A BANACH GELFAND TRIPLE
motivated by and useful for time-Frequency
An easy path to distribution theory
also suitable for engineers

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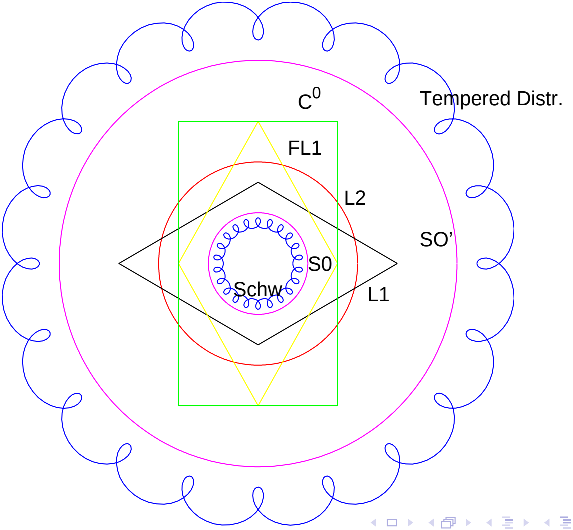
OVERVIEW

THE GOAL OF THIS PRESENTATION IS TO CONVEY THE CONCEPTS OF **Gelfand Triples**, IN PARTICULAR BANACH GELFAND TRIPELS, BUT ALSO BANACH FRAMES BY DESCRIBING THEM AND SHOW THEIR USEFULNESS IN THE CONTEXT OF MATHEMATICAL ANALYSIS, IN PARTICULAR TIME-FREQUENCY ANALYSIS

- Recall some concepts from linear algebra, especially that of a *generating system*, a *linear independent* set of vectors, and that of the dual vector space;
- already in the context of Hilbert spaces the question arises: *what is a correct generalization of these concepts?*
- Banach Gelfand Triple (comparable to rigged Hilbert spaces) are one way out;

A suitable Banach space of test functions & distributions



... there is an implicit message:

Aside from the various technical terms coming up I hope to convey implicitly a few other messages, closely related to my *view of Harmonic Analysis* as the sub-field of functional analysis which has to do with group actions:

- staying with Banach spaces and their duals one can do amazing things (avoiding topological vector spaces, Lebesgue integration, or Schwartz distribution theory);
- alongside with the norm topology just the very natural w^*-topology, just in the form of pointwise convergence of functionals, for the dual space has to be kept in mind (allowing the use of non-reflexive Banach spaces);
- diagrams and operator descriptions allow to naturally generalize concepts from finite dimensional theory up to the category of Banach Gelfand triples.

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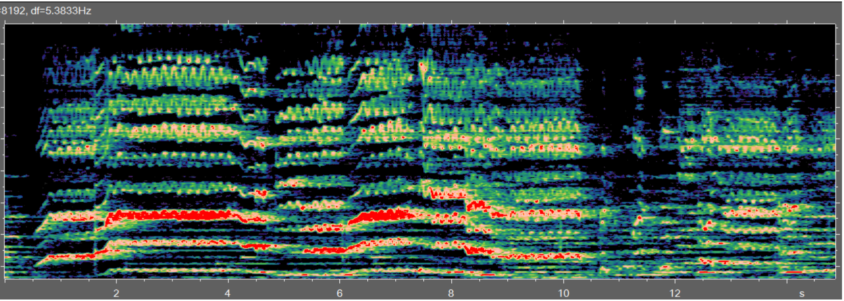
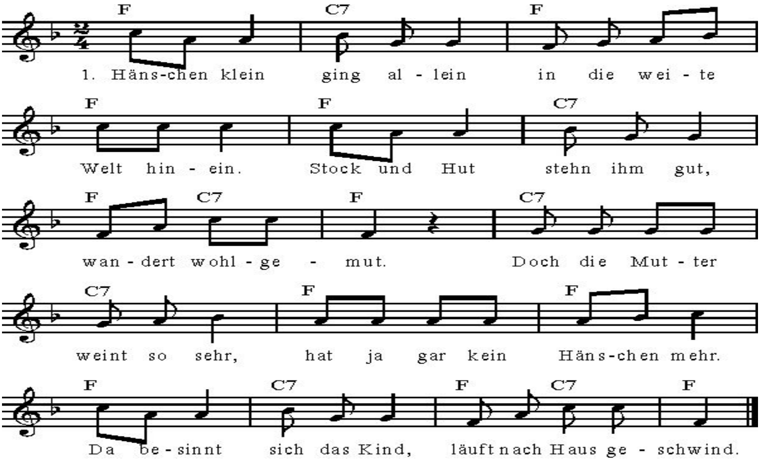
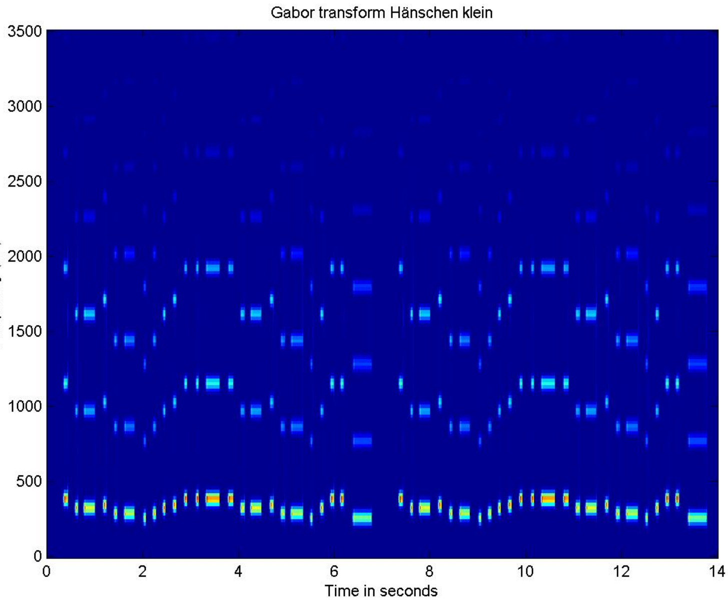
A picture of the singing of Pavarotti



compared to musical score ...



The analysis of a synthetic sound example



The two main difficulties in Gabor Analysis

There has been a very important influence on the understanding of signal expansions by the work of Denis Gabor (1946) which is related to the discussion of bases in vector spaces of signals. These difficulties are related to two short-comings which arise (in fact very naturally) in the context of Gabor analysis, if one has the "trivial" linear algebra situation in mind:

1. Given a **finite-dimensional** vector space it makes sense to mostly work with bases. They are either maximal (finite) sequence of vectors (by enlarging linear independent vectors), or by searching for minimal generating sets (obtained by removing elements from any given generating set);
2. While one has in the setting of signals of finite length both the basis of unit vectors resp. the orthogonal basis of pure frequencies characters are not anymore in the Hilbert space $\mathbf{L}^2(\mathbb{R})$ of signals(of finite energy).

Denis Gabor's suggestion II

We will discuss Gabor's suggestion of 1946 in more detail, but basically his reasoning was based on the following (good and interesting) idea: Using the Gauss-function as a building block, which is optimally concentrated in a TF-sense (it provides equality in the Heisenberg uncertainty) he suggested to represent *arbitrary signals/functions* as superpositions of TF-shifted version of the Gauss-functions along some lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$, with $a = b = 1$. His reasoning was based on the following intuition:

- If $a \cdot b > 1$ then one does not have sufficiently many building blocks in the Hilbert spaces in order to even approximate general functions in $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$. In fact, for *any* $g \in \mathbf{L}^2(\mathbb{R})$ (not just the Gauss function) the closed linear span of such a *Gabor family* is a proper subspace of $\mathbf{L}^2(\mathbb{R})$.
- If on the other hand $a \cdot b$ ¡ 1 then the corresponding family gets linear dependent, and therefore the representation would not be *unique* anymore.

Denis Gabor's suggestion III

- As a consequence of these two observations he came to the conclusion that the choice $a = 1 = b$ should be ideal, i.e. *should* allow for a unique representation of general $\mathbf{L}^2(\mathbb{R})$-functions f, and thus give the coefficients a very clear meaning.
- While the suggestion was ignored by most mathematicians (except for A.J.E.M. Janssen, who tried to give it a precise, mathematical meaning using distribution theory) until the late 70-th, the idea was developed further in the engineering community, typically with the comment that Gabor expansions are "unfortunatly" numerically instable, although quite intuitive (microtonal piano);

Denis Gabor's suggestion IV

Let me give an interpretation of the idea, and formulate (the wishes, first) in a strict mathematical language.

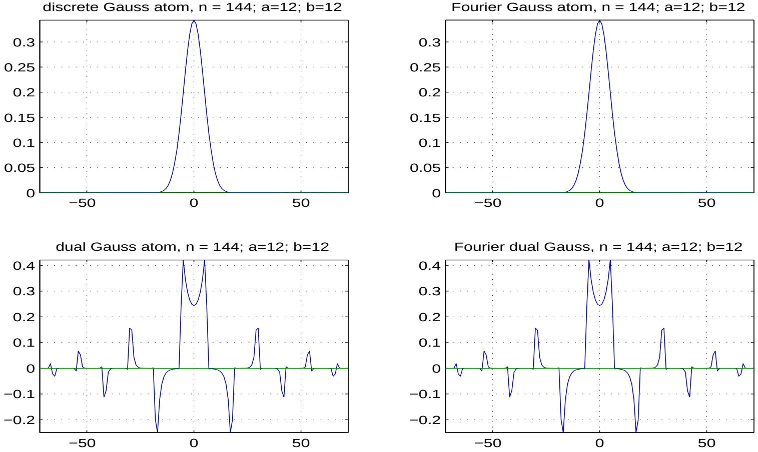
- It is obvious that the uniqueness of coefficients implies that (as in the linear algebra situation) the coefficient mapping assigning each $f \in \mathbf{L}^2(\mathbb{R})$ its coefficient in such a system would be linear. It is quite sure, that D. Gabor would have agreed (i.e. confirmed potential hopes) that those coefficients should be in $\ell^2(\mathbb{Z}^2)$ and that the coefficient mapping is also continuous. In fact, if small changes in the signal would introduce huge changes in the coefficients their usefulness would be very much spoiled. So in other words it is probably not overinterpreting D. Gabor if one says that most likely he was claiming that the suggested family should be a **Riesz-basis for the Hilbert space** $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$.

Denis Gabor's suggestion V

- One might argue that the price to be paid for the specific and interesting structure of the Gabor family would by just the fact that one is loosing orthogonality (obviously: two shifted Gaussians are never perpendicular to each other in $\mathbf{L}^2(\mathbb{R})$!), but otherwise there should be some invertible (Banach-space) operator on $\mathbf{L}^2(\mathbb{R})$ which allows to turn the Gabor family into an orthonormal basis. In fact, nowadays we would argue that one just has to the so called Löwdin orthogonalization, i.e. apply the inverse square root of the Gram-matrix of this system in order to obtain an orthonormal Riesz basis. If this was in fact possible one could even claim: replace the Gauss-function by some other (Schwartz) function in $\mathbf{L}^2(\mathbb{R})$, then you can even have an orthonormal basis of Gabor-type.
- one can do a MATLAB experiment, with interesting results.

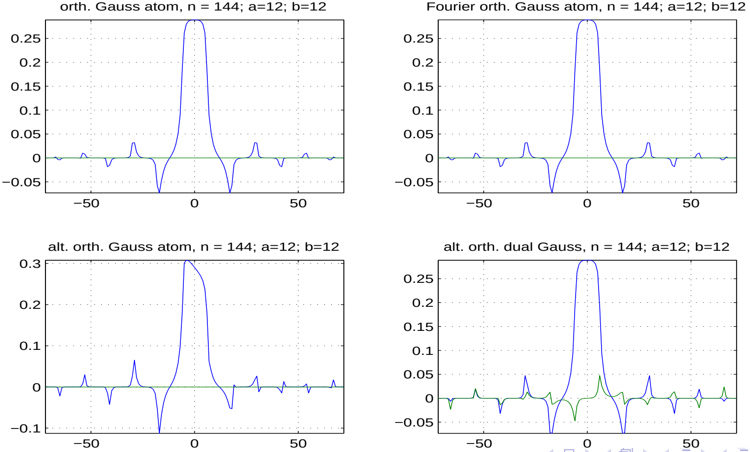
Finite Gabor families at critical density

Either the biorthogonal family is not well concentrated (and in fact one dimension is lost!), or ...



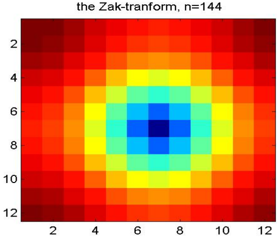
Finite Gabor families at critical density

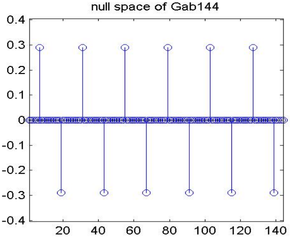
same for the (quasi-orthonormal) version (Löwdin). OR one takes the "symmetric Gauss function" (invariant under the flip-operation in MATLAB): orthogonalization then:



Finite Gabor families at critical density

The loss of one dimension (the orthogonal complement of the range is one-dimensional) is connected with the zero of the Zak-transform of the Gauss function at $(1/2, 1/2)$ (see left part of the picture);





What is known about Gabor families I

What we also know nowadays:

- For any value of a, b the corresponding Gabor families are *linear independent* in the classical sense (cf. **Heil's conjecture**), i.e. finite linear combinations are zero if and only if the sequence of coefficients is trivial;
- Gabor's family is dense in $\mathbf{L}^2(\mathbb{R})$, so finite linear combinations suffice in order to approximate arbitrary $\mathbf{L}^2(\mathbb{R})$-functions. HOWEVER, if one wants to obtain a better approximation the coefficients have to be recalculated from scratch, and the better the approximation, the higher the (cost, i.e. the) ℓ^2-norm of the corresponding coefficients will be;
- the set is not even minimal in that respect, one can even remove one element and still have the same property (however not two or more elements!)

What is known about Gabor families II

On the other hand we know positively:

- when $a = 1 = b$ then not every $f \in \mathbf{L}^2(\mathbb{R})$ can be represented using $\ell^2(\mathbb{Z}^2)$-coefficients [not even bounded coefficients suffice, according to A.J.E.M. Janssen];
- If $ab < 1$ the corresponding Gabor family is a stable generating system (a so-called *frame* for $\mathbf{L}^2(\mathbb{R})$);
- any element in the system can be expressed as a linear combination of the remaining ones, even when only ℓ^1-coefficients are allowed;
- For $ab < 1$ the so-called dual frame (providing the minimal ℓ^2-coefficients) is in fact another Schwartz-function, hence the whole procedure has good locality.

Lessons to be learned from this situation

If one tries to imitate the situation envisaged by D. Gabor in the setting of the finite discrete group $\mathbb{Z}_{144}$ one comes to the following conclusions/observations:

- if one takes the correct Gauss-function (which is invariant under the discrete Fourier transform) one is loosing in the critical case, $a = 12 = b$, one dimension, therefore one has to take - if the TF-structure is important - redundant families of TF-shifted discrete Gauss-functions;
- Having taken this decision it is natural to obtain coefficients using the strategy of *normal equations*, i.e. to go for the MNLSQ-solution of the problem. It turns out that this is possible (using the Moore-Penrose pseudo-inverse) and the resulting family is essentially another Gabor family, with the *dual generator*;
- In a similar way, making the lattice constants larger one can get fine linear independent Gabor families which have biorthogonal families which are again of a Gaborian structure.

Choose Generating System or Lin. Independence

One possible conclusion from the above problems is the observation that there might be some conflict with the wish of having a certain structure of a (Riesz) basis, combined with good numerical stability as well as good localization in time and frequency.
It turns out that one has to make a distinction, whether one wants to expand arbitrary signals (then generating systems with a decent redundancy are preferred over those with minimal redundancy but bad condition number!), or whether (a stable form of) linear independence is relevant: This is of course the case for (mobile/digital) communication.

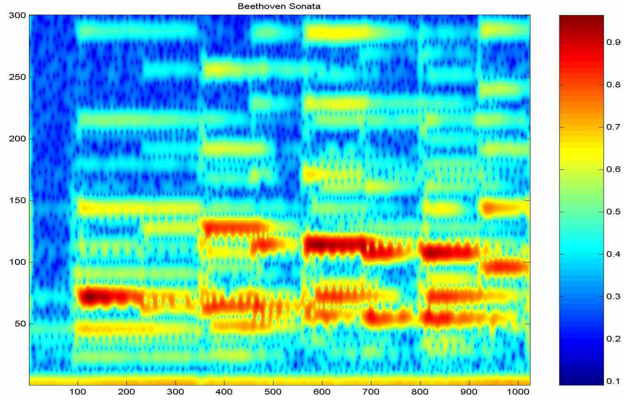
Moving to the continuous setting, e.g. $\mathbb{R}^d$

If one replaces the finite sequences with "function on the real line $\mathbb{R}$" or more general on Euclidian spaces $\mathbb{R}^n$ one faces extra challenges, and questions like the following arise:

- Is it true, that the synthesis mapping from $\ell^2(\mathbb{Z}^{2d})$ into $\mathbf{L}^2(\mathbb{R}^d)$ is continuous for every $g \in \mathbf{L}^2(\mathbb{R}^d)$? (NO!);
- Can one say that the (continuous) STFT (short-time Fourier transform) of $V_g(f) \in \mathbf{L}^2(\mathbb{R}^{2d})$ belongs to $\ell^2(\mathbb{Z}^{2d})$? (NO!);
- If a Gabor family defines an orthonormal (or just Riesz) basis for a given lattice, can we be sure that it is still a Riesz basis for lattices which are closeby? (NO, take $\mathbf{1}_{[0,1]}$ and the standard lattice $\mathbb{Z}^2$);
- What about Gabor expansions for $\mathbf{L}^p$-functions?

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

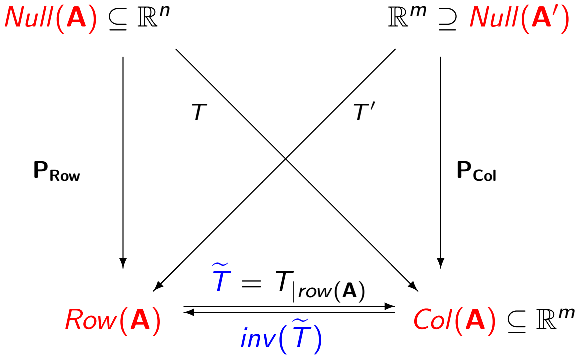
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda)g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t,\omega);$$

The Schrödinger Representation

For people in representation theory I could explain the spectrogram is just displaying to you a typical representation coefficient of the (projective) Schrödinger Representation of the (reduced) Heisenberg Group $\mathbb{H}^d$ (for $d = 1$).
According to Roger Howe this group has the phantastic "hinduistic multiplicity in one" property of allowing a variety of different looking but in fact mathematically equivalent representations (due to the von-Neumann uniqueness theorem), which indicates the connection to quantum mechanics, the theory of coherent states, and related topics (where e.g. rigged Hilbert spaces, the bras and kets appear already), where concepts as described below are in fact also helpful (to put expressions such as continuous integral representations on a firm mathematical ground); but we will start from known grounds...

Geometric interpretation of matrix multiplication



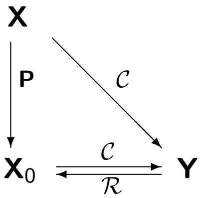
$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Matrices of maximal rank

We will be mostly interested (as models for Banach Frames and Riesz projection bases) in the situation of matrices of maximal ranks, i.e. in the situation where $r = \text{rank}(A) = max(m, n)$, where $A = (a_1, \cdots, a_k)$.
Then either the synthesis mapping $x \mapsto A * x = \sum_k x_k a_k$ has trivial kernel (i.e. the column vectors of A are a linear independent set, spanning the column-space of which is of dimension $r = n$), or the analysis mapping $y \mapsto A' * y = (\langle y, a_k \rangle)$ has trivial kernel, hence the column spaces equals the target space (or $r = m$), or the column vectors are a spanning set for $\mathbb{R}^m$.

.... continued

For *Riesz basic sequences* we have the following diagram:



Definition

A sequence (h_k) in a separable Hilbert space $\mathcal{H}$ is a *Riesz basis* for its closed linear span (sometimes also called a Riesz basic sequence) if for two constants $0 < D_1 \leq D_2 < \infty$,

$$D_1 \|c\|_{\ell^2}^2 \leq \Big\| \sum_k c_k h_k \Big\|_{\mathcal{H}}^2 \leq D_2 \|c\|_{\ell^2}^2, \qquad \forall c \in \ell^2 \tag{1}$$

A detail description of the concept of *Riesz basis* can be found in

Reflect also for a moment about daily actions:

We are calculating with all kind of numbers in our daily life. But just recall the most beautiful equation

$$e^{2\pi i} = 1.$$

It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

$$cos(2\pi) + i * sin(2\pi) = 1 \quad \text{in} \quad \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Existing examples of Gelfand Triples

So-called *Gelfand Triples* are already widely used in various fields of analysis. The prototypical example in the theory of PDE is certainly the *Schwartz Gelfand triple*, consisting of the space of test functions $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions, densely sitting inside of $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which in turn is embedded into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d). \tag{2}$$

Alternatively (e.g. for elliptic PDE) one used

$$\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{H}_s'(\mathbb{R}^d). \tag{3}$$

It is obtained via the Fourier transform form

$$\mathbf{L}^2_w(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2_w(\mathbb{R}^d)'. \tag{4}$$

What is a generating set in a Hilbert space

We teach in our linear algebra courses that the following properties are equivalent for a set of vectors $(f_i)_{i \in I}$ in $\mathbf{V}$:

1. The only vector perpendicular to a set of vectors is $\emptyset$;
2. Every $v \in \mathbf{V}$ is a linear combination of these vectors.

An attempt to transfer these ideas to the setting of Hilbert spaces one comes up with several different generalizations:

- a family is *total* if its linear combinations are dense;
- a family is a *frame* if there is a bounded linear mapping from $\mathcal{H}$ into $\ell^2(I)$ $f \mapsto \mathbf{c} = c(f) = (c_i)_{i \in I}$ such that

$$f = \sum_{i \in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{5}$$

The usual definition of frames

There is another, *equivalent* characterization of frames. First, it is an obvious consequence of the characterization given above, that

$$f = \sum_{i \in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{6}$$

implies that there exists $C, D > 0$ such that

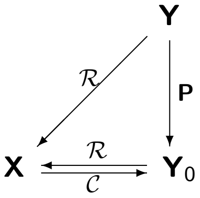
$$C\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{7}$$

For the converse observe that $Sf := \sum_{i \in I} \langle f, f_i \rangle f_i$ is a strictly positive definite operator and the *dual frame* $(\tilde{f}_i)$ satisfies

$$f = \sum_{i \in I} \langle f, \widetilde{f}_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \widetilde{f}_i$$

Frames and Riesz Bases: the Diagram

$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



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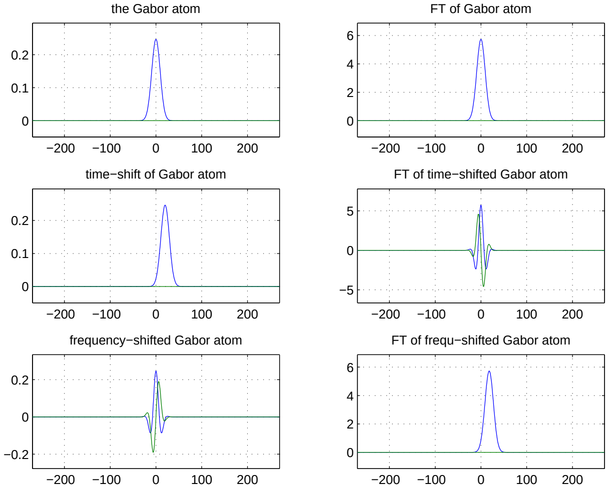
Denis Gabor's suggestion of 1946

There is one very interesting example (the prototypical problem going back to D. Gabor, 1946): Consider the family of all time-frequency shifted copies of a standard Gauss function $g_0(t) = e^{-\pi|t|^2}$ (which is invariant under the Fourier transform), and shifted along $\mathbb{Z}$ ($T_n f(z) = f(z-n)$) and shifted also in time along $\mathbb{Z}$ (the modulation operator is given by $M_k h(z) = \chi_k(z) \cdot h(z)$, where $\chi_k(z) = e^{2\pi ikz}$).
Although D. Gabor gave some heuristic arguments suggesting to expand every signal from $\mathbf{L}^2(\mathbb{R})$ in a unique way into a (double) series of such "Gabor atoms", a deeper mathematical analysis shows that we have the following problems (the basic analysis has been undertaken e.g. by A.J.E.M. Janssen in the early 80s):

TF-shifted Gaussians: Gabor families



Problems with the original suggestion

Even if one allows to replace the time shifts from along $\mathbb{Z}$ by time-shifts along $a\mathbb{Z}$ and accordingly frequency shifts along $b\mathbb{Z}$ one faces the following problems:

1. for $a \cdot b = 1$ (in particular $a = 1 = b$) one finds a *total* subset, which is not a frame nor Riesz-basis for $\mathbf{L}^2(\mathbb{R})$, which is redundant in the sense: after removing one element it is still total in $\mathbf{L}^2(\mathbb{R})$, while it is not total anymore after removal of more than one such element;
2. for $a \cdot b > 1$ one does not have anymore totalness, but a Riesz basic sequence for its closed linear span ($\subsetneq \mathbf{L}^2(\mathbb{R})$);
3. for $a \cdot b < 1$ one finds that the corresponding Gabor family is a *Gabor frame*: it is a redundant family allowing to expand $f \in \mathbf{L}^2(\mathbb{R})$ using ℓ^2-coefficients (but one can remove infinitely many elements and still have this property!);

Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?);
- It is computed using the FFT (what is the connection);
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency (FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \mathbf{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{8}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{9}$$

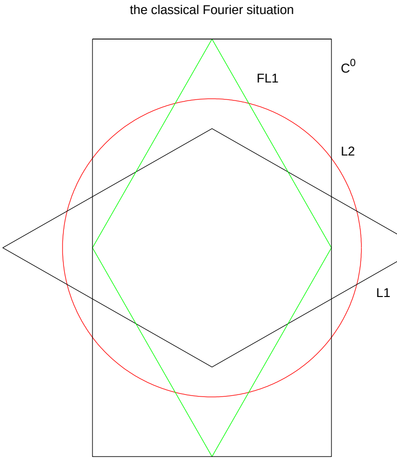
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$. One often speaks of Fourier analysis followed by Fourier inversion as a method to build f from the pure frequencies (Fourier synthesis).

The classical situation with Fourier

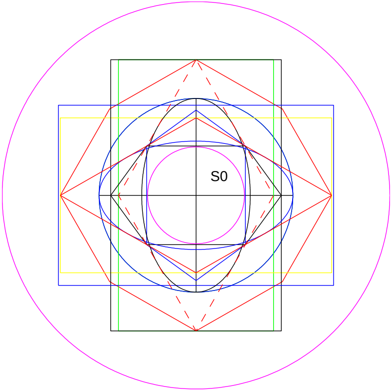
Unfortunately the Fourier transform does not behave well with respect to $\mathbf{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \mathbf{L}^1(\mathbb{R}^d)$ we have $\hat{f} \in \mathbf{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in \mathbf{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\mathbf{L}^1(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d)$ is isometric in the $\mathbf{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\mathbf{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\hat{f}$.

A schematic description of the situation



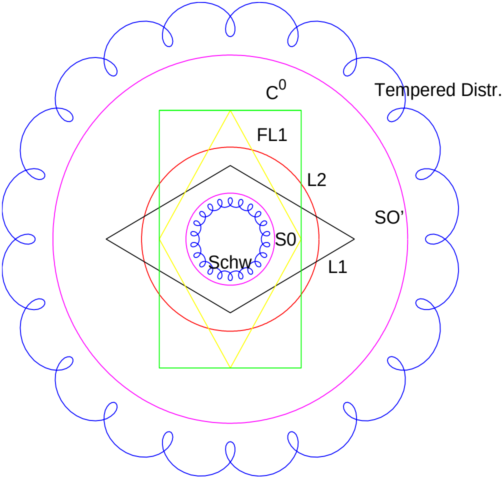
A schematic description (more details/spaces)



The way out: Test Functions and Generalized Functions

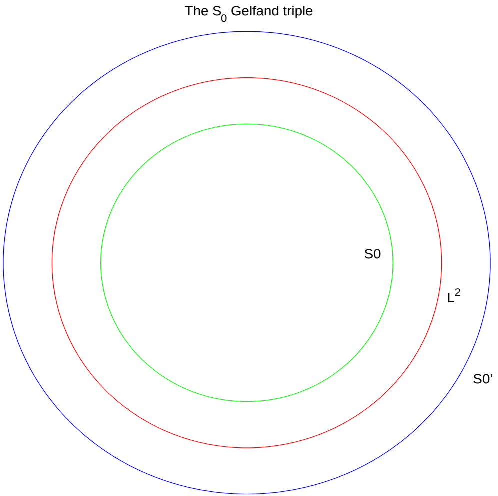
The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

A schematic description of the situation



The Banach space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ or $(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1}))$, which will serve our purpose. Its dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.


The S0 Gelfand triple
S0
L2
S0'

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\mathbf{M}^1 = \mathbf{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as **B** the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{U}) = \mathbf{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{10}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

FACTS

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):

Theorem

Assume that for $g \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor frame operator

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

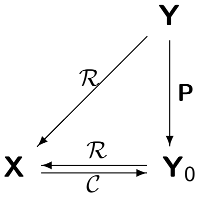
is invertible as an operator on $\mathbf{L}^2(\mathbb{R}^d)$, then it is also invertible on $\mathbf{S}_0(\mathbb{R}^d)$ and in fact on $\mathbf{S}_0'(\mathbb{R}^d)$.
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S is (resp. extends to) an isomorphism of the Gelfand triple automorphism for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

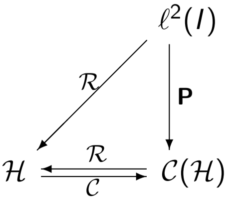
It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

Frames and Riesz Bases: the Diagram

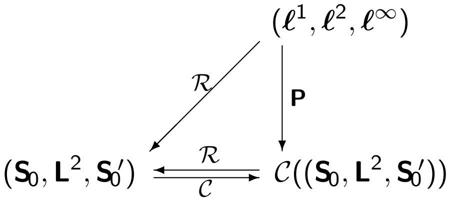
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ. Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping $\mathcal{C}$ from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse $\mathcal{R}$ (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.
In practice it means, that the dual Gabor atom $\tilde{g}$ is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.

Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathbf{S}_0(\mathbb{R}^d)$ (typically one will take g and $\tilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity (> Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ)*, with* $g \in \mathbf{S}_0(\mathbb{R}^d)$ *defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the* $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$*-sense on both parameters.*

Invertibility, Surjectivity and Injectivity

In another, very recent paper, Charly Groechenig has discovered that there is another analogy to the finite dimensional case: There one has: A square matrix is invertible if and only if it is surjective or injective (the other property then follows automatically).
We have a similar situation here (systematically describe in Charly's paper):
`K.Grchenig: Gabor frames without inequalities, Int. Math. Res. Not. IMRN, No.23, (2007).`

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\mathbf{B} = \mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$.

Theorem

There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. *This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. *Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators* $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

Kernel Theorem for general operators in $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$

Theorem

If K is a bounded operator from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathbf{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \mathbf{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

Kernel Theorem II: Hilbert Schmidt Operators

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels. The complete picture can be best expressed by a unitary Gelfand triple isomorphism. First the innermost shell:

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S \rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\mathbf{L}^2(\mathbb{R}^{2d})$ on the kernels. An operator T has a kernel in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ if and only if the T maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$, boundedly, but continuously also from w^*-topology into the norm topology of $\mathbf{S}_0(\mathbb{R}^d)$.

Kernel Theorem III

Remark: Note that for such regularizing kernels in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ the usual identification. Recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x, y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

Note that $\delta_y \in \mathbf{S}_0'(\mathbb{R}^d)$ implies that $K(\delta_y) \in \mathbf{S}_0(\mathbb{R}^d)$ by the regularizing properties of K, hence the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')).$$

AN IMPORTANT TECHNICAL warning!!

How should we realize these various BGT-mappings?
Recall: How can we check numerically that $e^{2\pi i} = 1$??
Note: we can only do our computations (e.g. multiplication, division etc.) properly in the rational domain $\mathbb{Q}$, we get to $\mathbb{R}$ by approximation, and then to the complex numbers applying "the correct rules" (for pairs of real numbers).
In the BGT context it means: All the (partial) Fourier transforms, integrals etc. only have to be meaningful at the $\mathbf{S}_0$-level!!! (no Lebesgue even!), typically isometric in the $\mathbf{L}^2$-sense, and extend by duality considerations to $\mathbf{S}_0'$ when necessary, using w^*-continuity!
The Fourier transform is a good example (think of Fourier inversion and summability methods), similar arguments apply to the transition from the integral kernel of a linear mapping to its Kohn-Nirenberg symbol., e.g..

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
Therefore it is interesting to check what the w^*-convergence looks like:

Lemma

For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ a sequence σ_n is w^*-convergent to σ_0 if and only the spectrograms $V_g(\sigma_n)$ converge uniformly over compact sets to the spectrogram $V_g(\sigma_0)$.

The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

The w^*- topology: dense subfamilies

From the practical point of view this means, that one has to look at the spectrograms of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution $V_g(\sigma_0)$ over compact sets.
The approximation of elements from $\mathbf{S}_0'(\mathbb{R}^d)$ takes place by a bounded sequence.
Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^*-dense subsets of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.
Let us look at some concrete examples: Test-functions, finite discrete measures $\mu = \sum_i c_i \delta_{t_i}$, trigonometric polynomials $q(t) = \sum_i a_i e^{2\pi i \omega_i t}$, or discrete AND periodic measures (this class is invariant under the generalized Fourier transform and can be realized computationally using the FFT).

The w^*- topology: approximation strategies

- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \to \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an $\mathbf{L}^1$-Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite disrete sequence using quasi-interpolation (N. Kaiblinger): $Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x)$.