NuHAG Numerical Harmonic Analysis Group

Anti-Wick Calculus and Gabor Multipliers, Functional Analytic Tools and Computations

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What are Gabor Multipliers?

This talk is about **Gabor multipliers** and their continuous analogue, which could be called STFT (= short-time Fourier transform) multipliers, known in the literature by the name of **Anti-Wick symbolic calculus**, because every such multiplier on phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ gives us a linear operator with some desired (to some extent) properties.

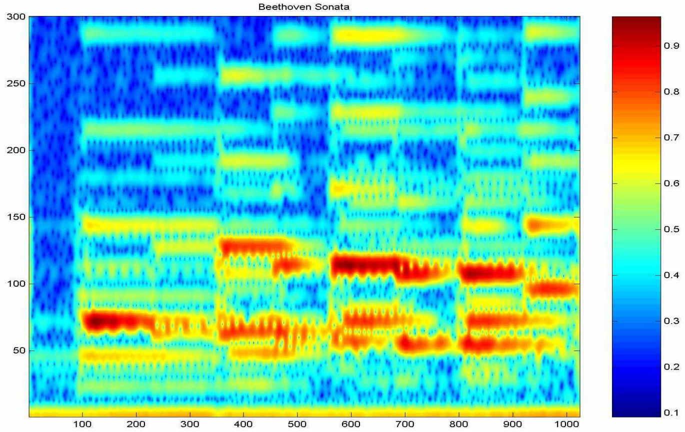
By analogy with *Fourier multipliers* (multiplying the Fourier coefficients by a sequence of numbers, as e.g. in summability theory) or *wavelet multipliers* (operators which have a diagonal representation in some of the well-known wavelet ONBs) this looks like a harmless questions!

Let us first take a look at images, conveying the spirit of the task to be discussed.

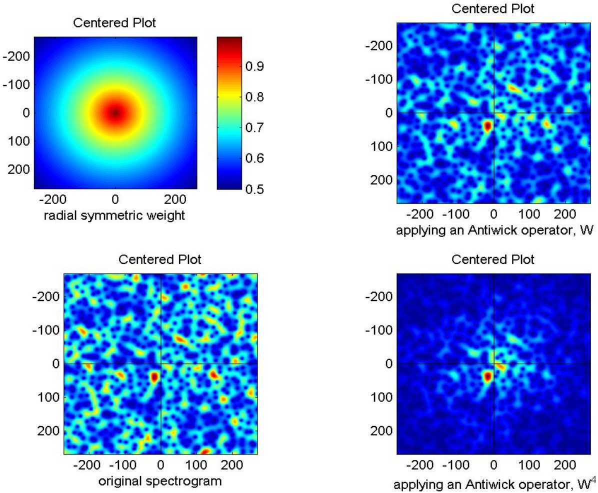
Some hints to the literature

[Lerner, Nicolas] The Wick calculus of pseudo-differential operators and energy estimates. [9]
[Guentner, E.] Wick quantization and asymptotic morphisms [8]
[Boggiatto, Paolo; Cordero, E.; Gröchenig, Karlheinz] Generalized anti-Wick operators with symbols in distributional Sobolev spaces [1]
[Cordero, Elena;Rodino, Luigi] Wick calculus: a time-frequency approach [2]
[Lerner, N.] Some facts about the Wick calculus. [10]
[Gröchenig, Karlheinz;Toft, Joachim] The range of localization operators and lifting theorems for modulation and Bargmann-Fock spaces [6]
[Gröchenig, Karlheinz;Toft, Joachim] Isomorphism properties of Toeplitz operators in time-frequency analysis [7]

Gabor Analysis: Beethoven Piano Sonata

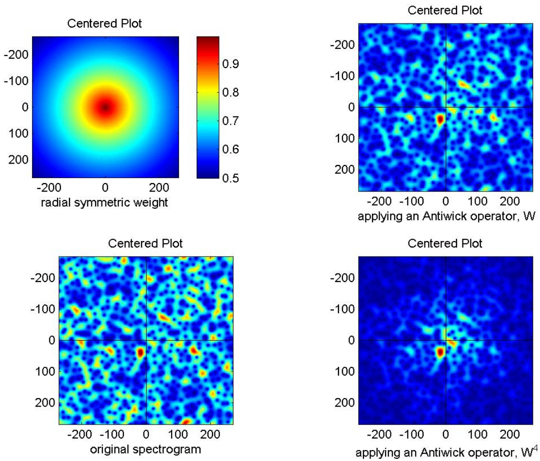


Gabor Analysis: Pictorial: radial filtering

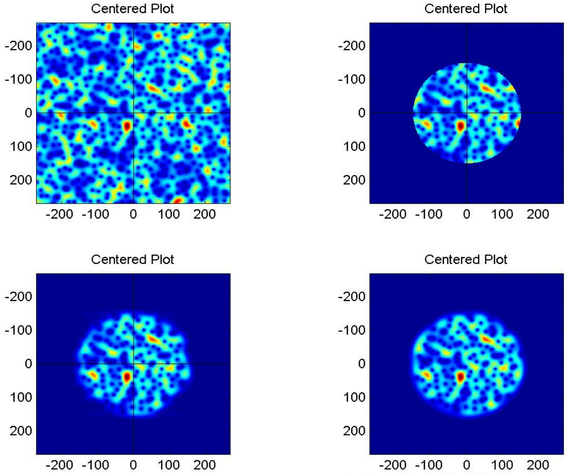


Gabor Analysis: Pictorial I

Applying a radial symmetric multiplier produces from $\mathbf{L}^2(\mathbb{R}^d)$ the so-called Shubin classes $Q_s(\mathbb{R}^d)$:

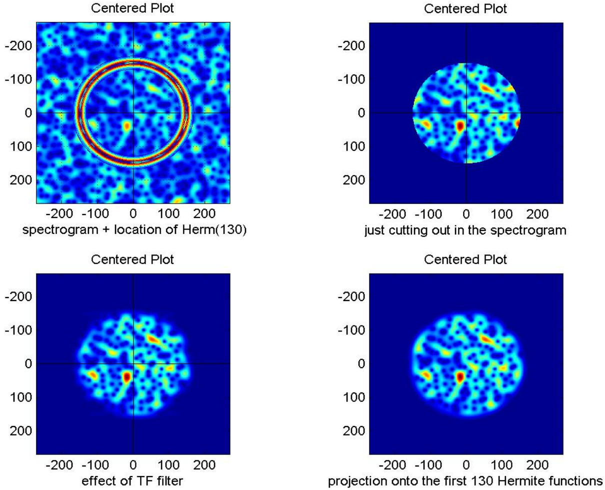


Gabor Analysis: Pictorial II



projection onto the first 130 Hermite functions

Gabor Analysis: Pictorial III



General Aspects of Gabor Analysis

Gabor Analysis is concerned with discrete expansions of functions or (tempered) distributions in the form of sums of building blocks, which are obtained from a single *Gabor atom*, typically a Gaussian window, by applying TF-shifts along some lattice in phase space. It is thus (even for functions of a *continuous variable*) a discrete expansions, often viewed as *discretizations* of the *coherent states expansion* used in theoretical physics.

In contrast to wavelet analysis, where sophisticated constructions allow to discretize the continuous wavelet transform and work with discrete *orthogonal expansions* of wavelet type, the *Balian-Low principle* prohibits the existence of orthonormal systems obtained by TF-shifts of a single atom which is well-concentrated in the TF-sense.

General Aspects of Gabor Analysis II

Gabor analysis can be described over general locally compact Abelian groups, and is this a branch of abstract harmonic analysis, making crucial use of the underlying Abelian group (of time- or space-) translation operators a family of functions on a group G, as well as the frequency multiplication operators (multiplication by pure frequencies resp. characters from $\widehat{G}$.)

This means that *Gabor multipliers* can be studied in a variety of concrete cases. The first test case are of course Gabor systems generated by a Gaussian atom, along the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, viewed as operators on $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$. But it makes also sense to consider Gabor multipliers on $\mathbb{R}^d$ (at least for $d = 1, 2, 3$), where they are used in practice to produce *slowly time-variant filters*. It is also possible to build Gabor multipliers in the finite-discrete setting, where in fact eigenvectors and eigenvalues of the corresponding matrices are easily built.

General Aspects of Gabor Analysis III

In the last years many aspects of Gabor analysis have been well understood. The role of certain function spaces (namely the so-called *modulation spaces* introduced in the early 80's has) become clear, in order to describe boundedness of operators arising in this context. In fact, a theory of *Banach Gelfand triples* has been developed meanwhile, which forms the backbone for the description of many questions arising in Gabor analysis. First of all it is about the boundedness of analysis and synthesis operators, the choice of good windows, or the questions of continuous dependence (e.g. of dual atoms) on the ingredients, i.e. the window and the lattice used to build the Gabor family.
At the same time we have witnessed (and also pushed for it) the improvement on the algorithmic side. While Gabor analysis was considered an interesting approach it was for a long time viewed as *computationally intensive and instable* task.
One can say, that this problem has been overcome by now.

General Aspects of Gabor Analysis IV

- **Linear Aspects**: dual frame = pseudo-inverse, Gabor Riesz basis: biorthogonal system
- **Algebraic Aspects**: lattice (Abelian Groups) act on a Hilbert space of signals via some *projective* representation, *commutation property* of Gabor frame operator, i.e. $[\pi(\lambda), S] == 0$, imply specific sparsity structure of S.
- **Functional Analytic features**: Convergence of double sums (sums over lattice) appear to be complicated (Bessel condition, unconditional convergence, etc.)
- Specific properties of the acting **Weyl-Heisenberg group** (!phase factors) or the validity of Poisson's formula for the symplectic Fourier transform. (contrast to wavelets);

Main Questions in Gabor Analysis I

Following the suggestion of Denis Gabor ([4]) the focus was for a long time on the question: Given a (Gabor) window g, i.e. a smooth function which is well localized near the origin, what can one say about the pairs of lattice constants such that the family of TF-translates, usually denoted by $G(g, a, b)$ is a Gabor frame, i.e. allows to span all of the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ (in a stable way, i.e. using only $\ell^2(\mathbb{Z}^{2d})$-coefficients).
The study of this question is typically focussing on the realization of a pair of *frame inequalities* of the form: There exists $A, B > 0$ such that the family (g_λ) satisfies for all $f \in \mathbf{L}^2(\mathbb{R}^d)$

$$A\|f\|^2 \leq \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \leq B\|f\|^2 \qquad (1)$$

Main Questions in Gabor Analysis II

Of course this questions is known to be equivalent to the invertibility of the so-called *frame operator* S defined as

$$Sf = \sum_{lainLa} \langle f, g_\lambda \rangle g_\lambda. \quad (2)$$

The fact that in the *regular case* Λ is a discrete subgroup of the additive group $G \times \hat{G}$ (resp. of *phase space*) implies that $S = S_{gLam}$ satisfies important commutation relations, i.e.

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda. \quad (3)$$

This in turn implies to fact that the dual frame of a regular Gabor frame is again a Gabor from, generated by the *canonical dual* atom $\tilde{g} = S^{-1}(g)$, which normally inherits good TF-localization properties of g. In fact, (3) is equivalent to the so-called *Janssen representation* of the Gabor frame operator which has other important consequences.

Main Questions in Gabor Analysis IIb

More recently this basic questions has been answered in a number of different ways, e.g. for totally positive functions Gröchenig and Stöckler ([5]), generalizing the classical statement for the Gauss function due to Seip/Wallstein ([12]) and Lyubarskii ([11]) respectively, namely $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ is OK if and only if $ab < 1$.
But it is also interesting to find out under which conditions alternative windows can be used. For example, in the 1D-case the so-called painless approach turned out to be useful, which works fine for compactly supported windows (also good for computations), with the extra condition that the frame-operator is just a simple (invertible) multiplication operator.
What is crucial in all these situations is the following:

Main Questions in Gabor Analysis IIc

- One is using a Gabor atom g which is just in $\mathcal{S}(\mathbb{R}^d)$ or any of the modulation spaces $\mathbf{M}^1_{v_s}(\mathbb{R}^d)$, for some $s \geq 0$;
- One works with general lattices $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, of all kinds, e.g. separable lattices $\Lambda = \Lambda_1 \times \Lambda_2$, with $\Lambda_1 \lhd \mathbb{R}^d$ and $\Lambda_2 \lhd \widehat{\mathbb{R}}^d$, or $\Lambda = \Lambda_1 \times \cdots \times \Lambda_d$ in the pairs of variables $(t_k, \omega_k), 1 \leq k \leq d$.
- One allows multi-window systems, i.e. instead of a *single generator* g one allows for a finite set $\{g_1, \ldots, g_L\}$ of atoms, which again are moved along the TF-lattice;
- ask for regular (meaning Λ is a discrete lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$) Gabor (multi-window) systems which are *Gabor frames* or *Gabor Riesz bases*.

Dual Gabor Windows

Assume that the family $(g_\lambda)_{\lambda \in \Lambda}$ is a Gabor frame, than the family $(\tilde{g}_\lambda)_{\lambda \in \Lambda}$, with $\tilde{g} = S^{-1} g$ is the (canonical Gabor) frame, hence

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) \tilde{g} = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) \tilde{g} \rangle \pi(\lambda) g \quad \forall f \in \mathbf{L}^2(\mathbb{R}^d).$$

or in a more compact notation

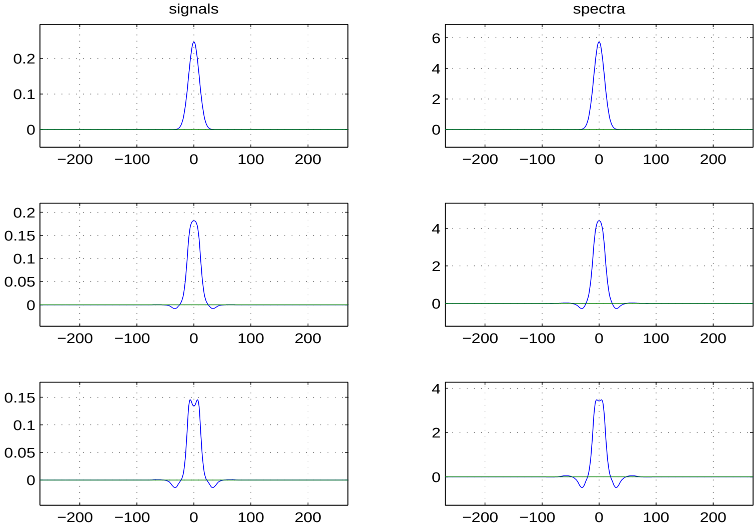
$$f = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \tilde{g}_\lambda = \sum_{\lambda \in \Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda .$$

or using the (canonical) tight Gabor atom $h := S^{-1/2} g$ gives

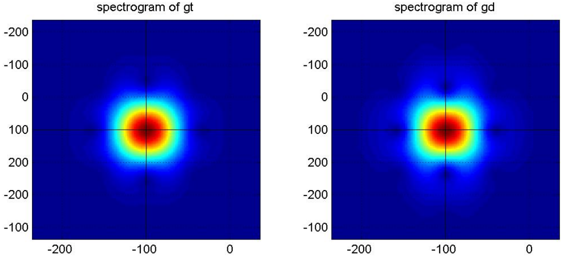
$$f = \sum_{\lambda \in \Lambda} \langle f, h_\lambda \rangle h_\lambda .$$

Dual and tight Gabor atoms I

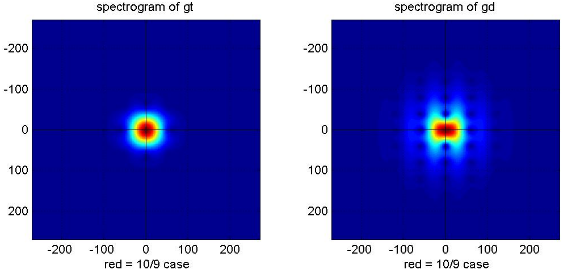
Let us visualize the situation:



Dual and tight Gabor atoms II

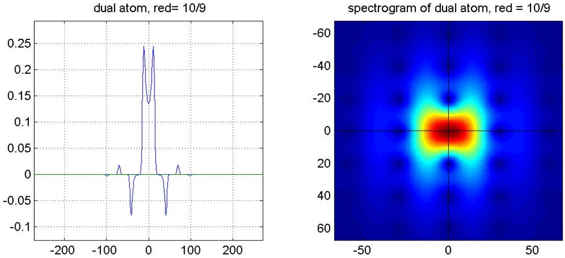


Dual and tight Gabor atoms III



Dual and tight Gabor atoms IV

Showing the dual window at more critical situation:



Tasks in Gabor Analysis

There is a long list of possible task and applications for Gabor analysis:

- Expand a given signal into a sum of atoms sitting at the lattice points (micro-tonal piano making "any sound");
- Compute the dual window $\tilde{g}$ needed for getting the (minimal norm) coefficients for such a (non-unique) *atomic decomposition*;
- use Gabor multipliers to perform a so-called *time-invariant* filter (this is more or less what an audio-engineer does!);
- find the best approximation to a given system (Hilbert Schmidt operator) via a Gabor multiplier based on a given Gabor system; etc. etc. ...

The audio-engineer's work: Gabor multipliers

Digital realization of Gabor Multiplier

Mathematical Tools for TF-analysis

1. **Plancherel Theorem**, showing that the (ordinary resp. the symplectic) Fourier transform is a unitary mapping on $\mathbf{L}^2(\mathbb{R}^d)$ resp. $\mathbf{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Verbally: *Every signal is a (continuous) superposition of pure frequencies.*
2. **Kernel Theorem**: The Hilbert-Schmidt operators are exactly the integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels (!unitarily).
3. **Spreading representation** Every $\mathcal{HS}$-operator T has a spreading representation, $\exists \eta(T) \in \mathbf{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$:

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(T)(\lambda)\pi(\lambda), \quad i.e.$$

4. The symplectic Fourier transform $\kappa(T) = \mathcal{F}_s(\eta(T))$ is known as **Kohn-Nirenberg** symbol in the theory of pseudo-differential operators;

Real world applications of those tools

1. **MP3** is based on the (short-time) Fourier transform and lossy compression (loss cannot be heard due to *masking effects*);
2. The modelling of **wireless channels** (mobile communication between the base station and your mobile phone, when traveling by train or on the highway: multipath propagation + Doppler shift) as an *underspread operators*;
3. Equalization at the end of channel is based on **channel estimation** (can be viewed as a sampling of the KN-symbol) using *pilot tones*, followed by **channel decoding** (via a suitable Gabor multiplier, or a more general [inverse] pseudo-differential operator);

Various Banach Gelfand triples

1. $(\ell^1, \ell^2, \ell^\infty)$, $(\ell^2_w, \ell^2, \ell^2_w{}')$ (weighted ℓ^2-spaces); $(\mathbf{L}^2_w, \mathbf{L}^2, \mathbf{L}^2_w{}')$
2. $(\mathbf{A}(\mathbb{T}), \mathbf{L}^2(\mathbb{T}), \mathbf{PM}(\mathbb{T}))$ (isomorphic to $(\ell^1, \ell^2, \ell^\infty)$ via Fourier bases); $(\mathbf{C}(\mathbb{T}), \mathbf{L}^2(\mathbb{T}), \mathbf{M}(\mathbb{T}))$
3. $(\mathcal{H}_s, \mathbf{L}^2, \mathcal{H}_{-s})$ (rigged Hilbert space of Sobolev spaces) $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$
4. $(\mathbf{Q}_s, \mathbf{L}^2, \mathbf{Q}_{-s})(\mathbb{R}^d)$ (rigged Hilbert space of Shubin spaces, isomorphic to $(\ell^2_w, \ell^2, \ell^2_w{}')$ via Hermite ONB);
5. $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ (isomorphic to $(\ell^1, \ell^2, \ell^\infty)$ via Wilson bases or local Fourier bases);
6. $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ (isomorphic to $(\ell^1, \ell^2, \ell^\infty)$ via Wilson bases on $\mathbb{R}^{2d}$), $(\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^\infty)$ (Schatten classes);

Of course one can look at families obtained between the extreme cases using complex interpolation resp. embed them into larger families (of compatible Banach spaces).

Various Homomorphisms between BGTs

1. **Fourier Transform:** it is unitary automorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ (Poisson's formula, Shannon,...);
2. The identifications operators in $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ with their **kernels** in $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ is a unitary BGT-isomorphism.
3. The **Anti-Wick operator symbolic calculus** given by

$$W(t,\omega) \mapsto \mathbf{A}_W : f \mapsto C_g \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} W(\lambda) V_g(f)(\lambda) \pi(\lambda) g d\lambda$$

or $$\mathbf{A}_W := C_g \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} W(\lambda) \langle f, g_\lambda \rangle g_\lambda d\lambda = C_g \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} W(\lambda) P_\lambda d\lambda,$$

is a BGT-homomorphism from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ to both $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ and $(\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^\infty)$. as long as $g \in \mathbf{S}_0(\mathbb{R}^d)$.

Various Homomorphisms between BGTs: II

1. The STFT itself is (by definition) a retract from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ into $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, but it is also a BGT-retract from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$;
2. The Anti-Wick mapping from the *upper symbol* $W \mapsto \mathbf{A}_W$ is a BGT homomorphism from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ (or $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$) into $(\mathcal{J}^1, \mathcal{J}^2, \mathcal{J}^\infty)$;
3. For a fixed lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and $g \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor multiplier mapping from the upper symbol (w_λ) in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ to the Gabor multiplier

$$GM_{g,\Lambda,w}(f) := \sum_{\lambda \in \Lambda} w(\lambda) \langle f, \pi(\lambda) g \rangle g_\lambda = \sum_{\lambda \in \Lambda} w(\lambda) P_\lambda(f)$$

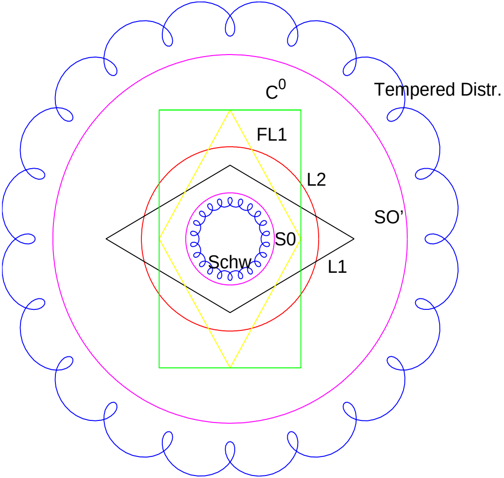
Various Homomorphisms between BGTs: III

- Assume that the (2d) Fourier transform $\mathcal{F}_\Lambda$ of $|V_g(g)(\lambda)|^2$ then the best approximation operator which assigns to every Hilbert Schmidt operator $T \in \mathcal{HS}$ the best (HS)-approximation by a Gabor multiplier of the above form is another BGT-homomorphism on $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$. The mapping from the operators $T \in (\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ to the lower symbol

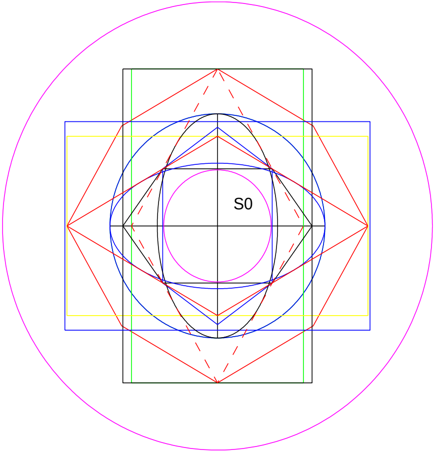
$$low_T(\lambda) := \langle T(g_\lambda), g_\lambda \rangle = \langle T, P_\lambda \rangle_{\mathcal{HS}}$$

as well as the upper symbol of the best approximant (describing the coefficient sequence of the best approximating Gabor multiplier) are mappings into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

A schematic description of the situation



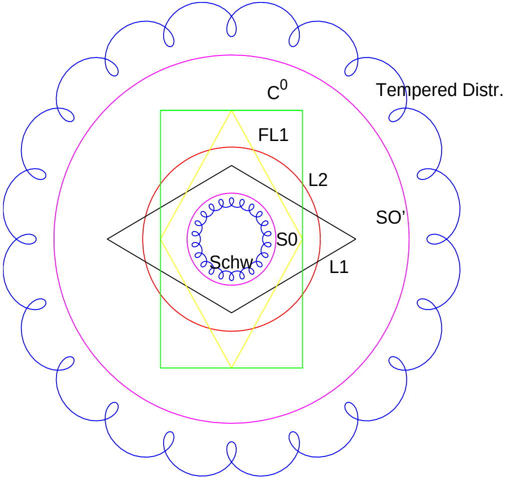
A schematic description of the situation



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

A schematic description of the situation

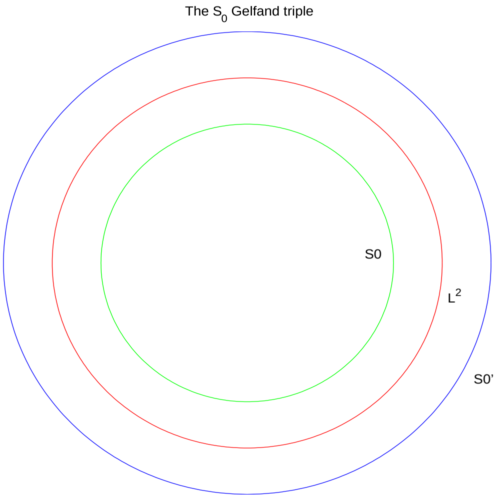


The Banach space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ or $(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1})$), which will serve our purpose. Its dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is correspondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The S_0-Banach Gelfand Triple



The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\mathbf{M}^1 = \mathbf{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as **B** the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{U}) = \mathbf{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{4}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

FACTS

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):

Theorem

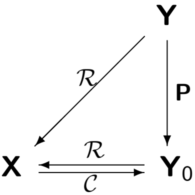
Assume that for $g \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor frame operator

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

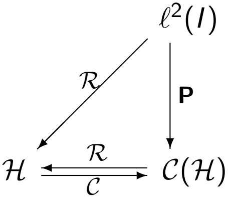
is invertible as an operator on $\mathbf{L}^2(\mathbb{R}^d)$, then it is also invertible on $\mathbf{S}_0(\mathbb{R}^d)$ and in fact on $\mathbf{S}_0'(\mathbb{R}^d)$.
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S is (resp. extends to) an isomorphism of the Gelfand triple automorphism for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

Fames and Riesz Bases: the Diagram

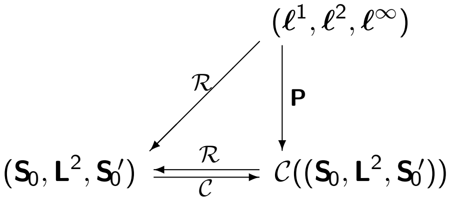
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ. Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping $\mathcal{C}$ from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse $\mathcal{R}$ (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.
In practice it means, that the dual Gabor atom $\tilde{g}$ is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.

Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathbf{S}_0(\mathbb{R}^d)$ (typically one will take g and $\tilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity (> Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ)*, with* $g \in \mathbf{S}_0(\mathbb{R}^d)$ *defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the* $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$*-sense on both parameters.*

Invertibility, Surjectivity and Injectivity

In another, very recent paper, Charly Groechenig has discovered that there is another analogy to the finite dimensional case: There one has: A square matrix is invertible if and only if it is surjective or injective (the other property then follows automatically).
We have a similar situation here (systematically describe in Charly's paper):
`K.Gröchenig: Gabor frames without inequalities, Int. Math. Res. Not. IMRN, No.23, (2007).`

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\mathbf{B} = \mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$.

Theorem

There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. *This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. *Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators* $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

The concept of REALIZIBILITY/computability

When working with continuous functions (say on $\mathbb{R}^d$) one has the problem, that even after fine sampling one can use only finitely many samples within (for example) a computer program. In finite time (meaning seconds or minutes) one can do only finitely many matrix operators (say matrix inversion), so the question is, whether one **can approximate the underlying truth** using the information obtained in such a way.
In the work with Darian Onchis on multi-window spline type space we have thus develop the concept of *[constructive] realizability*.

Realizable constructive approaches vs. constructive

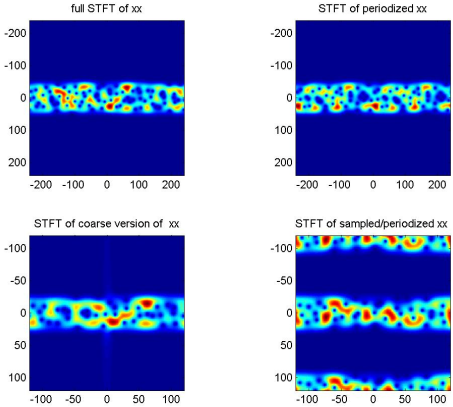
We will illustrate this by the following:
Let us look at the spline-type spaces (e.g. cubic splines on $\mathbb{R}$) in $\mathbf{L}^p$-spaces, for $p \neq 2$. Then this spaces are closed subspaces of $(\mathbf{L}^p(\mathbb{R}), \|\cdot\|_p)$, which is uniformly convex, hence there is a best approximation to any given $f \in \mathbf{L}^p(\mathbb{R})$ by an element from the corresponding spline-type space. But this is not a constructive approach.
I contrast we have quite explicit methods (so-called constructive approximation) to obtain the (orthogonal) approximation in the case $p = 2$, and the corresponding operator can be shown to be also (uniformly) bounded on the family of $\mathbf{L}^p$-spaces, for the full range of $1 \leq p \leq \infty$.
See [3] and an upcoming Springer BRIEF for discussions of the problems arising with this, for the multi-window case.

Main aspect: Factorize through finite dimensions

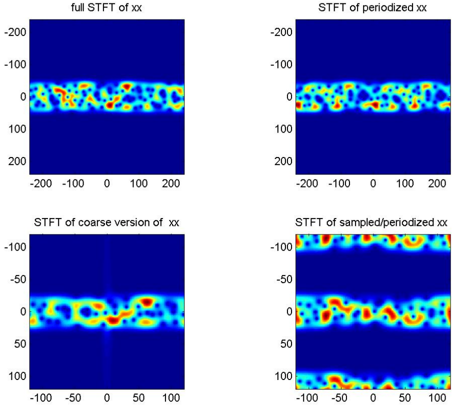
The general idea - at least in the context of Gabor analysis - of this refined approach (merging concepts of functional analysis and numerical analysis) is to approximate a continuous problem by the *corresponding problem* in the finite context, i.e. on a suitably chosen finite Abelian group, with the need to describe also the process of returning from discrete/finite data to the continuous domain. Usually this is done by quasi-interpolation. The most important special case of quasi-interpolation is piecewise linear interpolation, (quasi-interpolation using linear splines), while the operator $Q(d) = \sum d_\lambda T_\lambda \varphi$ is not interpolating anymore (e.g. for $d = (f(\lambda)))$ for the case of cubic B-splines.

The role of w^*-convergence

A sequence (σ_n) in $\mathbf{S}_0'(\mathbb{R}^d)$ is convergent to $\sigma_0 \in \mathbf{S}_0'(\mathbb{R}^d)$ in the w^*-topology if (def!) $\sigma_n(f) \to \sigma_0(f)$ for $f \in \mathbf{S}_0(\mathbb{R}^d)$.



The role of w^*-convergence



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