BANACH FRAMES and BANACH GELFAND TRIPLES

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

Sept. 27th, 2010
Journees d'Analyse a Bordeaux

Goals of this lecture

- show that the usual generalizations of linear algebra concepts to the Hilbert space case (namely linear independence and totality) are inappropriate in many cases;
- that frames and Riesz bases (for subspaces) are the right generalization to Hilbert spaces;
- that Hilbert spaces are themselves a too narrow concept and should be replaced Banach Gelfand Triples, ideally isomorphic to the canonical ones $(\ell^1, \ell^2, \ell^\infty)$;
- Describing the situation of frames or Riesz bases via commutative diagrams allows to extend this notation to BGTs;
- Demonstrate by examples (Fourier transform, kernel theorem) that this viewpoint brings us very close to the finite-dimensional setting!

OVERVIEW over this lecture 60- MINUTES

- This is a talk about the frames viewing frames as RETRACTS, i.e. a construction which can be done in any *category*;
- about the ubiquity of Banach Gelfand Triples ;
- provides a setting very similar to the finite dimensional setting;
- showing how easy they are to use;
- showing some applications in Fourier Analysis;
- indicating its relevance for numerical applications;
- and for teaching purposes;
- that it is a good vehicle to transfer algebraic facts (over finite Abelian group to the setting of LCA groups);
- perhaps change your view on Fourier Analysis.

ANALYSIS: Calculating with all kind of numbers

We teach in our courses that there is a huge variety of *NUMBERS*, but for our daily life rationals, reals and complex numbers suffice. The most beautiful equation

$$e^{2\pi i} = 1.$$

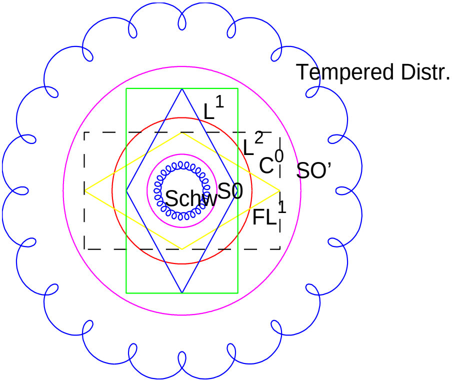
It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

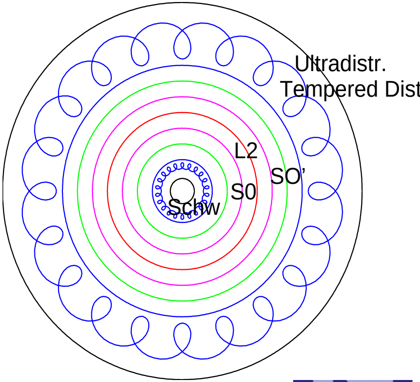
$$cos(2\pi) + i * sin(2\pi) = 1 \quad \text{in} \quad \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

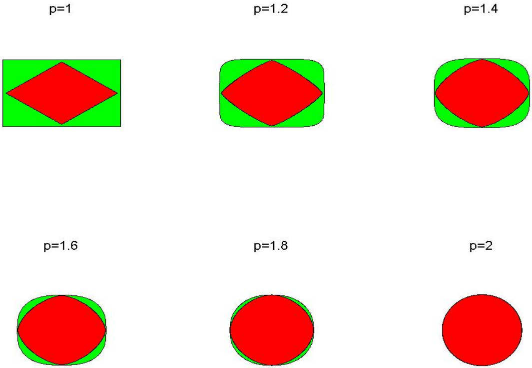
ANALYSIS: Spaces used to describe the Fourier Transform



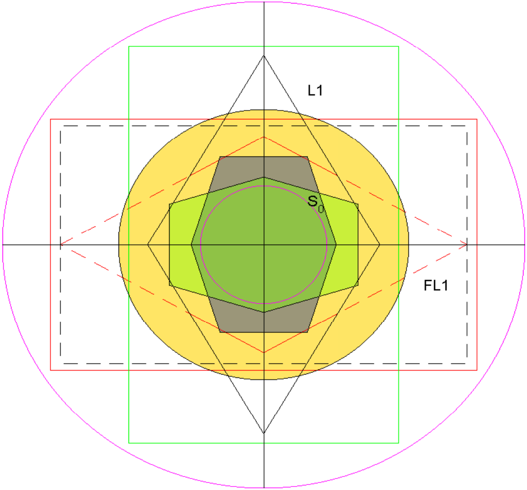


Hausdorff Young Theorem for the Fourier Transform

$$\mathcal{F}\mathbf{L}^p(\mathbb{R}^d) \subseteq \mathbf{L}^q(\mathbb{R}^d), \quad 1 \leq p \leq 2, \; \frac{1}{q} + \frac{1}{p} = 1.$$



Wiener Amalgam spaces, Wiener Algebra, etc.



Frames in Hilbert Spaces: Classical Approach

Definition

A family $(f_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \qquad (1)$$

It is well known that condition (1) is satisfied if and only if the so-called frame operator is invertible, which is given by

Definition

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for} \quad f \in \mathcal{H},$$

Frames in Hilbert Spaces: Classical Approach II

The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\widetilde{f_i})_{i \in I}$, defined by $\widetilde{f_i} := S^{-1}(f_i)$ has the property that one has for $f \in \mathcal{H}$:

Definition

$$f = \sum_{i \in I} \langle f, \widetilde{f_i} \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \widetilde{f_i} \tag{2}$$

Moreover, applying S^{-1} to this equation one finds that the family $(\widetilde{f_i})_{i \in I}$ is in fact a frame, whose frame operator is just S^{-1}, and consequently the "second dual frame" is just the original one.

Frames in Hilbert Spaces: Approach III

Since S is *positive definite* in this case we can also get to a more symmetric expression by defining $h_i = S^{-1/2} f_i$. In this case one has

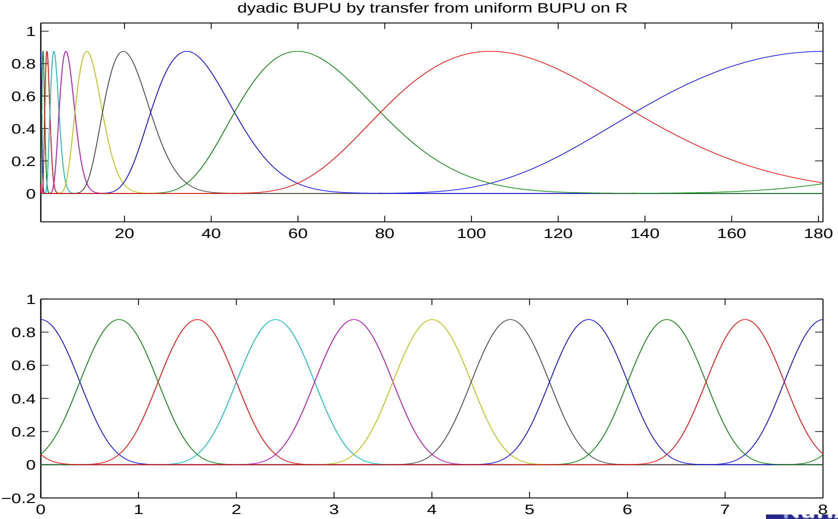
$$f = \sum_{i \in I} \langle f, h_i \rangle h_i \quad \text{for all } f \in \mathcal{H}. \tag{3}$$

The family $(h_i)_{i \in I}$ defined in this way is called the *canonical tight frame* associated to the given family $(g_i)_{i \in I}$. It is in some sense the closest tight frame to the given family $(f_i)_{i \in I}$.

Where did frames come up? Historical views:

I think there is a historical reason for frames to pop up in the setting of separable Hilbert spaces $\mathcal{H}$. The first and fundamental paper was by Duffin and Schaeffer ([**?**]) which gained popularity in the "painless" paper by Daubechies, Grossmann and Y. Meyer ([**?**]). It gives explicit constructions of tight Wavelet as well as Gabor frames. For the wavelet case such dual pairs are are also known due to the work of Frazier-Jawerth, see [**?**, **?**]. Such characterizations (e.g. via atomic decompositions, with control of the coefficients) can in fact seen as prerunners of the concept of Banach frames to be discussed below.
These methods are closely related to the Fourier description of function spaces (going back to H. Triebel and J. Peetre) using *dyadic partitions of unity* on the Fourier transform side.

Dyadic Partitions of Unity and Besov spaces



Where did frames come up? Historical views II

The construction of orthonormal wavelets (in particular the first constructions by Y. Meyer and Lemarie, and subsequently the famous papers by Ingrid Daubechies), with prescribed degree of smoothness and even compact support makes a big difference to the Gabor case.
In fact, the Balian-Low theorem prohibits the existence of (Riesz- or) orthogonal Gabor bases with well TF-localized atoms, hence one has to be content with Gabor frames (for signal expansions) or Gabor Riesz basic sequences (for mobile communication such as OFDM).
This also brings up a connection to filter banks, which in the case of Gabor frames has been studied extensively by H. Bölcskei and coauthors (see [**?**]).

LINEAR ALGEBRA: Gilbert Strang's FOUR SPACES

Let us recall the *standard linear algebra situation*. Given some $m \times n$ -matrix $\mathbf{A}$ we view it as a collection of *column* resp. as a collection of *row vectors*. We have:

$$\textbf{row-rank}(\mathbf{A}) = \textbf{column-rank}(\mathbf{A})$$

Each *homogeneous linear system of equations* can be expressed in the form of scalar products[1] we find that

$$\textbf{Null}(\mathbf{A}) = \quad \textbf{Rowspace}(\mathbf{A})^{\perp}$$

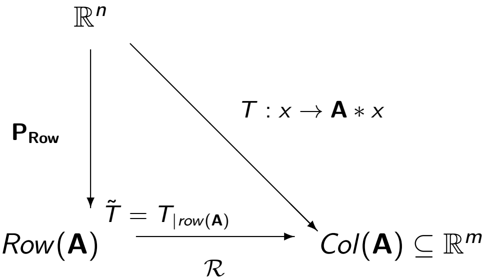
and of course (by reasons of symmetry) for $\mathbf{A}' := conj(A^t)$:

$$\textbf{Null}(\mathbf{A}') = \quad \textbf{Colspace}(\mathbf{A})^{\perp}$$

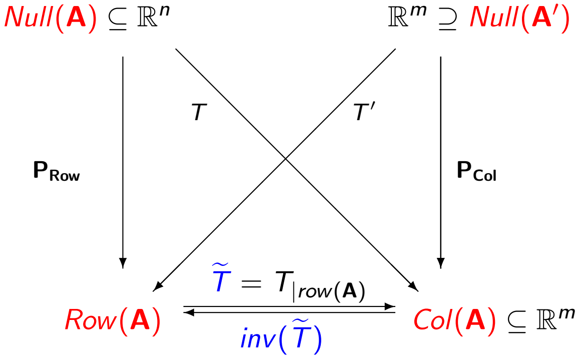
[1] Think of $3x + 4y + 5z = 0$ is just another way to say that the vector $\mathbf{x} = [x, y, z]$ satisfies $\langle \mathbf{x}, [3, 4, 5] \rangle = 0$.

Geometric interpretation of matrix multiplication

Since *clearly* the restriction of the linear mapping $x \mapsto \mathbf{A} * x$



Geometric interpretation of matrix multiplication



$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Four spaces and the SVD

The **SVD** (the so-called Singular Value Decomposition) of a matrix, described in the MATLAB helpful as a way to write A as

$$A = U * S * V'$$

, where the columns of U form an ON-Basis in $\mathbb{R}^m$ and the columns of V form an ON-basis for $\mathbb{R}^n$, and S is a (rectangular) diagonal matrix containing the non-negative *singular values* (σ_k) of A. We have $\sigma_1 \geq \sigma_2 \ldots \sigma_r > 0$, for $r = rank(A)$, while $\sigma_s = 0$ for $s > r$. In standard description we have for A and $pinv(A) = A^+$:

$$A * x = \sum_{k=1}^{r} \sigma_k \langle x, v_k \rangle u_k, \quad A^+ * y = \sum_{k=1}^{r} \frac{1}{\sigma_k} \langle y, u_k \rangle v_k.$$

Generally known facts in this situation

The Four Spaces are well known from LINEAR ALGEBRA, e.g. in the **dimension formulas**:

ROW-Rank of **A** *equals COLUMN-Rank of* **A**.

The defect (i.e. the dimension of the Null-space of **A***) plus the dimension of the range space of* **A** *(i.e. the column space of* **A***) equals the dimension of the domain space* $\mathbb{R}^n$. Or in terms of linear, homogeneous equations: The dimension of set of all solution to the homogeneous linear equations equals the number of variables minus the dimension of the column space of **A**.
The SVD also shows, that the *isomorphism between the Row-space and the Column-space* can be described by a diagonal matrix, if suitable orthonormal basis for these spaces are used.

Consequences of the SVD

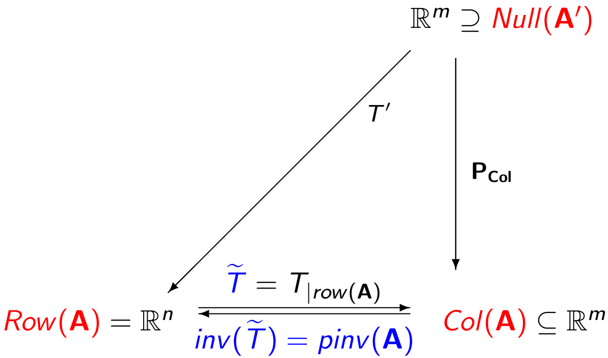
We can describe the quality of the isomorphism $\widetilde{T}$ by looking at its condition number, which is σ_1/σ_r, the so-called **Kato-condition number** of T.

It is not surprising that for **normal matrices** with $A' * A = A * A'$ one can even have diagonalization, i.e. one can choose $U = V$, because

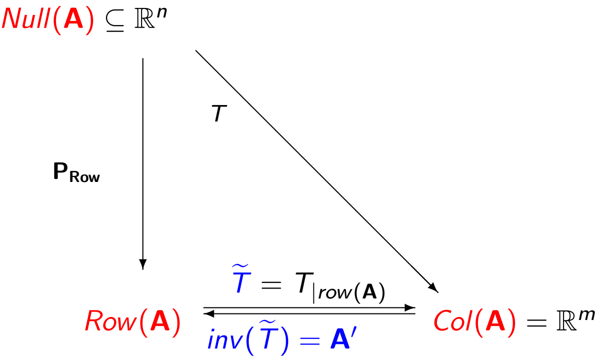
$$Null(A) =_{always} \; Null(A' * A) = Null(A * A') = Null(A').$$

The most interesting cases appear if a matrix has maximal rank, i.e. if $rank(\mathbf{A}) = min(m, n)$, or equivalently if one of the two Null-spaces is trivial. Then we have either linear independent columns of $\mathbf{A}$ (injectivity of T >> RIESZ BASIS for subspaces) or the columns of $\mathbf{A}$ span all of $\mathbb{R}^m$ (i.e. $Null(A') = \{0\}$): FRAME SETTING!

Geometric interpretation: linear independent set > R.B.

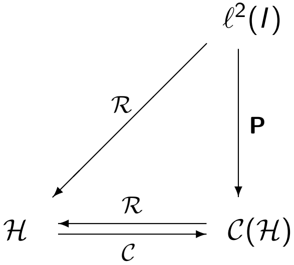


Geometric interpretation: generating set > FRAME



The frame diagram for Hilbert spaces:

If we consider **A** as a collection of column vectors, then the role of **A**$'$ is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (1).

Riesz basic sequences in Hilbert spaces:

The diagram for a Riesz basis (for a subspace), nowadays called a Riesz basic sequence looks quite the same.
In fact, from an abstract sequence there is no! difference, just like there is no difference (from an abstract viewpoint) between a matrix $\mathbf{A}$ and the transpose matrix $\mathbf{A}'$.
However, it makes a lot of sense to think that in one case the collection of vectors (making up a Riesz BS) spans the (closed) subspace of $\mathcal{H}$ by just taking all the infinite linear combinations (series) with ℓ^2-coefficients.
In this way the synthesis mapping $\mathbf{c} \mapsto \sum_i c_i g_i$ from $\ell^2(I)$ into the closed linear span is *surjective*, while in the frame case the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ from $\mathcal{H}$ into $\ell^2(I)$ is injective (with bounded inverse).

Frames versus atomic decompositions

Although the *definition of frames in Hilbert spaces* emphasizes the aspect, that the frame elements define (via the Riesz representation theorem) an injective analysis mapping, the usefulness of frame theory rather comes from the fact that frames allow for atomic decompositions of arbitrary elements $f \in \mathcal{H}$.
One could even replace the lower frame bound inequality in the definition of frames by assuming that one has a Bessel sequence (i.e. that the upper frame bound is valid) with the property that the synthesis mapping from $\ell^2(I)$ into $\mathcal{H}$, given by $\mathbf{c} \mapsto \sum_i c_i g_i$ is *surjective* onto *all of* $\mathcal{H}$.
Analogously one can find Riesz bases interesting (just like linear independent sets) because they allow to uniquely determine the coefficients of f in their closed linear span on that closed subspace of $\mathcal{H}$.

A hierarchy of conditions 1

While the following conditions are equivalent in the case of a finite dimensional vector space (we discuss the frame-like situation) one has to put more assumptions in the case of separable Hilbert spaces and even more in the case of Banach spaces.
Note that one has in the case of an infinite-dimensional Hilbert space: A set of vectors $(f_i)_{i \in I}$ is total in $\mathcal{H}$ if and only if the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ is injective. In contrast to the frame condition nothing is said about a series expansion, and in fact for better approximation of $f \in \mathcal{H}$ a completely different finite linear combination of $g_i's$ can be used, without any control on the ℓ^2-norm of the corresponding coefficients.
THEREFORE one has to make the assumption that the range of the coefficient mapping has to be a *closed subspace* of $\ell^2(I)$ in the discussion of *frames in Hilbert spaces*.

A hierarchy of conditions 2

In the case of Banach spaces one even has to go one step further. Taking the norm equivalence between some Banach space norm and a corresponding sequence space norm in a suitable Banach space of sequences over the index set I (replacing $\ell^2(I)$ for the Hilbert space) is not enough!
In fact, making such a definition would come back to the assumption that the coefficient mapping $\mathcal{C} : f \mapsto (\langle f, g_i \rangle)$ allows to identify with some closed subspace of that Banach space of sequences. Although in principle this might be a useful concept it would not cover typical operations, such as taking Gabor coefficients and applying localization or thresholding, as the modified sequence is then typically not in the range of the sampled STFT, but resynthesis should work!

A hierarchy of conditions 3

What one really needs in order to have the diagram is the identification of the Banach space under consideration (modulation space, or Besov-Triebel-Lozirkin space in the case of wavelet frames) with a close and complemented subspace of a larger space of sequences (taking the abstract position of $\ell^2(I)$.
To assume the existence of a left inverse to the coefficient mapping allows to establish this fact in a natural way. Assume that $\mathcal{R}$ is the left inverse to $\mathcal{C}$. Then $\mathcal{C} \circ \mathcal{R}$ is providing the projection operator (the orthogonal projection in the case of $\ell^2(I)$, if the canonical dual frame is used for synthesis) onto the range of $\mathcal{C}$. The converse is an easy exercise: starting from a projection followed by the inverse on the range one easily obtains a right inverse operator $\mathcal{R}$.

A hierarchy of conditions 4

The above situation (assuming the validity of a diagram and the existence of the reconstruction mapping) is part of the definition of Banach frames as given by K. Gröchenig in [**?**].
Having the classical situation in mind, and the *spirit of frames in the Hilbert spaces case* one should however add two more conditions:
In order to avoid trivial examples of Banach frames one should assume that the associated Banach space $(\mathbf{B} \| \cdot \|_{\mathbf{B}})$ of sequences should be assumed to be solid, i.e. satisfy that $|a_i| \leq |b_i|$ for all $i \in I$ and $b \in B$ implies $a \in \mathbf{B}$ and $\|a\|_{\mathbf{B}} \leq \|b\|_{\mathbf{B}}$.
Then one could identify the reconstruction mapping $\mathcal{R}$ with the collection of images of unit vectors $h_i := \mathcal{R}(\vec{e}_i)$, where $\vec{e}_i$ is the unit vector at $i \in I$. Moreover, unconditional convergence of a series of the form $\sum_i c_i h_i$ would be automatic.

A hierarchy of conditions 6

Instead of going into this detail (including potentially the suggestion to talk about unconditional Banach frames) I would like to emphasize another aspect of the theory of Banach frames. According to *my personal opinion* it is not very interesting to discuss individual Banach frames, or the existence of *some Banach frames* with respect to *some abstract Banach space of sequences*, even if the above additional criteria apply.

The *interesting cases* concern situations, where the coefficient and synthesis mapping concern a whole family of related Banach spaces, the setting of Banach Gelfand triples being the minimal (and most natural) instance of such a situation.

A comparison: As the family, consisting of father, mother and the child is the foundation of our social system, Banach Gelfand Triples are the prototype of *families*, sometimes *scales of Banach spaces*, the "child" being of course our beloved Hilbert space.

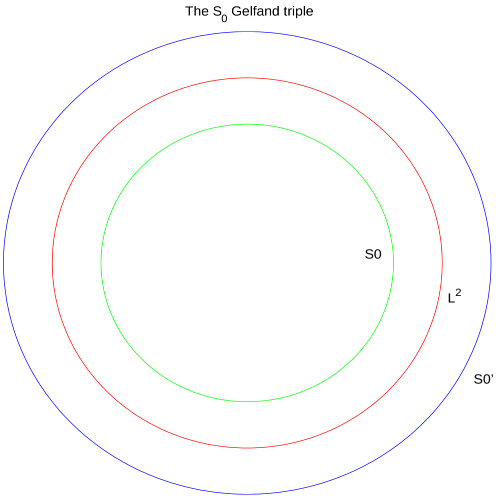
Banach Gelfand Triples and Rigged Hilbert space

The next term to be introduced are Banach Gelfand Triples.
There exists already and established terminology concerning triples of spaces, such as the Schwartz triple consisting of the spaces $(\mathcal{S}, \mathbf{L}^2, \mathcal{S}')(\mathbb{R}^d)$, or triples of weighted Hilbert spaces, such as $(\mathbf{L}^2_w, \mathbf{L}^2, \mathbf{L}^2_{1/w})$, where $w(t) = (1 + |t|^2)^{s/2}$ for some $s > 0$, which is - via the Fourier transform isomorphic to another ("Hilbertian") Gelfand Triple of the form $(\mathcal{H}_s, \mathbf{L}^2, \mathcal{H}_s{}')$, with a Sobolev space and its dual space being used e.g. in order to describe the behaviour of elliptic partial differential operators.
The point to be made is that suitable Banach spaces, in fact imitating the prototypical Banach Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ allows to obtain a surprisingly large number of results resembling the finite dimensional situation.

A Classical Example related to Fourier Series

There is a well known and classical example related to the more general setting I want to describe, which - as so many things - go back to N. Wiener. He introduced (within $\mathbf{L}^2(\mathbb{T})$) the space $\left(\mathbf{A}(\mathbb{T}), \|\cdot\|_{\mathbf{A}}\right)$ of absolutely convergent Fourier series. Of course this space sits inside of $\left(\mathbf{L}^2(\mathbb{T}), \|\cdot\|_2\right)$ as a dense subspace, with the norm $\|f\|_{\mathbf{A}} := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|$.
Later on the discussion about Fourier series and generalized functions led (as I believe naturally) to the concept of pseudo-measures, which are either the elements of the dual of $\left(\mathbf{A}(\mathbb{T}), \|\cdot\|_{\mathbf{A}}\right)$, or the (generalized) inverse Fourier transforms of bounded sequences, i.e. $\mathcal{F}^{-1}(\ell^\infty(\mathbb{Z}))$.
In other words, this extended view on the Fourier analysis operator $\mathcal{C} : f \mapsto (\widehat{f}(n)_{n\in\mathbb{Z}})$ on the BGT $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})$ into $(\ell^1, \ell^2, \ell^\infty)$ is the prototype of what we will call a BGT-isomorphism.

The visualization of a Banach Gelfand Triple



Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?);
- It is computed using the FFT (what is the connection);
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency (FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \mathbf{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{4}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{5}$$

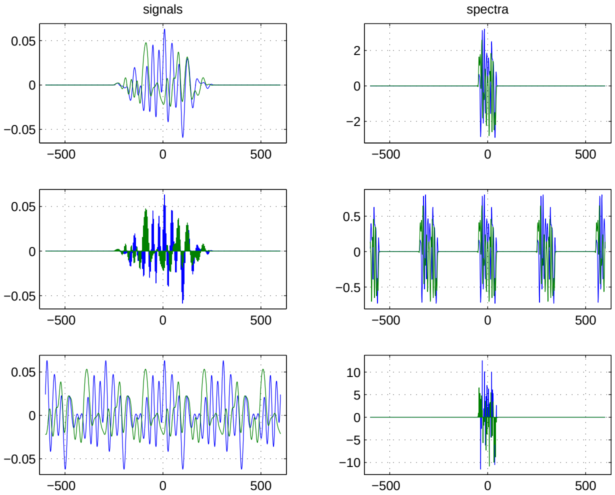
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$. One often speaks of Fourier analysis being the first step, and the Fourier inversion as a method to build f from the pure frequencies (we talk of Fourier synthesis).

The classical situation with Fourier

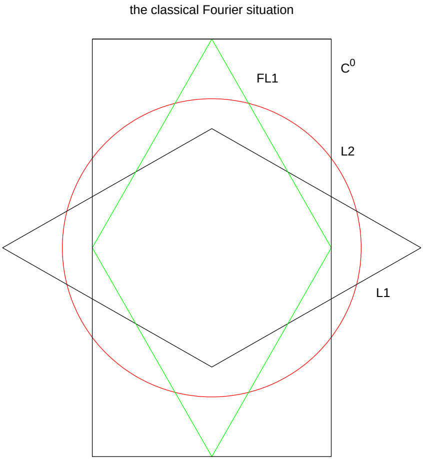
Unfortunately the Fourier transform does not behave well with respect to $\mathbf{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \mathbf{L}^1(\mathbb{R}^d)$ we have $\hat{f} \in \mathbf{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\mathbf{L}^1(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d)$ is isometric in the $\mathbf{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\mathbf{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\hat{f}$.

Effects of Sampling and Periodization: Poisson's formula



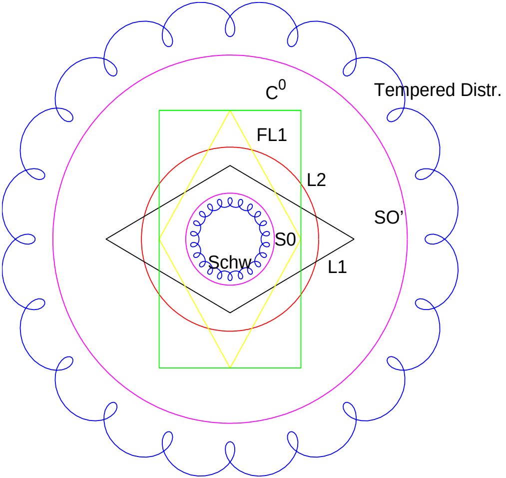
A schematic description of the situation



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. But there are easier alternatives.

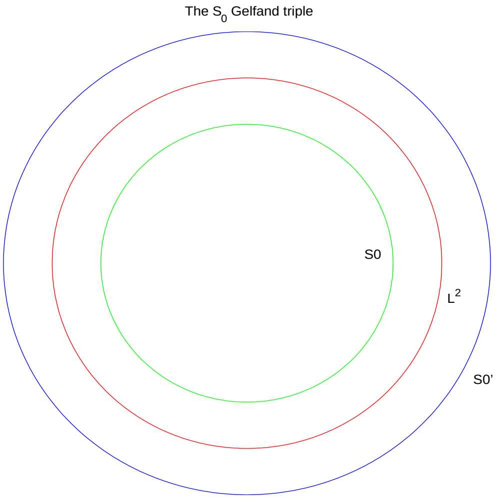
A schematic description of the situation



The Banach space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ or $(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1})$), which will serve our purpose. Its dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to Banach Gelfand Triples, mostly related to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

repeated: SOGELFTR



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

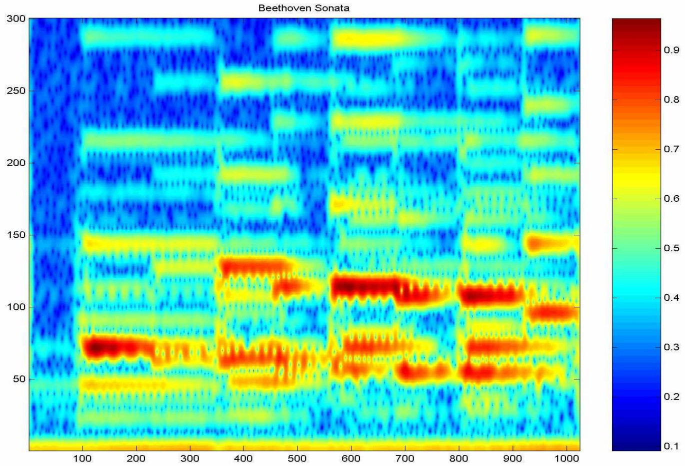
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\mathbf{M}^1 = \mathbf{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).
There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) = \mathbf{M}^0_{1,1}(\mathbb{R}^d)$.

Basic properties of $\mathbf{M}^\infty(\mathbb{R}^d) = \mathbf{S}_0'(\mathbb{R}^d)$

It is probably no surprise to learn that the dual space of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$, i.e. $\mathbf{S}_0'(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts $\pi(\lambda), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

As an amalgam space one has $\mathbf{S}_0'(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)' = \mathbf{W}(\mathcal{F}\mathbf{L}^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $\mathbf{M}^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in \mathbf{S}_0(\mathbb{R}^d)$).

Consequently norm convergence in $\mathbf{S}_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain atomic characterizations of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT. – Hifi recordings!

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\mathbf{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{T}) = \mathbf{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.

For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the "ax+b"-group) on finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler's formula) to cos and sin functions in order to build a Wilson ONB for $\mathbf{L}^2(\mathbb{R}^d)$.

In this way another BGT-isomorphism between $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{6}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.
This is a typical case, where we can see, that the w^*-continuity plays a role, and where the fact that $\delta_x \in \mathbf{S}_0'(\mathbb{R}^d)$ as well as $\chi_s \in \mathbf{S}_0'(\mathbb{R}^d)$ are important.
In the STFT-domain the w^*-convergence has a particular meaning: a sequence σ_n is w^*-convergent to σ_0 if $V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda)$ uniformly over compact subsets of the TF-plane (for one or any $g \in \mathbf{S}_0(\mathbb{R}^d)$).

Classical Results in Banach Space Theory

Wiener's inversion theorem:

Theorem

Assume that $h \in \mathbf{A}(\mathbb{T})$ is free of zeros, i.e. that $h(t) \neq 0$ for all $t \in \mathbb{T}$. Then the function $g(t) := 1/h(t)$ belongs to $\mathbf{A}(\mathbb{T})$ as well.

The proof of this theorem is one of the nice applications of a spectral calculus with methods from Banach algebra theory.
This result can be reinterpreted in our context as a results which states:
Assume that the pointwise multiplication operator $f \mapsto h \cdot f$ is invertible as an operator on $\left(\mathbf{L}^2(\mathbb{T}), \|\cdot\|_2\right)$, and also a BGT-morphism on $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})$ (equivalent to the assumption $h \in \mathbf{A}(\mathbb{T})$!), then it is also continuously invertible as BGT-morphism.

SOGTr-results in Banach Triple terminology

In the setting of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ a quite similar results is due to Gröchenig and coauthors:

Theorem

Assume that for some $g \in \mathbf{S}_0$ the Gabor frame operator $S : f \mapsto \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda$ is invertible at the Hilbert space level, then S defines automatically an automorphism of the BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$. Equivalently, when $g \in \mathbf{S}_0$ generates a Gabor frame (g_λ), then the dual frame (of the form $(\widetilde{g}_\lambda)$) is also generated by the element $\widetilde{g} = S^{-1}(g) \in \mathbf{S}_0$.

The first version of this result has been based on matrix-valued versions of Wiener's inversion theorem, while the final result (due to Gröchenig and Leinert, see [?]) makes use of the concept of symmetry in Banach algebras and Hulanicki's Lemma.

Classical Results in Banach Space Theory 2

Theorem (Theorem by S. Banach)

Assume that a linear mapping between two Banach spaces is continuous, and invertible as a mapping between sets, then it is automatically an isomorphism of Banach spaces, i.e. the inverse mapping is automatically linear and continuous.

So we have invertibility only in a more comprehensive category, and want to conclude invertibility in the given smaller (or richer) category of objects.

Other relations to the finite-dimensional case

The paper [?]: Gabor frames without inequalities Int. Math. Res. Not. IMRN, No.23, (2007) contains another collection of statements, showing the strong analogy between a finite-dimensional setting and the setting of Banach Gelfand triples: The main result (Theorem 3.1) of that paper shows, that the Gabor frame condition (which at first sight looks just like a two-sided norm condition) is in fact equivalent to injectivity of the analysis mapping (however at the "outer level", i.e. from $\mathbf{S}_0'(\mathbb{R}^d)$ into $\ell^\infty(\mathbb{Z}^d)$), while it is also equivalent to surjectivity of the synthesis mapping, but this time from $\ell^1(\mathbb{Z}^d)$ onto $\mathbf{S}_0(\mathbb{R}^d)$.

Kernel Theorem for general operators in $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$

Theorem

If K is a bounded operator from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathbf{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \mathbf{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

Kernel Theorem II: Hilbert Schmidt Operators

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels. The complete picture can be best expressed by a unitary Gelfand triple isomorphism. First the innermost shell:

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\mathbf{L}^2(\mathbb{R}^{2d})$ on the kernels. An operator T has a kernel in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ if and only if the T maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$, boundedly, but continuously also from w^*-topology into the norm topology of $\mathbf{S}_0(\mathbb{R}^d)$.

Kernel Theorem III

Remark: Note that for such regularizing kernels in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ the usual identification. Recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

Note that $\delta_y \in \mathbf{S}_0'(\mathbb{R}^d)$ implies that $K(\delta_y) \in \mathbf{S}_0(\mathbb{R}^d)$ by the regularizing properties of K, hence the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

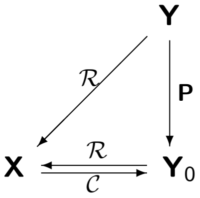
$$(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')).$$

AN IMPORTANT TECHNICAL warning!!

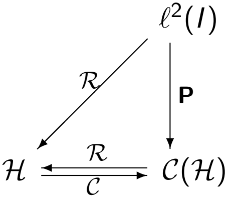
How should we realize these various BGT-mappings?
Recall: How can we check numerically that $e^{2\pi i} = 1$??
Note: we can only do our computations (e.g. multiplication, division etc.) properly in the rational domain $\mathbb{Q}$, we get to $\mathbb{R}$ by approximation, and then to the complex numbers applying "the correct rules" (for pairs of real numbers).
In the BGT context it means: All the (partial) Fourier transforms, integrals etc. only have to be meaningful at the $\mathbf{S}_0$-level!!! (no Lebesgue even!), typically isometric in the $\mathbf{L}^2$-sense, and extend by duality considerations to $\mathbf{S}_0'$ when necessary, using w^*-continuity!
The Fourier transform is a good example (think of Fourier inversion and summability methods), similar arguments apply to the transition from the integral kernel of a linear mapping to its Kohn-Nirenberg symbol., e.g..

Frames and Riesz Bases: the Diagram

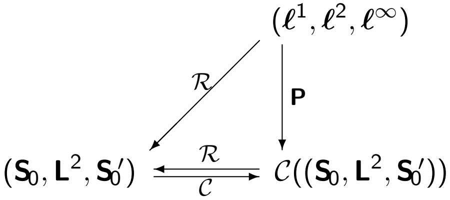
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ. Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping $\mathcal{C}$ from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse $\mathcal{R}$ (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.
In practice it means, that the dual Gabor atom $\tilde{g}$ is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.

Gabor frame matrix representations

Much in the same way as basis in $\mathbb{C}^n$ are used in order to describe linear mappings as matrices we can also use Gabor frame expansions in order to describe (and analyze resp. better understand) certain linear operators T (slowly variant channels, operators in Sjoestrand's class, connected with another family of modulation spaces) by their frame matrix expansion. Working (for convenience) with a Gabor frame with atom $g \in \mathbf{S}_0(\mathbb{R}^d)$ (e.g. Gaussian atom, with $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$), and form for $\lambda, \mu \in \Lambda$ the infinite matrix

$$a_{\lambda,\mu} := [T(\pi(\lambda)g)](\pi(\mu)g).$$

This makes sense even if T maps only $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{S}_0'(\mathbb{R}^d)$!

Gabor frame matrix representations II

For any good Gabor family (tight or not) the mapping $T \mapsto \mathbf{A} = (a_{\lambda,\mu})$ is it self defining a frame representation, hence a retract diagram, from the operator BGT $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ into the $(\ell^1, \ell^2, \ell^\infty)$ over $\mathbb{Z}^{2d}$!
In other words, we can recognize whether an operator is regularizing, i.e. maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$ (with w^*-continuity) if and only if the matrix has coefficients in $\ell^1(\mathbb{Z}^{2d})$.
Note however, that invertibility of T is NOT equivalent to invertibility of $\mathbf{A}$! (one has to take the pseudo-inverse).

The Spreading Representation

The kernel theorem corresponds of course to the fact that every linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^n$ can be represented by a uniquely determined matrix $\mathbf{A}$, whose columns $\mathbf{a}_k$ are the images $T(\vec{e}_k)$. When we identify $\mathbb{C}^N$ with $\ell^2(\mathbf{Z}_N)$ (as it is suitable when interpreting the FFT as a unitary mapping on $\mathbb{C}^N$) there is another way to represent every linear mapping: we have exactly N cyclic shift operators and (via the FFT) the same number of frequency shifts, so we have exactly N^2 TF-shifts on $\ell^2(\mathbf{Z}_N)$. They even form an orthonormal system with respect to the Frobenius norm, coming from the scalar product

$$\langle \mathbf{A}, \mathbf{B} \rangle_{Frob} := \sum_{k,j} a_{k,j} \bar{b}_{k,j} = trace(A * B')$$

This relationship is called the spreading representation of the linear mapping T resp. of the matrix $\mathbf{A}$. It can be thought as a kind of operator version of the Fourier transform.

The unitary spreading BGT-isomorphism

Theorem

There is a natural (unitary) Banach Gelfand triple isomorphism, called the spreading mapping, which assigns to operators T from $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ the function or distribution $\eta(T) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. It is uniquely determined by the fact that $T = \pi(\lambda) = M_\omega T_t$ corresponds to $\delta_{t,\omega}$.

Via the symplectic Fourier transform, which is of course another unitary BGT-automorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ we arrive at the Kohn-Nirenberg calculus for pseudo-differential operators. In other words, the mapping $T \mapsto \sigma_T = \mathcal{F}_{symp}\eta(T)$ is another unitary BGT isomorphism (onto $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$, again).

Consequences of the Spreading Representation

The analogy between the ordinary Fourier transform for functions (and distributions) with the spreading representation of operators (from nice to most general within our context) has interesting consequences.

We know that Λ-periodic distributions are exactly the ones having a Fourier transform supported on the orthogonal lattice $\Lambda^\perp$, and periodizing an $\mathbf{L}^1$-function corresponds to sampling its FT.

For operators this means: an operator T commutes with all operators $\pi(\Lambda)$, for some $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, if and only if $\mathrm{supp}(\eta(T)) \subset \Lambda^\circ$, the adjoint lattice. The Gabor frame operator is the Λ-periodization of $P_g : f \mapsto \langle f, g \rangle g$, hence $\eta(S)$ is obtained by multiplying $\eta(P_g) = V_g(g)$ pointwise by $⊔⊔_{\Lambda^\circ} = \sum_{\lambda^\circ \in \Lambda^\circ} \delta_{\lambda^\circ}$.

Consequences of the Spreading Representation 2

This observation is essentially explaining the Janssen representation of the Gabor frame operator (see [**?**]).
Another analogy is the understanding that there is a class of so-called underspread operators, which are well suited to model slowly varying communication channels (e.g. between the basis station and your mobile phone, while you are sitting in the - fast moving - train).
These operators have a known and very limited support of their spreading distributions (maximal time- and Doppler shift on the basis of physical considerations), which can be used to "sample" the operator (pilot tones, channel identification) and subsequently decode (invert) it (approximately).

Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathbf{S}_0(\mathbb{R}^d)$ (typically one will take g and $\tilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity ($>$ Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ), with $g \in \mathbf{S}_0(\mathbb{R}^d)$ defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$-sense on both parameters.

THANK YOU!

Thank you for your attention!

Most of the referred papers of NuHAG can be downloaded from
http://www.univie.ac.at/nuhag-php/bibtex/

Furthermore there are various talks given in the last few years on related topics (e.g. Gelfand triples), that can be found by searching by title or by name in
http://www.univie.ac.at/nuhag-php/nuhag_talks/

Selection of bibliographic items, see www.nuhag.eu