NuHAG Numerical Harmonic Analysis Group

Foundations of Computational Time-Frequency Analysis

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

July 6th, 2011
FoCM 2011, BUDAPEST

FoCM Context and Motivation

When reading papers in electric engineering (e.g. concerning problem in communication theory) or in talking to engineers it often taken for granted that *signals have to be of finite energy* (in order to justify the work within $\boldsymbol{L}^2(\mathbb{R})$ or $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$), or to say: when we have to work with the computer all the information comes in the form of sequences in $\ell^2(\mathbb{Z})$ resp. even finite sequences, and hence e.g. the more complicated Fourier transform (originally given as an integral transform, with a highly oscillator kernel has to be replaced by the (fast) version of the DFT (discrete Fourier transform), the well-known FFT!
There are also all kinds of heuristic transition from e.g. the case of periodic functions expanded into Fourier series to the Fourier transform, or from the Fourier coefficients defined by integrals to the finite DFT.

Structure of the talk

1. Motivating QUESTIONS: Fourier transforms, time-variant systems, spline-type spaces;
2. The TOOLS: The setting of Banach Gelfand Triples, specifically $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$;
3. How the properties of these spaces and their topologies can be put to good use in the FoCM spirit;
4. What kind of results we can derive resp. work on at NuHAG (www.nuhag.eu);

[original] ABSTRACT

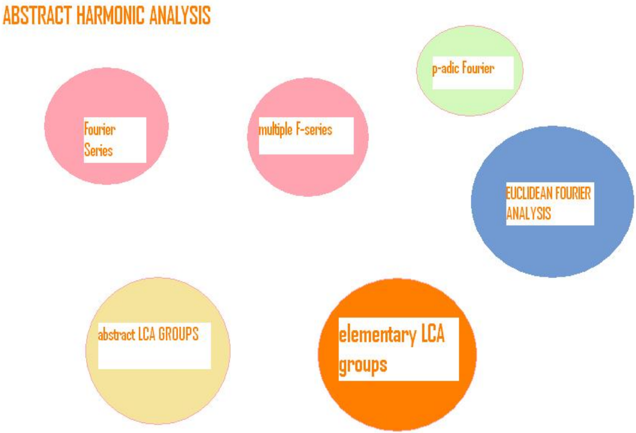
In the last two decades Gabor Analysis and Time-Frequency Analysis in general have made significant progress. Gabor Analysis over LCA groups is in principle well understood, while computational methods have been established which allow to realize at least the most important aspects of Gabor analysis (computation of dual windows, realization of Gabor multipliers, best approximation of a given matrix by an operator of this type, etc.). We also have a number of results indicating not only the robustness of Gabor systems with respect to perturbations (change of the atom, or small modification of the TF-lattice used), but also that the continuous problem (over Euclidean spaces $\mathbb{R}^d$) can be approximated - at least in an asymptotic sense and for test functions from the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ - by finite models.

The talk is going to describe the setting of Banach Gelfand Triples as the appropriate frame-work for the description of these approximation processes, indicates existing results and methods used in this field already now, and describes the demand for further research in order to improve from qualitative asymptotic to quantitative results, in the spirit of approximation theory. So whatever should be computed (e.g. a dual Gabor atom, the action of a pseudo-differential operator on an L^2-function, etc.) one should have tools to describe, how a realizable (by actual computation, using finitely many matrices etc.) approximation can be achieved by suitable, hopefully at least suboptimal, procedures, which allow to compute the entity under consideration up to a given $\varepsilon > 0$, or up to a given relative error in some appropriate norm (such as a Sobolev norm or a Shubin norm). It will be demonstrated that the interplay between functional analysis, harmonic and numerical analysis and approximation theory can provide such methods.

What AHA is providing, and what not!

Coming from *Abstract Harmonic Analysis* I am used to work with general LCA (locally compact Abelian) groups, in the spirit of Andre Weil, who has correctly propagated the claim that this is the appropriate setting for doing Fourier Analysis.
This view-point allows to use a *unified language* for the classical theory of Fourier series with one and many variables, but also of Fourier transforms over Euclidean spaces or the DFT (resp. FFT, the Fast Fourier Transform, for periodic and discrete signals).
While this analogy allows to transfer statements concerning functions or measures over such groups (resp. their duals) from one setting into the other setting it is not immediately useful when it comes to make problems in a continuous setting accessible to computations (using vectors of finite length).

The classical view on the Fourier Transform



Goals of this lecture

The goal of this lecture is - very much in the spirit of FoCM - to initiate a more detailed discussion about the use of finite methods in order to come up with valid approximation to the continuous situation. Typically we want to compute the value of a linear functional, or some linear operator on a function spaces, and in order to do so we have to come up with a *realizable* way of finding an approximate answer (in the given norm), with a prescribed small error.

Note that the *effectiveness* required here is different from the concept of *constructive approximation*, where a *concrete prescription* of steps which might at the end of the day NOT be realizable is good enough.

The Fourier transform as a prototypical example

We all know that the (classical) Fourier transform is well defined on the space $\boldsymbol{L}^1(\mathbb{R}^d)$ of Lebesgue integrable functions via

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{1}$$

The inverse Fourier transform then has the form

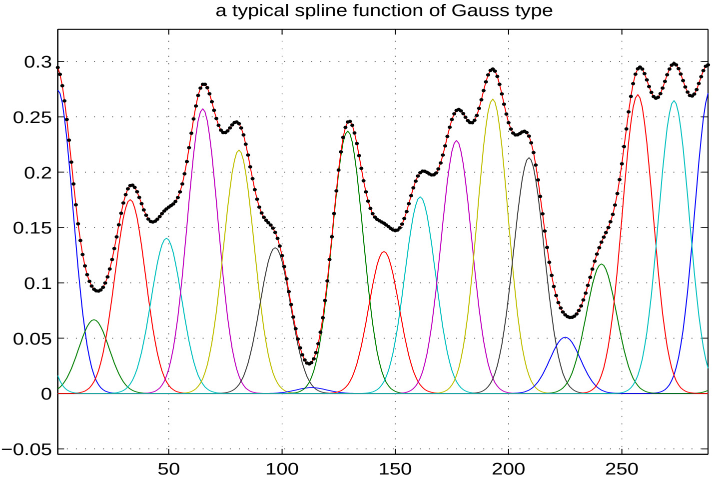
$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{2}$$

But what is an efficient way of calculating the Fourier (or its inverse)? Is it really enough (at least for decent functions) to sample the function f sufficiently fine and put the resulting vector into the FFT-routine. Moreover, is it OK to assume that the output is just a sampled sequence of $\hat{f}$??

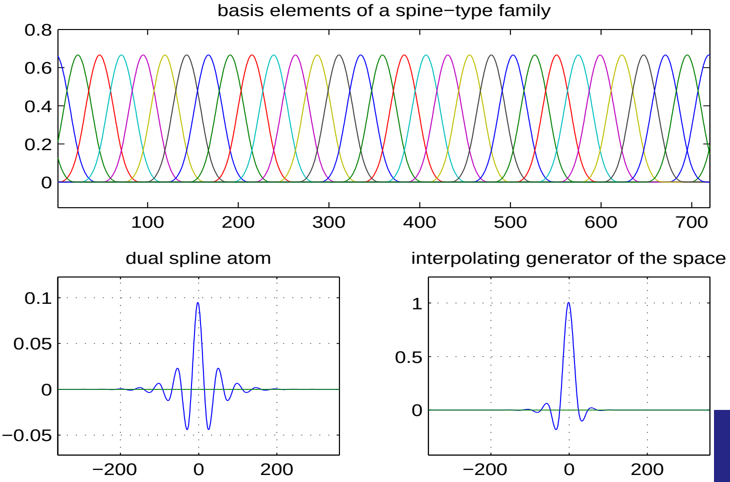
How to simulate slowly time-variant channels?

Another, more sophisticated problem is the question of simulating and understanding the behaviour of *slowly time variant channels*. This are more or less *moving average operators*, but not with a constant, but rather a slowly changing profile, so to say a time-dependent lowpass-filter. Obviously one expects that it behaves locally like a convolution operator with constant profile, but also they cannot be diagonalized by the strictly by the Fourier transform. So how can we model them, describe them matheamtically, simulate their behaviour (in order to find good transmission schemes, ways for channel identification and channel decoding, using pilote tones).
As I learned from an engineer (Werner Kozek) this has a lot to do with the Kohn-Nirenberg representation of an operator resp. its symplectic Fourier transform, the so-called spreading function.

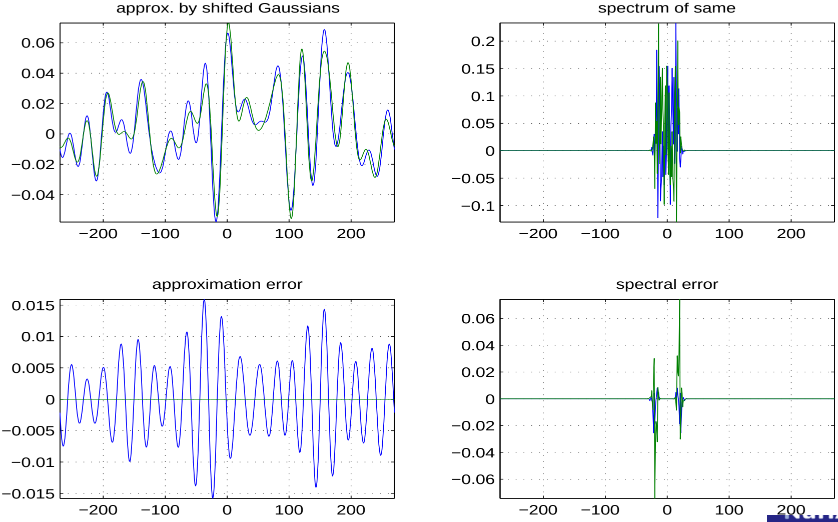
Spline-type spaces with Gaussian kernel



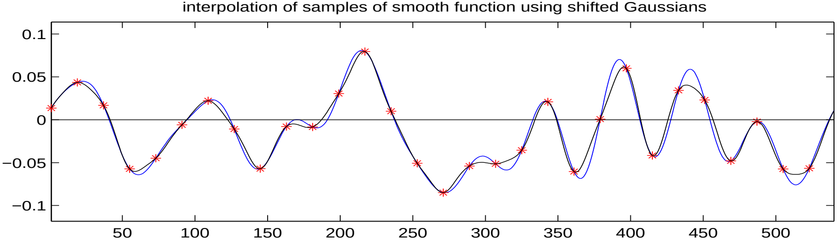
Spline-type functions: dual and interpolating atom

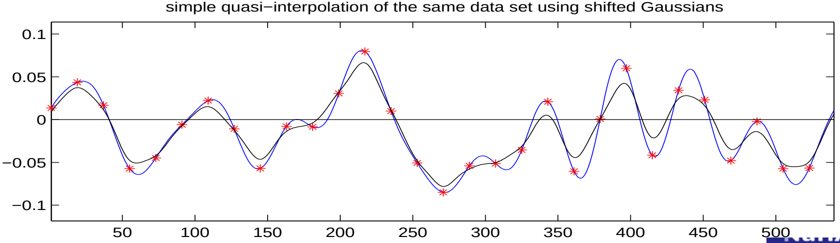


Spline-type spaces: best approx. of smooth functions



Spline-type functions: dual and interpolating atom





Multi-Window-Spline-type spaces

Natural generalization (joint work with Darian Onchis, [?]): *finitely generated shift-invariant spaces*, also called in our context *multi-window spline-type spaces*.

There are closed formulas for the projection operator onto such a spline-type space, and constructive descriptions of an iterative algorithm allowing recovery of functions in such spaces from irregular samples, but they are not realizable as they stand.

So let us describe the existing results in this direction shortly next. However for this we will need a certain (generally useful) Banach space of functions, called $\boldsymbol{S}_0(\mathbb{R}^d)$.

What kind of approximation can we go for?

The question concerning the way in which the realizable computations should approximate the continuous problem is by no means a trivial one, and the answer depends very much on the problem at hand.
Among the most typical methods to realizably approximate the action of a linear functional or a given operator on some input signal/function is of course to generate a sequence of operators resp. functionals which converge in the strong operator topology to the desired limit, just like (finite) Riemannian sums approximate the integral.

The w^*- topology: approximation strategies

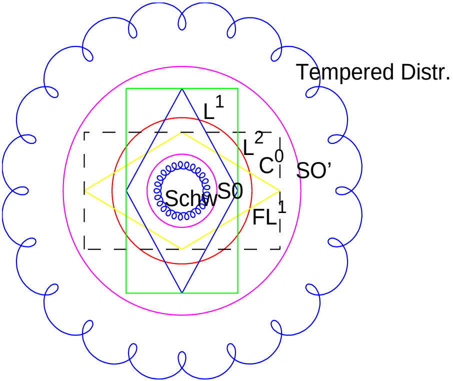
- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \to \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an $\boldsymbol{L}^1$-Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite discrete sequence using quasi-interpolation (N. Kaiblinger):

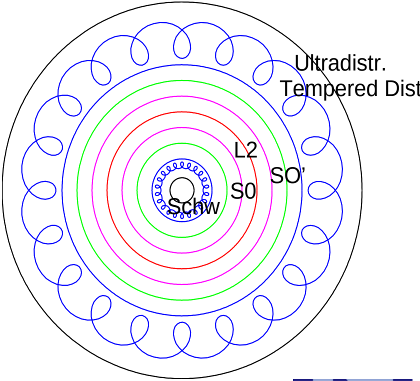
$$Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x).$$

Goals of this lecture

- that Hilbert spaces are themselves a too narrow concept and should be replaced Banach Gelfand Triples, ideally isomorphic to the canonical ones $(\ell^1, \ell^2, \ell^\infty)$;
- Demonstrate by examples (Fourier transform, kernel theorem) that this viewpoint brings us very close to the finite-dimensional setting!
- We could go on and show that the usual generalizations of linear algebra concepts to the Hilbert space case (namely linear independence and totality) are inappropriate in many cases and should be replaced by frame and Riesz basis, in fact by commutative diagrams in the category of BGTRs.

ANALYSIS: Spaces used to describe the Fourier Transform

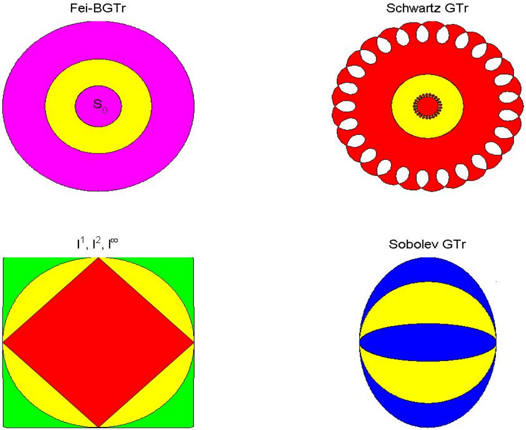




Banach Gelfand Triples and Rigged Hilbert space

The next term to be introduced are Banach Gelfand Triples.
There exists already and established terminology concerning triples of spaces, such as the Schwartz triple consisting of the spaces $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')(\mathbb{R}^d)$, or triples of weighted Hilbert spaces, such as $(\boldsymbol{L}^2_w, \boldsymbol{L}^2, \boldsymbol{L}^2_{1/w})$, where $w(t) = (1 + |t|^2)^{s/2}$ for some $s > 0$, which is - via the Fourier transform isomorphic to another ("Hilbertian") Gelfand Triple of the form $(\mathcal{H}_s, \boldsymbol{L}^2, \mathcal{H}_s{}')$, with a Sobolev space and its dual space being used e.g. in order to describe the behaviour of elliptic partial differential operators.
The point to be made is that suitable Banach spaces, in fact imitating the prototypical Banach Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ allows to obtain a surprisingly large number of results resembling the finite dimensional situation.

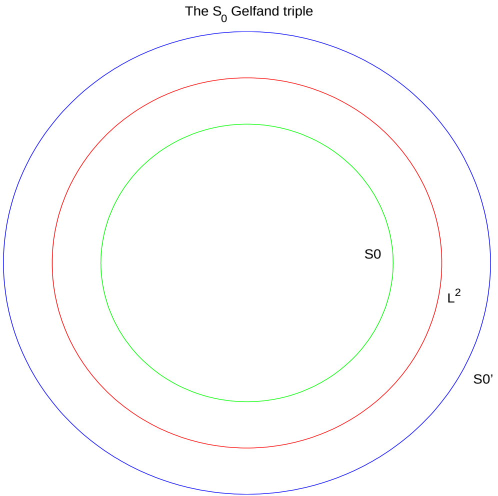
Different Gelfand Triples



A Classical Example related to Fourier Series

There is a well known and classical example related to the more general setting I want to describe, which - as so many things - go back to N. Wiener. He introduced (within $\boldsymbol{L}^2(\mathbb{T})$) the space $(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$ of absolutely convergent Fourier series. Of course this space sits inside of $(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2)$ as a dense subspace, with the norm $\|f\|_{\boldsymbol{A}} := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|$.
Later on the discussion about Fourier series and generalized functions led (as I believe naturally) to the concept of pseudo-measures, which are either the elements of the dual of $(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}})$, or the (generalized) inverse Fourier transforms of bounded sequences, i.e. $\mathcal{F}^{-1}(\boldsymbol{\ell}^\infty(\mathbb{Z}))$.
In other words, this extended view on the Fourier analysis operator $\mathcal{C} : f \mapsto (\widehat{f}(n)_{n\in\mathbb{Z}})$ on the BGT $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})$ into $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$ is the prototype of what we will call a BGT-isomorphism.

The visualization of a Banach Gelfand Triple



Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?);
- It is computed using the FFT (what is the connection);
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency (FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{3}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{4}$$

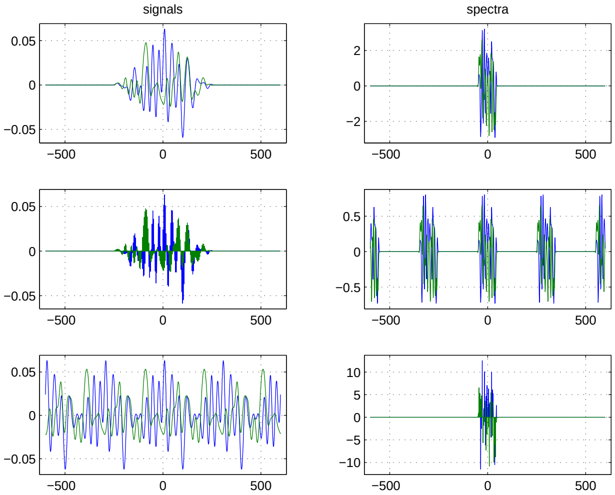
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$. One speaks of Fourier analysis as the first step, and Fourier inversion as a method to build f from the pure frequencies : Fourier synthesis.

The classical situation with Fourier

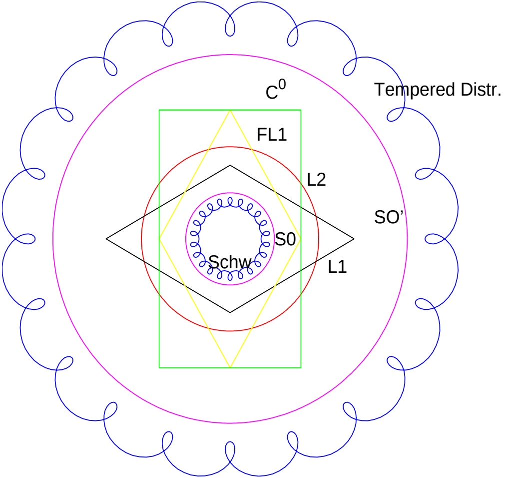
Unfortunately the Fourier transform does not behave well with respect to $\boldsymbol{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ we have $\hat{f} \in \boldsymbol{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^d)$ is isometric in the $\boldsymbol{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\boldsymbol{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\hat{f}$.

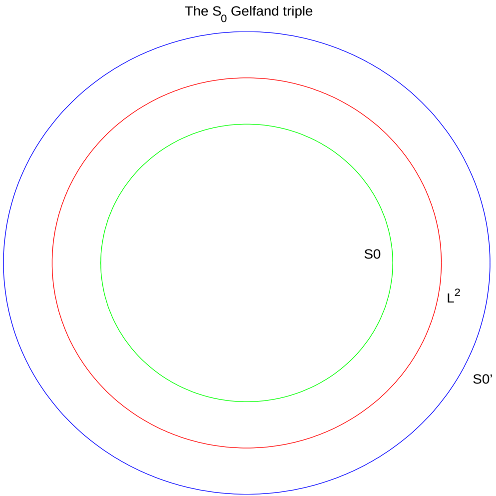
Effects of Sampling and Periodization: Poisson's formula



A schematic description of the situation



repeated: SOGELFTR



ANALYSIS: Calculating with all kind of numbers

We teach in our courses that there is a huge variety of *NUMBERS*, but for our daily life rationals, reals and complex numbers suffice. The most beautiful equation

$$e^{2\pi i} = 1.$$

It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

$$cos(2\pi) + i * sin(2\pi) = 1 \quad \text{in} \quad \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

The Banach space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ or $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1}))$, which will serve our purpose. Its dual space $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to Banach Gelfand Triples, mostly related to $(S_0, L^2, S_0')(\mathbb{R}^d)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

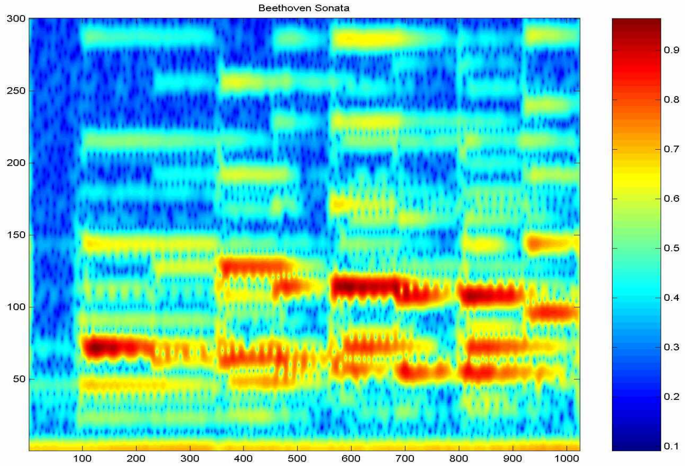
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{M}^0_{1,1}(\mathbb{R}^d)$.

Basic properties of $M^\infty(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$

The dual space of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$, i.e. $S_0'(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (resp. local pseudo-measures) which is isometrically invariant under all time-frequency shifts. As an amalgam space one has

$$S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)' = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d),$$

the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in S_0(\mathbb{R}^d)$). Consequently norm convergence in $S_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain atomic characterizations of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT. – Hifi recordings!

BANACH GELFAND TRIPLES: a new category

Definition

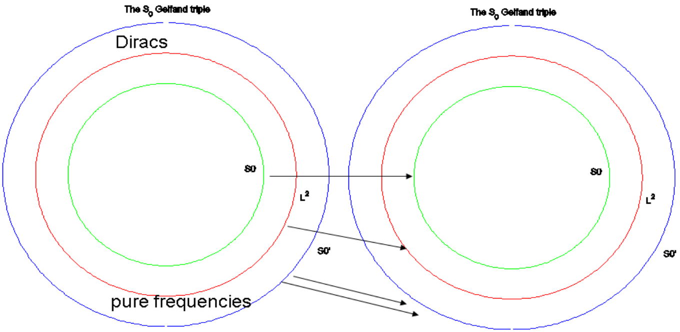
A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

Gelfand triple mapping



Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The BGT (S_0, L^2, S_0') and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.
For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the "ax+b"-group) on finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler's formula) to cos and sin functions in order to build a Wilson ONB for $\boldsymbol{L}^2(\mathbb{R}^d)$.
In this way another BGT-isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{5}$$

is valid for $(f,g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.
This is a typical case, where we can see, that the w^*-continuity plays a role, and where the fact that $\delta_x \in \mathbf{S}_0'(\mathbb{R}^d)$ as well as $\chi_s \in \mathbf{S}_0'(\mathbb{R}^d)$ are important.
In the STFT-domain the w^*-convergence has a particular meaning: a sequence σ_n is w^*-convergent to σ_0 if $V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda)$ uniformly over compact subsets of the TF-plane (for one or any $g \in \mathbf{S}_0(\mathbb{R}^d)$).

Kernel Theorem for general operators in $\mathcal{L}(S_0, S_0')$

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

Kernel Theorem II: Hilbert Schmidt Operators

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels. The complete picture can be best expressed by a unitary Gelfand triple isomorphism. First the innermost shell:

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels. An operator T has a kernel in $K \in \boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the T maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, boundedly, but continuously also from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Kernel Theorem III

Remark: Note that for such regularizing kernels in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ the usual identification. Recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x, y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

Note that $\delta_y \in \mathbf{S}_0'(\mathbb{R}^d)$ implies that $K(\delta_y) \in \mathbf{S}_0(\mathbb{R}^d)$ by the regularizing properties of K, hence the pointwise evaluation makes sense.

With this understanding the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')).$$

AN IMPORTANT TECHNICAL warning!!

How should we realize these various BGT-mappings?
Recall: How can we check numerically that $e^{2\pi i} = 1$??
Note: we can only do our computations (e.g. multiplication, division etc.) properly in the rational domain $\mathbb{Q}$, we get to $\mathbb{R}$ by approximation, and then to the complex numbers applying "the correct rules" (for pairs of real numbers).
In the BGT context it means: All the (partial) Fourier transforms, integrals etc. only have to be meaningful at the $\boldsymbol{S}_0$-level!!! (no Lebesgue even!), typically isometric in the $\boldsymbol{L}^2$-sense, and extend by duality considerations to $\boldsymbol{S}_0'$ when necessary, using w^*-continuity!
The Fourier transform is a good example (think of Fourier inversion and summability methods), similar arguments apply to the transition from the integral kernel of a linear mapping to its Kohn-Nirenberg symbol., e.g..

The Spreading Representation

The kernel theorem corresponds of course to the fact that every linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^n$ can be represented by a uniquely determined matrix $\mathbf{A}$, whose columns $\mathbf{a}_k$ are the images $T(\vec{e}_k)$. When we identify $\mathbb{C}^N$ with $\ell^2(\mathbf{Z}_N)$ (as it is suitable when interpreting the FFT as a unitary mapping on $\mathbb{C}^N$) there is another way to represent every linear mapping: we have exactly N cyclic shift operators and (via the FFT) the same number of frequency shifts, so we have exactly N^2 TF-shifts on $\ell^2(\mathbf{Z}_N)$. They even form an orthonormal system with respect to the Frobenius scal.prod.:

$$\langle \mathbf{A}, \mathbf{B} \rangle_{Frob} := \sum_{k,j} a_{k,j} \bar{b}_{k,j} = trace(A * B')$$

This relationship is called the spreading representation of the linear mapping T resp. of the matrix $\mathbf{A}$. It is a kind of operator version of the Fourier transform.

The unitary spreading BGT-isomorphism

Theorem

There is a natural (unitary) Banach Gelfand triple isomorphism, called the spreading mapping, which assigns to operators T from $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ the function or distribution $\eta(T) \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$. It is uniquely determined by the fact that $T = \pi(\lambda) = M_\omega T_t$ corresponds to $\delta_{t,\omega}$.

Via the symplectic Fourier transform, which is of course another unitary BGT-automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ we arrive at the Kohn-Nirenberg calculus for pseudo-differential operators. In other words, the mapping $T \mapsto \sigma_T = \mathcal{F}_{symp}\eta(T)$ is another unitary BGT isomorphism (onto $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$, again).

Consequences of the Spreading Representation

The analogy between the ordinary Fourier transform for functions (and distributions) with the spreading representation of operators (from nice to most general within our context) has interesting consequences.
We know that Λ-periodic distributions are exactly the ones having a Fourier transform supported on the orthogonal lattice $\Lambda^\perp$, and periodizing an $\boldsymbol{L}^1$-function corresponds to sampling its FT.
For operators this means: an operator T commutes with all operators $\pi(\Lambda)$, for some $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, if and only if $\mathrm{supp}(\eta(T)) \subset \Lambda^\circ$, the adjoint lattice. The Gabor frame operator is the Λ-periodization of $P_g : f \mapsto \langle f, g \rangle g$, hence $\eta(S)$ is obtained by multiplying $\eta(P_g) = V_g(g)$ pointwise by $⊔⊔_{\Lambda^\circ} = \sum_{\lambda^\circ \in \Lambda^\circ} \delta_{\lambda^\circ}$.

Consequences of the Spreading Representation 2

This observation is essentially explaining the Janssen representation of the Gabor frame operator (see [**?**]).
Another analogy is the understanding that there is a class of so-called underspread operators, which are well suited to model slowly varying communication channels (e.g. between the basis station and your mobile phone, while you are sitting in the - fast moving - train).
These operators have a known and very limited support of their spreading distributions (maximal time- and Doppler shift on the basis of physical considerations), which can be used to "sample" the operator (pilot tones, channel identification) and subsequently decode (invert) it (approximately).

Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\boldsymbol{S}_0(\mathbb{R}^d)$ (typically one will take g and $\widetilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity (> Balian-Low)

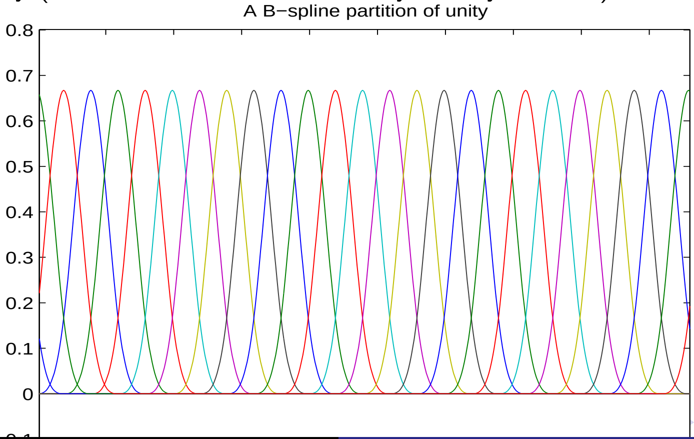
In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$-sense on both parameters.

Quasi-Interpolation in the Fourier algebra I: BUPUs

For the definition of spline-quasi-interpolation we will need the (very useful!) concept of BUPUs, something well known to everybody (maybe not by this name): Bounded Uniform Partitions of Unity (which can be made arbitrarily fine by dilation):



Quasi-Interpolation in the Fourier algebra II

Result with Norbert Kaiblinger ([?]).

Theorem

Let $\widehat{\psi} \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ and suppose that $\widehat{\psi}(k) = \delta_{k,0}$ for $k \in \mathbb{Z}$. Then for all $f \in A$ we have $\|Q_h f - f\|_A \to 0$ as $h \to 0$.

The Thm. 1 is a principle prerequisite for proving the second main result, the quasi-interpolation in $\boldsymbol{S}_0$.

Theorem

Let $\psi \in \boldsymbol{S}_0$ and suppose that $\widehat{\psi}(k) = \delta_{k,0}$ for $k \in \mathbb{Z}$. Then for all $f \in \boldsymbol{S}_0$ we have $\|Q_h f - f\|_{\boldsymbol{S}_0} \to 0$ as $h \to 0$.

Quasi-Interpolation: Consequences

What makes the quasi-interpolation results so useful (aside from the fact that more or less all the useful Banach spaces of functions and distibutions which are isometrically TF-invariant are inside of $\boldsymbol{S}_0'(\mathbb{R}^d)$, even if they do not consist of ordinary, locally Lebesgue integrable functions!), is the fact that it allows a "constructive way" of approximating an (abstract) linear functional $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ by discrete measures, in fact finite discrete measures, in the w^*-mode.
In fact, the adjoint of the quasi-interpolation operator is the operator

$$D_\Psi(\sigma) := \sum_{i \in I} \sigma(\psi_i)\delta_{x_i}.$$

where we can say, it assigns the point mass at the center the amplitude which describes the strength (effectiveness) of the action of σ near x_i, by applying σ to ψ (living near x_i!).

Multi-window spline-type spaces (with D. Onchis)

The results in this direction are a combination of the following ingredients:

- *abstract harmonic analysis* (operators commuting with translation, FFT-based constructive descriptions of the operator to be inverted);
- the proper choice of function spaces in order to correctly describe the problem and in order to carry out the error analysis in the right way (simple and useful for applications);
- then study to *realizability* aspect, with the idea of trying to keep also an eye not only on asymptotic results but on approximation theoretic features (computational costs versus size of the error);
- reduction to finite computations is demonstrated, with valid error estimates in the $\boldsymbol{S}_0$-sense.

Applications to Gabor multipliers

With the tools indicated we can find good, realizable approximations of the action of Gabor multipliers for example. Assume that a pair (g, Λ), with some $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and some lattice $\boldsymbol{A}(\mathbb{Z}^{2d})$ we have to (approximately, in the $\boldsymbol{S}_0$-sense) fine a dual atom, which can be used to synthesize/reconstruct a signal from the sampled STFT $(V_g(f)(\lambda))_{\lambda\in\Lambda}$. Of course one has to use the control of the reconstruction error (in the operator norm on either $\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d)$ or $\boldsymbol{S}_0'(\mathbb{R}^d)$ by the $\boldsymbol{S}_0$-error of the approximate dual atom).

Since it is known that general STFT-multipliers (arising also in the so-called Anti-Wick calculus) can be approximated well by Gabor multipliers we are thus also able to calculate the action of those, typically localization operators.

Best approximation of Hilbert Schmidt operators

The question of best approximation of a Hilbert Schmidt operator by a Gabor multipliers can make use of the same framework. In fact, we can never completely describe the operator, but we can find the most important of its Gabor frame matrix coefficients, i.e. (good approximate values for)

$$\langle Tg_\lambda, g_{\lambda'} \rangle, \lambda, \lambda' \in \Lambda.$$

From these information we can compute the coefficient required in order to do a best approximation of this operator by Gabor multipliers (typical application: try to approximately invert a Gabor multiplier by another Gabor multiplier).
In this context it is interesting to note that the Kohn-Nirenberg mapping allows to equivalently reformulate this problem with a spline-type approximation problem over phase-space. The spline-generator is then the KNS-symbol of the rank one operator $f \mapsto \langle f, g \rangle g$, which is in $\boldsymbol{S}_0(\mathbb{R}^{2d})$, of $g \in \boldsymbol{S}_0(\mathbb{R}^d)$!

What are the properties needed?

One can pin down the possibility of making use of the specific space $\boldsymbol{S}_0(\mathbb{R}^d)$ and its dual, also for numerical applications, essentially to a relatively small number of rather important and useful properties of the BGTR based on $\boldsymbol{S}_0(\mathbb{R}^d)$.

- The space $\boldsymbol{S}_0(\mathbb{R}^d)$ is invariant under many operations, including the Fourier transform, TF-shifts, as well as convolution by $\boldsymbol{L}^1(\mathbb{R}^d)$-functions g and multiplications by functions from $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ (Fourier algebra);
- For every lattice Λ in $\mathbb{R}^d$ the Λ-periodization of the functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ are continuous, in fact they even belong to Wiener's Algebra $\left(\boldsymbol{A}(\mathbb{T}), \|\cdot\|_{\boldsymbol{A}}\right)$.
- the restriction of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ to a discrete subgroup Λ belongs to $\ell^1(\Lambda)$.

In which way are the properties used?

- It is clear that the decay of functions can be used to sample only over sufficiently large compact domains and recover (in the $\boldsymbol{S}_0$-norm!) the function using quasi-interpolation methods. In fact, the Wiener amalgam characterization of $\boldsymbol{S}_0(\mathbb{R}^d)$ implies that any function is the *absolutely* convergent sum over the pieces living over the blocks of uniform size near the lattice points $k \in \mathbb{Z}^d$;
- x
- x

What kind of results, by the S_0-norm?

Let us discuss the case of spline-type spaces. Why should we be interested in a good recovery of the projection operator onto the spline-type space by providing a good approximation in the $\boldsymbol{S}_0(\mathbb{R}^d)$-norm?
The answer is based on the observation, that the projection operator is motivated by the Hilbert space case, but not restricted to the case of the $\boldsymbol{L}^2$-norm. In fact, nobody(!) is doing the best approximation in the $\boldsymbol{L}^p$-sense, for some fixed(!) $p > 1$, despite the well known fact that the unit balls of such spaces are uniformly convex.
On is rather happy to see that the orthogonal projection (which can be well desribed and computed, using PINV) is also stable in the $\boldsymbol{L}^p$-setting. But when we do recovery of functions from irregular samples the membership of spline function f in some $\boldsymbol{L}^p(\mathbb{R}^d)$ is a useful additional fact, but cannot be part of the algorithm.

THANK YOU!

Thank you for your attention!

Most of the referred papers of NuHAG can be downloaded from
`http://www.univie.ac.at/nuhag-php/bibtex/`

The work of the author is partially supported by the EU network UnlocX (Uncertainty and Localization) as well as the NFN SISE (FWF, 2009-2014).

Furthermore there are various talks given in the last few years on related topics (e.g. Gelfand triples), that can be found by searching by title or by name in
`http://www.univie.ac.at/nuhag-php/nuhag_talks/`

Type of Questions that can be treated

1. Realization of the Fourier transform using FFT methods;
2. Computing best approximation of decent functions from spline-type spaces;
3. Iterative reconstruction of functions in spline-type spaces from irregular samples;
4. Realizing the action of operators (e.g. slowly time-variant channels) using their Kohn-Nirenberg or spreading representations;
5. (future) Solving linear pseudo-differential equations;
6. Realizing Gabor multipliers or STFT-multipliers;
7. Computing the inverse of an invertible, slowly time-variant channel (patent!);
8. Calculating the best approximation of a given HS-operator by Gabor multipliers of a given type;

The BASIC ingredients of the approach

The $\boldsymbol{S}_0$-Banach-Gelfand-Triple setting allows as to approximate general objects of interest (distributions having bounded STFT) in the w^*-sense by any kind of nice objects, such as

- test functions, or compactly supported, continuous functions;
- finite discrete measures, but also
- discrete and periodic measures

Recall, that the $\boldsymbol{S}_0'$-norm of $\sigma \in \boldsymbol{S}_0'$ is equivalent to

$$\sup_{\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g(\sigma)(\lambda)|$$

while w^*-convergence of a sequence (or net) $(\sigma_n)_{n\geq 1}$ towards σ_0 is equivalent to the compact/open convergence of $V_g(\sigma_n)$ towards the limit $V_g(\sigma_)$.

Selection of bibliographic items, see www.nuhag.eu

THERE will be a workshop in Marburg (MACHA11), August 22-26th! for those who want to learn more about Gabor Analysis carried out using MATLAB.