplier in the plane:

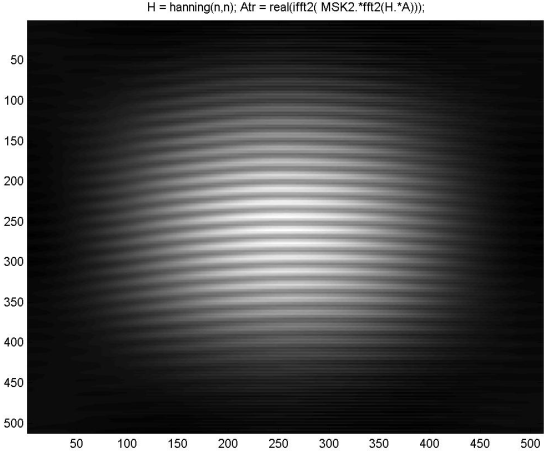


# Gabor Analysis in our kid's daily live (MP3)



# The audio-engineer's work: Gabor multipliers

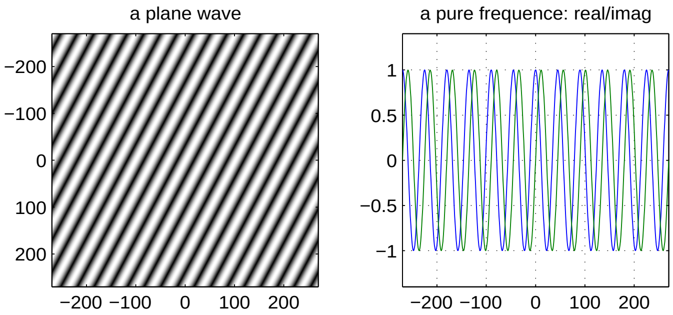
# Pending application: Removing Fringes in Astronomy ? as part of an ongoing ESO project in AUSTRIA.



# Motivation for compactness of musical description

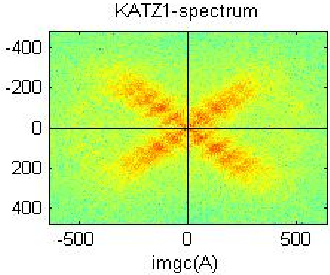
1. It is localized (as opposed to the global Fourier transform)
2. Its building blocks are localized pure frequencies, hence approximate eigenvectors to slowly variant systems;
3. recall that the pure frequencies are a complete system of eigenvectors for the (commutative algebra) of translation operators;
4. one has to choose whether one wants to have redundant and generating families (frames), OR undersampled, linear independent families (Riesz bases), and one cannot have both, except with other undesirable properties (Balian-Low principle)

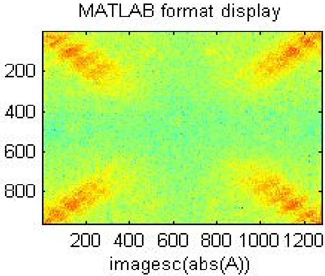
# 2D-Gabor Transform: Plane Waves



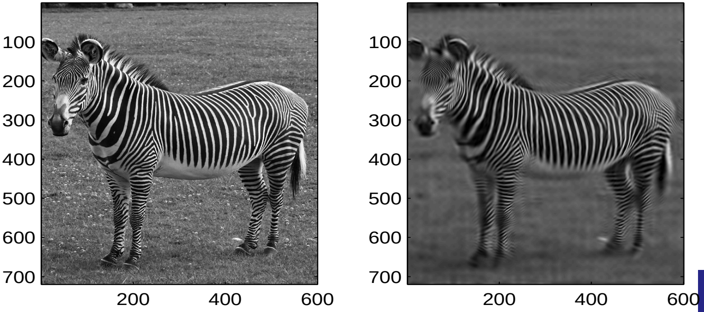
# 2D-Gabor Analysis: Test Images

# 2D-Gabor Transform: Test-Images 2

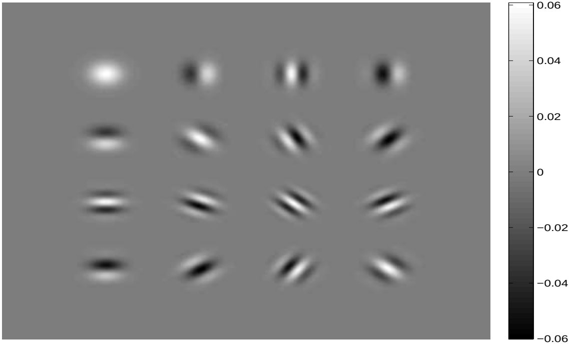


# Image Compression: a Test Image



# Showing the elementary $2D$-building blocks



# Thanks!

THANK you for your attention

maybe you visit www.nuhag.eu

and checkout other talks and related material. hgfei

# The setting of finite Abelian grous

We recall that we have for any finite Abelian Group $G$ the character group

$$\hat{G} = \{\chi \,|\, |\chi(x)| \equiv 1, \chi(x+y) = \chi(x)\cdot\chi(y), \quad x, y, G\}$$

Let us assume for this session that $G$ is a FINITE ABELIAN GROUP (so in principle it is a product of cyclic groups $\mathbb{Z}_N$). On of the basic observations, providing a starting point to time-frequency analysis, is the introduction of time-frequency shift operators

$$\pi(x,\chi) : f \mapsto M_\chi T_x f$$

where $T_x f(y) = f(y-x)$ and $M_\chi = \chi \cdot f$.
Engineers write $M_\omega$ for $\chi(t) = \chi_\omega(t) = e^{2\pi i \omega t}$, and call them pure frequencies.

# The setting of finite Abelian grous

Some algebra: [THE REST is for people with interest in GROUP theory and ALGEBRA, it is part of a Talk at AMS, JAN. 2010]

**Theorem**

1. *Both the time- resp. frequency shifts for an Abelian group (isomorphic to $G \times \widehat{G}$);*
2. *they don't commute in general, but satisfy the canonical commutation relations (with phase factors, related to the eigenvalues of the characters under shift operators);*
3. $(t, \chi) \mapsto \pi(t, \chi) := M_\chi T_x$ *is a so-called projective representation of phase space, i.e. the Abelian group* $G \times \widehat{G}$.
4. *There is a natural extension of $\pi$ which makes it an ordinary unitary representation of the reduced Heisenberg group* $G \times \widehat{G} \times \mathbb{T}$ *on* $\ell^2(G)$ *(which is irreducible);*

# Phase space points as operators

So we have now for a group of cardinality $\#G = N$ exactly $N^2$ TF-shift matrices, within the matrix algebra over $G$, which has exactly this dimension $N^2$. In fact, they form an orthonormal basis (w.r.t. the Hilbert-Schmidt resp. Frobenius scalar product).

**Theorem (spreading representation)**

*Every matrix has a representation of the form*

$$A = \sum_{G \times \widehat{G}} c_\lambda \pi(\lambda),$$

*with*

$$c_\lambda = \mathrm{trace}(A * \pi(\lambda)^*)/N^2.$$

The spreading function $A \mapsto c = c(A)$ is unitary from the Hilbert Schmidt operators to $\ell^2(G \times \widehat{G})$. It's symplectic Fourier transform is just the Kohn-Nirenberg symbol of the operator $A$.

# Phase space points as operators

Even if not all TF-shifts commute which each others, there are considerable large groups of commuting operators (we have seen the group of all time- and all frequency shifts are such lattices of order $N$, each)

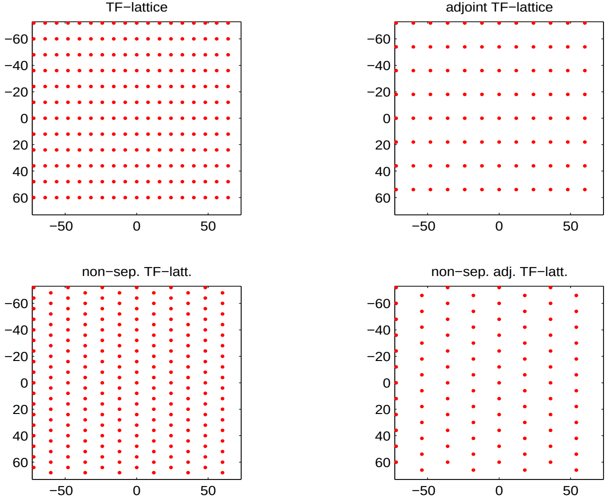
**Lemma**

*For each subgroup (lattice) $\Lambda$ within the group $G \times \widehat{G}$ (viewed as Abelian group) there is a corresponding group $\Lambda^\circ$, with* $\#\Lambda \cdot \#\Lambda = N^2$,*with*

$$\Lambda^\circ = \{\lambda^\circ \,|\, \pi(\lambda^\circ)\pi(\lambda) = \pi(\lambda)\pi(\lambda^\circ) \ \forall \lambda \in \Lambda\}$$

The *symplectic* version of Poisson's formula is made exactly such that it maps the indicator function of $\Lambda$ onto (a multiple of) the indicator function of $\Lambda^\circ$ (and vice versa, or equivalently: $\Lambda^{\circ\circ} = \Lambda$).

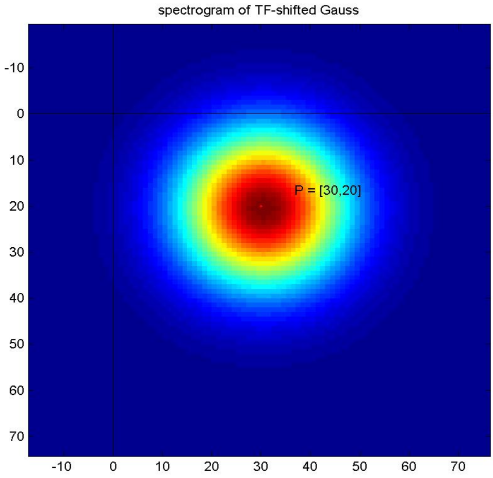
# Lattices and Adjoint Lattices



# Phase space points as operators

Now that we have an identification of points in phase space $G \times \widehat{G}$ and operators is given, we can produce for any finite or countable set of points a family of vectors in the Hilbert space $\ell^2(G)$ by applying the corresponding operators to a given vector (usually called atom or window), namely $g_\lambda = \pi(\lambda)g$. If we take a Gauss function then the spectrogram of $g_\lambda$ will be clearly concentrated around the point $\lambda$. Thus we expect that it makes sense to cover the whole TF-plane in a reasonable we.
For this (algebraically oriented) talk we restrict our attention to finite subgroups $\Lambda$ of the (Abelian) group $G \times \widehat{G}$.

# TF-picture of atoms in general position



# (regular) Gabor families: Frames and Riesz bases

**Definition**

Given a pair $(g, \Lambda)$, consisting of a vector $g \in \ell^2(G)$ and a lattice $\Lambda \lhd G \times \widehat{G}$ we call the family $(g_\lambda)_{\lambda \in \Lambda}$ a Gabor frame, if the family spans all of $\ell^2(G)$. It is called a Gaborian Riesz basis (resp. Riesz basic sequence) if it is a linear independent set.

There are - for people in numerical analysis - quality measures for the quality of such families, in the sense of a conditioning of the problem, thus being a quotient of two relevant singular values of associated operators, we don't go into details here.
Both situations are of practical relevance!

# Usefulness and applications of Gabor frames:

The question of Gabor frames is of interest, when a signal (say some audio signal, or some image, cf. introduction) is to be *decomposed* into meaningful elementary building blocks, somehow like *transcription*. Ideally the distribution of *energy* in the signal goes over into an equivalent energy distribution. AND WHAT can we do with this:
a) contributions may be irrelevant (or disturbing) and can be eliminated (the bird contributing to the open air classical concert): denoising of signals
b) signals can be separated in a TF-situation
c) unimportant, small contributions can be omitted (+ masking effect): allows for efficient lossy compression schemes >> MP3.

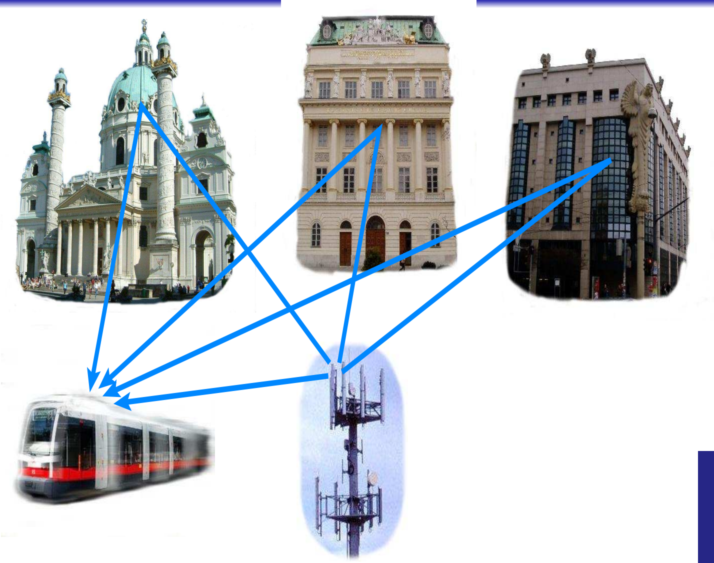
# The audio-engineer's work: Gabor multipliers

# Applications of Gabor Riesz bases:

Of course Gabor Riesz bases (for subspaces) will correspond to lattices $\Lambda$ with at most $N$ points. Ideally, the Gram matrix of the corresponding system is diagonal dominant (there is the so-called piano-reconstruction theorem).
They are very useful in mobile communication. The fact, that smooth envelopes (as used for Gabor frames), multiplied with pure frequencies are at least approximate eigenvectors for so-called *slowly varying channels* makes them useful for mobile communication. The physical assumption of limited multi-path propagation (variable kernels over time) and Doppler (due to movement) related to underspread operators, i.e. to matrices whose spreading function is supported on a given rectangular domain.

# Mobile Communication

# Mobile Communication



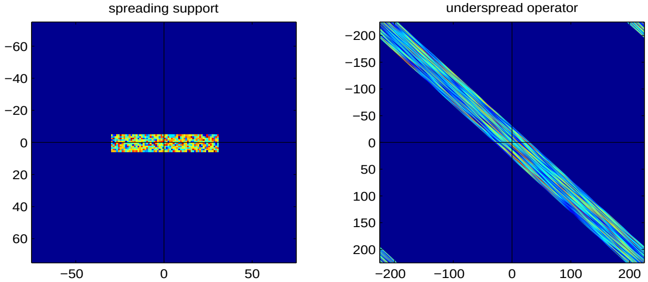
# Applications of Gabor Riesz bases:

The information, encoded as a collection of coefficients which we will call $(c_{\lambda^\circ})$ are used to form a linear combination of the elements of our Gaborian Riesz basis. I.e. the sender *plays slowly a melody on the piano*.

Assume we are able to estimate the approximate eigenvalues $(d_{\lambda^\circ}$ of the involved building blocks $(g_{\lambda^\circ}$, the approximate eigenvector property of these building blocks implies that the receiver obtains $\sum_{\lambda^\circ} c_{\lambda^\circ} d_{\lambda^\circ} g_{\lambda^\circ}$. Knowing the factors $(d_{\lambda^\circ})$ (by sending so-called pilot tones) and the biorthogonal basis the receiver can then (approximately) recover the set of coefficients $(c_{\lambda^\circ})$ sent by the sender.

In other words, *the receiver listens to the music behind a wall, knowing e.g. that higher frequencies are absorbed more (or less) then others and figures out, what has been played.*

# Underspread operators



# Pilot tones and Channel estimation

Underspread operators can be viewed as the analogue of band-limited signals. As we know they can be recovered (i.e. completely desribed) by knowing sufficiently many samples on a (sufficiently dense) lattice $\Lambda$.
The specific properties of such underspread operators in turn makes the idea of channel estimation on the basis of pilot tones work: sending station and receiver agree on certain Gabor atoms to be sent regularly. The receiver can recognize from their distortion (+noise) at least approximately what the channel (= operator) might be so that the inversion of the channel (decoding) can be done more efficiently.

# Periodic operators and spreading support on grids

It is a well know principle in signal processing resp. Fourier transform theory, that periodizing of a function corresponds to sampling of its Fourier transform (and vice versa).
At the mathematical basis of this observation is Poisson's formula, which in the discrete setting corresponds to the fact that the $G$-Fourier Transform of the indicator function of $\Lambda \lhd G$ is $\#\Lambda \cdot \Lambda^{\perp}$.
Reading sampling as multiplication with such a Shah-comb (and convolution with a Shah-comb as periodization) one can easily understand this connection (also closely related to the group theoretical view on Shannon's sampling theorem, also known as Kluvanek's theorem in the LCA setting).

# The Gabor frame operator:

The crucial operator for he determination, whether $(g, \Lambda)$ generates a Gabor frame or not is the question, whether to so-called *frame operator* is positive definite (or possibly singular, otherwise).

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda .$$

The most important fact to be known about this operator is a commutation relation:

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S \quad \forall \lambda \in \Lambda .$$

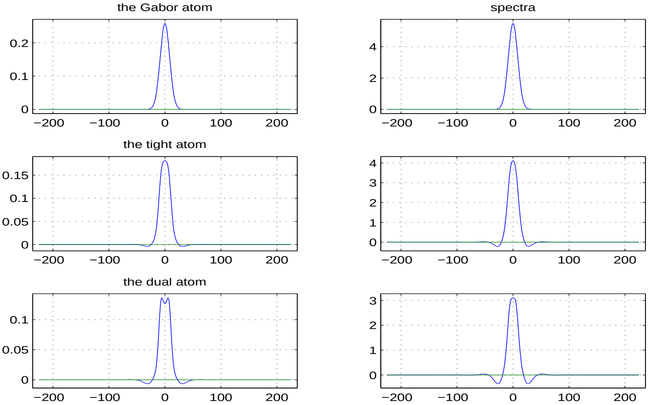
# The (canonical) dual Gabor frame

This greatly simplifies the calculation of (minimal norm) coefficients for the given signal. In fact, it is found that the solution $\tilde{g}$ of the simple (positive definite) linear equation

$$S\tilde{g} = g \quad \text{resp.} \quad \tilde{g} = S^{-1}g,$$

spans the *dual Gabor frame*. In fact FFT-based methods can be applied to efficiently calculate these coefficients, once $\tilde{g}$ is given. Sometimes alternative sets of coefficients are equally useful.
For the solution of the above equations various iterative methods, e.g. conjugate gradients, can be applied .

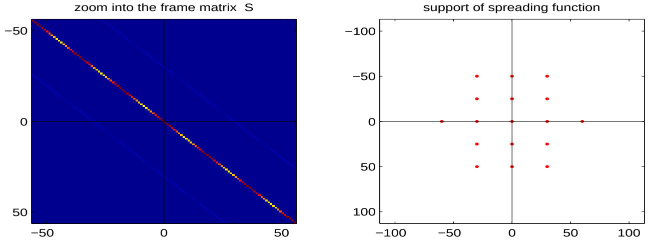
# A generic, high redundancy frame in the plane



# Janssen's representation of the frame operator

If we look at the operator $S$ (but also $S^{-1}$ or $S^{-1/2}$) they are commute with all the TF-shifts from the generating lattice $\Lambda$. This property is reflected by an equivalent property in the spreading domain: certain plane waves leave the spreading symbol invariant, or equivalently, *only the spreading coefficients* over the so-called *adjoint lattice* $\Lambda^\circ$ can be non-zero.
This also implies that the dual Gabor atom $\tilde{g}$ is just a linear combination of elements from the adjoint orbit. This gives an explicit sparsity of the frame operator and all the polynomials of the frame operator and hence a fast realization of conjugate gradients.

# A generic, high redundancy frame in the plane



# The relevant (non-commutative) Algebras

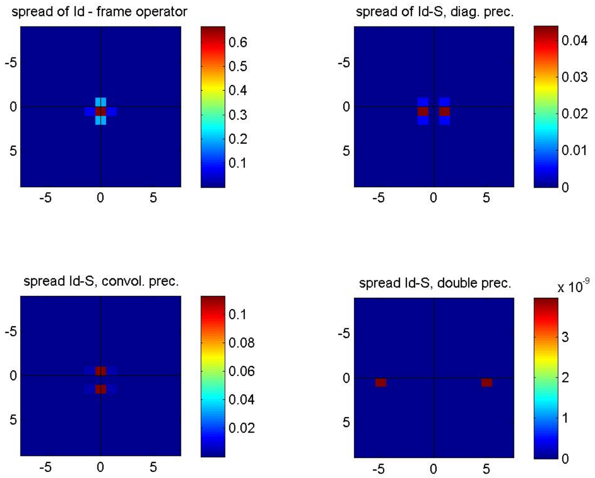
**Theorem**

*The set of all matrices commuting with all TF-shifts from the lattice $\Lambda$ are exactly those who are having a spreading support within $\Lambda^\circ$. Hence also the inverse of the operator $S^{-1}$ belongs to the same algebra.*
*This algebra is commutative if and only if $\Lambda^\circ \subset \Lambda$ (e.g. $ab|N$, the so-called integer oversampling case). In this case FFT2-based methods can be used, which are expressed using the so-called Zak-transform.*

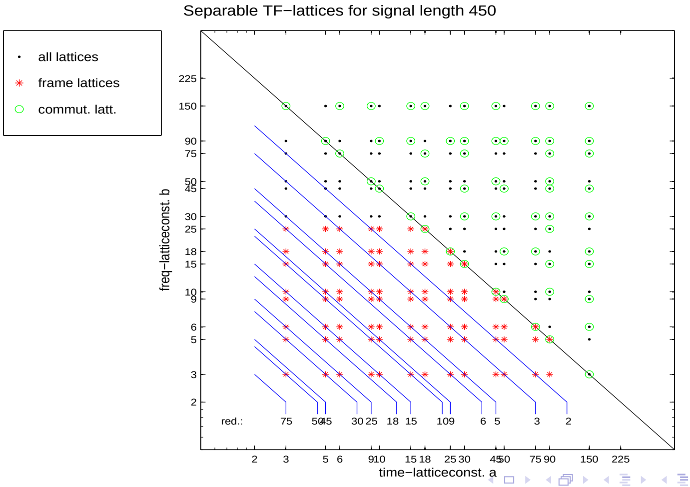
# Useful COMMUTATIVE subalgebras

This algebra is also in the background of a method called *double preconditioning*. Typically the frame matrix will either be diagonal dominant or very close to a convolution matrix (not these are related to commutative subgroups of the group $\Lambda$, viewed as a family of TF-shift operators).
For "nice atoms" it is a good option in order to get good approximation to the inverse frame operator (hence $\tilde{g}$) to apply both the inverse of the diagonal part and afterwards the inverse of the convolution part to $g$.
In this way the modified frame matrix will have a Janssen representation with a dominant Identity contribution.

# The effect of double preconditioning in the spreading domain



# A generic, high redundancy frame in the plane



# Continuous dependence of lattice constants

The last structure allows to explain and demonstrate a number of interesting features of Gabor families, such as

- The dual (and also the tight atom) depond continuously on the lattice constants; if $g$ is Fourier invariant and $a = b$ then the dual atom is also Fourier invariant;
- There is a mirror symmetry of along the (critical) midline: above it we have undersampled lattices with $\#\Lambda \leq N$ and below we have $\#\Lambda \geq N$, which is exactly the correspondence between $\Lambda$ (with $(a, b)$) and $\Lambda^\circ$ (with $(N/b, N/a)$).
- One can have biorthogonal families in the undersampled case, and dual frames in the undersampled case

It is clear, that one actually would like to build an arbitrary signal $f$ given the pair $(g, \Lambda)$ (in the frame case), or at least do the best approximation of $f$ by linear combinations from the Gabor family in the Riesz basis case. In both cases one has a number of choices, but the canonical one (related to `PINV` resp. to the associated MNLSQ-problem is the one usually preferred.
The appropriate coefficients are then obtained by taking scalar products with respect to the correspondig "dual" family, which is numerically efficiently implemented by doing a sampled STFT (using FFT-based methods).

# Wexler-Raz and Ron-Shen Theorems

**Theorem**

*(1) (Wexler-Raz): The family generated by $(g, \Lambda)$ is a Gabor frame, if and only if the family $(g, \Lambda^\circ)$ is a Gaborian Riesz basis. Moreover, the generator $\tilde{g}$ of the dual frame coincides (up to a normalizing factor) with the generator of the biorthogonal system of the Riesz basis built from $(g, \Lambda^\circ)$.*
*(2) (Ron-Shen): If we look at the corresponding bounds for the frame operator resp. the Gramian matrix the are (again up to a fixed scaling, depeding on the size of the fundamental domain of $\Lambda$) exactly equal, hence their condition numbers are exactly the same.*

From an abstract view-point there is a so-called Morita-Equivalence of Banach algebras in the background (cf. the work of Franz Luef).

# Operating on the audio signal: filter banks

# Finally let us operate on the Gabor coefficients

**Definition**

Let $g_1, g_2$ be two $L^2$-functions, $\Lambda$ a TF-lattice for $\mathbb{R}^d$, i.e. a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Furthermore let $\mathbf{m} = (\mathbf{m}(\lambda))_{\lambda \in \Lambda}$ be a complex-valued sequence on $\Lambda$. Then the **Gabor multiplier** associated to the triple $(g_1, g_2, \Lambda)$ with (*strong* or) **upper symbol m** is given as

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda \in \Lambda} m(\lambda) \langle f, \pi(\lambda) g_1 \rangle \, \pi(\lambda) g_2. \quad (2)$$

$g_1$ is called the *analysis* window, and $g_2$ is the synthesis window. If $g_1 = g_2$ and $\mathbf{m}$ is real-valued, then the Gabor multiplier is self-adjoint. Since the constant function $\mathbf{m} \equiv 1$ is mapped into the Identity operator if $g_1 = g_2$ is a $\Lambda$-tight Gabor atom this is often the preferred choice.

## The family of projection operators $(P_\lambda)$

### Theorem

*Assume that* $(g,\Lambda)$ *generates an* $S_0$*-Gabor frame for* $\mathbf{L}^2(\mathbb{R}^d)$*, with* $\|g\|_2 = 1$*, and write* $P_\lambda$ *for the projection* $f \mapsto \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$*.*
*i) Then the family* $(P_\lambda)_{\lambda\in\Lambda}$ *is a Riesz basis for its closed linear span within the Hilbert space* $\mathcal{HS}$ *of all Hilbert-Schmidt operators on* $\mathbf{L}^2(\mathbb{R}^d)$ *if and only if the function* $H(s)$*, defined as the* $\Lambda$*-Fourier transform of* $\left(|STFT_g(g)(\lambda)|^2\right)_{\lambda\in\Lambda}$ *is does not have zeros.*
*ii) An operator* $T$ *belongs to the closed linear span of this Riesz basis if and only if it belongs to* $\mathcal{GM}_2$*, the space of Gabor multiplier with* $\ell^2(\Lambda)$*-symbol.*
*iii) The canonical biorthogonal family to* $(P_\lambda)_{\lambda\in\Lambda}$ *is of the form* $(Q_\lambda)_{\lambda\in\Lambda}$*,*

$$Q_\lambda = \pi(\lambda)\circ Q \circ \pi^{-1}(\lambda) \text{ for } \lambda\in\Lambda,$$

*for a uniquely determined Gabor multiplier* $Q \in \mathcal{B}$*.*
*iv) The best approximation of* $T \in \mathcal{HS}$ *by Gabor multipliers based on the pair* $(g,\Lambda)$ *is of the form*

# The family of projection operators $(P_\lambda)$

We make use here of the fact, that $\pi \otimes \pi^*$ (coming from a projective representation on $\ell^2(G)$) is now a true unitary representation of $G \times \widehat{G}$ on $\mathcal{HS}$.
The Kohn-Nirenberg (recall: the symplectic Fourier version of he spreading symbol) is turning this representation of matrices into a subrepresentation of the regular representation of $G \times \widehat{G}$: The KN-symbols of Gabor multipliers are just linear combinations of translates along $\Lambda$ of the KNS of the rank-one operators $f \mapsto \langle f, g \rangle g$. In other words, the best $\mathcal{HS}$-approximation problem for matrices is equivalent to the problem of best approximation of function in $\mathbf{L}^2(G \times \widehat{G})$ by the elements of some spline-type space (so again groups and FFT-based methods can be employed).

# Equivalent representations and intertwining operators

Again a group theoretical view-point can be helpful.

**Theorem**

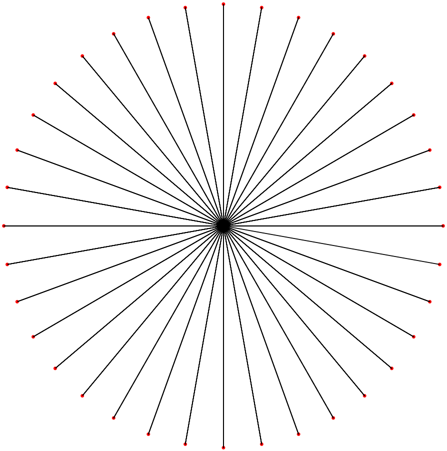
*The unitary transforms which identify a Hilbert Schmidt operator (as operator on $\mathbf{L}^2(G)$) with its matrix kernel (with the Frobenius norm), with its spreading symbol (in $\ell^2(G \times \widehat{G})$) or its Kohn-Nirenberg symbol are all intertwining with the natural action of phase space on each of these Hilbert spaces:*
*(i) $\pi \otimes \pi^*$ at the operator level*
*(ii) Multiplication with appropriate plane waves in the spreading domain (as a result of the commutation relations);*
*(iii) as ordinary shifts in the KNS-setting...*

# 1D versus 2D: a group theoretic view:

Whereas it is clear from the group theory that isomorphic Abelian groups will have "isomorphic Gabor theory" (atoms, frames, Riesz bases, dual atoms, etc.) in terms of code (for actual computation) or especially applications (images look really different from audio-signals) one has to think about it.
Just to mention: if one has a group of the form $\mathbb{Z}_p \times \mathbb{Z}_q$ with $p$ and $q$ relatively prime (e.g. $p = 256$ and $q = 243$) then we can map images of such a format (also Gabor atoms etc.) on corresponding functions of size $N = 256 \cdot 243 = 62208$, and use $1D$-MATLAB code for 2D applications.
Otherwise, of course, one is taking tensor products of $1D$-atoms.

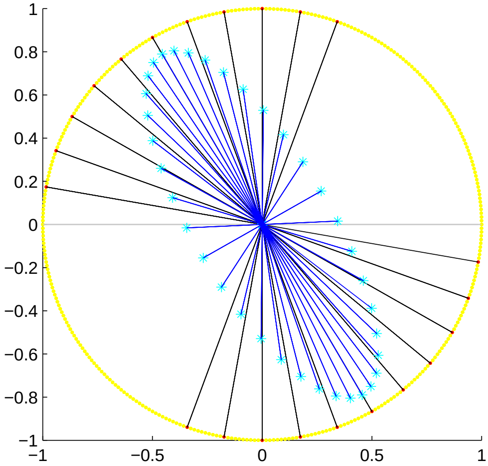
# A generic, high redundancy frame in the plane

a frame of redundancy 18 in the plane

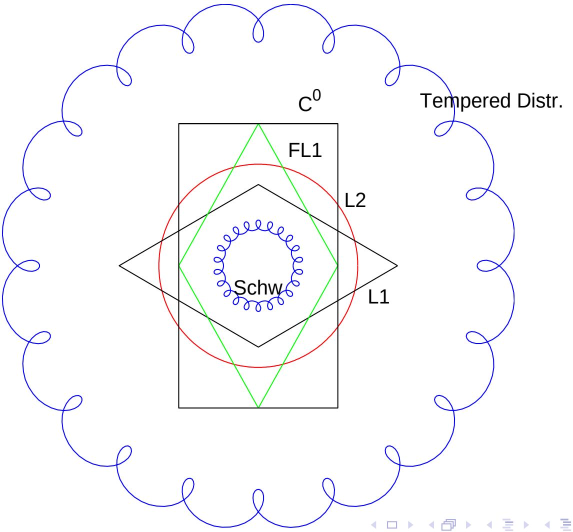


# The action of a corresponding frame multiplier

The effect of a frame multiplier in the plane:



# The classical setting of test functions & distributions



# A suitable Banach space of test functions & distributions