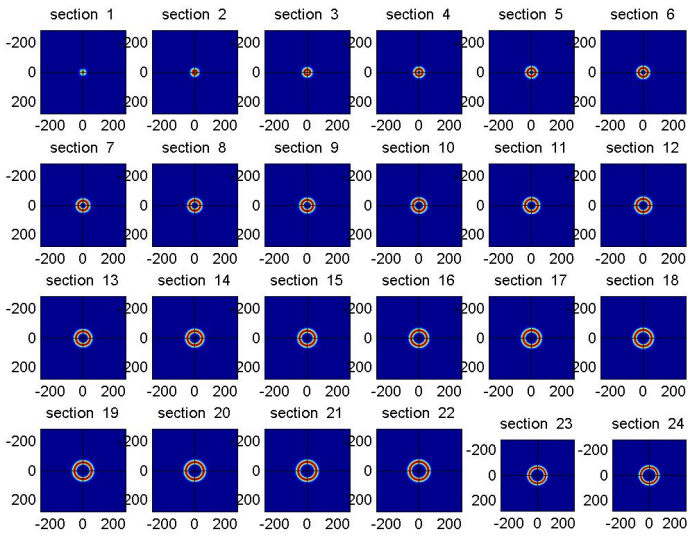
Landscape of Gabor Frames

INFORMAL NOTES BY HGFEI, RELATED TO THE RUNNING PROJECTS IN GABOR ANALYSIS, NAMELY UNLOCX AND THE EXPERIMENTS DONE BY RADU FOR THE HAMILTONIAN PROJECTS

The *general theory of Gabor analysis over finite Abelian groups* (with general windows and general lattices, described in a coordinate-free way) is described in the paper [4]. All the algebraic facts are described there, and all the results of that paper are more or less implemented for the case $G = \mathbb{Z}_N$ or other simple (concrete) finite groups. Recall that any finite group is a finite product of such groups, and that Gabor analysis over product groups can be reduced (using the Kronecker principle) to Gabor analysis over the factors, at least for separable atoms (such as multi-dimensional Gauss functions).

The Hermite-rotation makes use of the discrete Hermite functions, which are obtained numerically as eigenvectors (hence perfectly orthonormal in $\mathbb{C}^n$) of a suitable STFT-multiplier with radial symmetric weights.



1 What are good Gabor systems?

The discussion so far is focussing very much on individual aspects of concrete Gabor systems. A triple (g, a, b) is said to generate a Gabor frame if the family $(\pi(\lambda)g)$ of TF-shifted copies of the *Gabor atom* (or window) g along the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ is a frame for the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, which is known to be equivalent to a pair of *frame inequalities* (positiv *lower* and upper frame bounds) resp. the invertibility of the Gabor frame operator on the Hilbert space $\big(\boldsymbol{L}^2(\mathbb{R}^d),\, \|\cdot\|_2\big)$.

Well known and often repeated statements center around the impossibility of having a total system for $ab > 1$ for whatever $g \in \boldsymbol{L}^2(\mathbb{R}^d)$, or the fact that something like well-localized Gabor Riesz bases for the full Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ (the so-called *Balian-Low principle*) are impossible (for $ab = 1$, the so-called critical density, at least for "good functions" g, e.g. $g \in \boldsymbol{S}_0(\mathbb{R}^d)$).

On the positive side one reports on the famous results by Seip-Walsten ([17]) and Lyubarski ([12]) showing that for the Gauss-function g_0 the full complement (i.e. the open set of lattices with $ab < 1$) is OK, i.e. that the Gabor-Frame condition is valid for any such pair. Of course it is easy (by a simple transformation argument) that the same is true for any dilated Gauss-function (due to the special form of the set $ab < 1$). Very recently the family of functions for which such a statement is valid has been enormously increased to include all totally positive functions (see [9]).

On the other hand it is also known that even for indicator functions $g = \mathbf{1}_{[0,c]}$ the property of the so-called Janssen tie (all pair (a, b) which are OK) is quite complicated. (Janssen Tie paper, see [11]).

For our discussion below we would like to take a more pragmatic point of view. We will be happy to show how to obtain well-localized and useful systems, how to verify the frame conditions, how to compute (approximately) dual or tight windows, and how to use them e.g. in the context of Gabor multipliers.

2 The Overall Goal

The overall goal of our efforts is to help the applied scientist making use of Gabor-type or wavelet-like expansions (mostly under the assumption that the window is choosen according to some optimality criterion) in an optimal way. The main/original goal of WP3 was to find optimal discretization strategies which would/should allow replace the continuous family (say all TF-shifted copies of a given Gabor atom, or a shifted and dilated version of some mother wavelet) by a discrete/countable subcollection, chosen in an optimal way.

It was expected from the beginning, that the sampling should be *dense enough but not too dense*, because big wholes in the parameter space obviously will generate small lower frame bounds, hence a bad condition number, while too high density, even locally, is inefficient.

During the discussions and systematic experiments in the last half year various questions had to be settled. It became clear that the optimality of the atom may be spoiled by the discretization step. In fact, this is already easy to demonstrate and show in the Gaborian case, hence we have concentrated a lot on this. On the other hand the approach and perspective taken within the UnlocX project is teaching us a lot about Gabor analysis, providing thus new perspectives.

The main points are: **the overall system** (doing analysis, maybe some manipulation on the coefficients of the discrete system, say wavelet frame thresholding, and subsequent synthesis) should show **optimal locality**, *not just the starting ingredient*.

Hence the locality of the dual system, and the stability of the projection operator from the set of Hilbert Schmidt operators onto the set of corresponding Gabor multipliers. Hence the stability of this projection on the linear span of the projection operators

$$P_\lambda : f \mapsto P_\lambda(f) = \langle f, g_\lambda \rangle g_\lambda$$

is going to play a role in this process.

Consequently a number of figures of merits have been developed in the second half year of 2011, and their relative merits have been systematically exploited. Only the final choice will be reported in detail now.

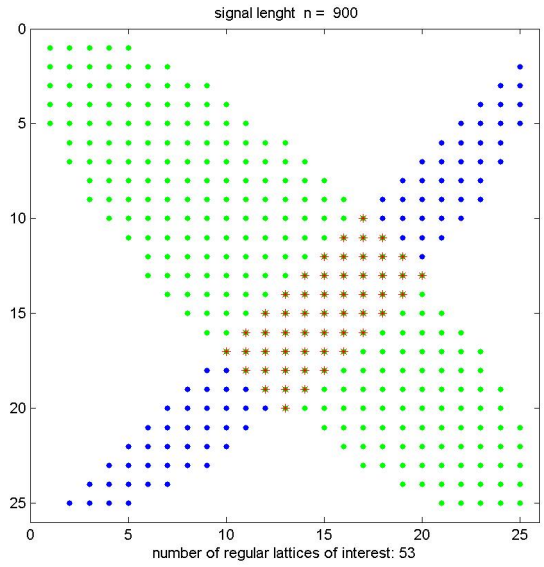
3 Gabor Territories

When we discuss the quality of Gabor families we expect that similar Gabor families are of a similar quality. There are various ways of describing similarity, but the most important one are related to suitable function space norms (from the family of modulation spaces, because these are the right spaces to describe TF-behavior of functions and distributions). As it is meanwhile well known, good localization is described in terms of the modulation spaces $\boldsymbol{M}^1_{v_s}(\mathbb{R}^d)$, mostly because the elements if these spaces are suitable atoms and the $\boldsymbol{M}^1_{v_s}$-norm allows to describe/control the operator norm of analysis and synthesis operators for a wide class of (generalized) modulation spaces. For simplicity of presentation we choose the case $s = 0$, so we will mostly work with $\boldsymbol{M}^1(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$ in order to present the principles. Adaptation to more general weighted space $\boldsymbol{M}^1_w(\mathbb{R}^d)$, for submultiplicative weights $w(x)$ on $\mathbb{R}^d$.

The prototypical results in this direction are provided by the paper [2], where it is shown that the dual window $\tilde{g}$ is in the same class (say $\boldsymbol{M}^1_{v_s}(\mathbb{R}^d)$) as the window itself, and that it depends continuously on both the atom and the lattice (described by the $2d \times 2d-$ matrix describing the lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$). As a consequence one can claim, that pairs (g, Λ) which are close to a given pair (h, Λ_0) will behave similarly (similar frame bounds, similar figures of merit of all kinds).

3.1 Convenient Territories of Separable Lattices

Although part of the focus of our presentation will be to include also non-separable lattices in the discussion let us first discuss the




```
[INFO INFOc] = demojans(g, red1, red2, maxex, 1);
resp.
[INFO INFOc] = demojans(g, 1,4, 5,'plot',100);
```

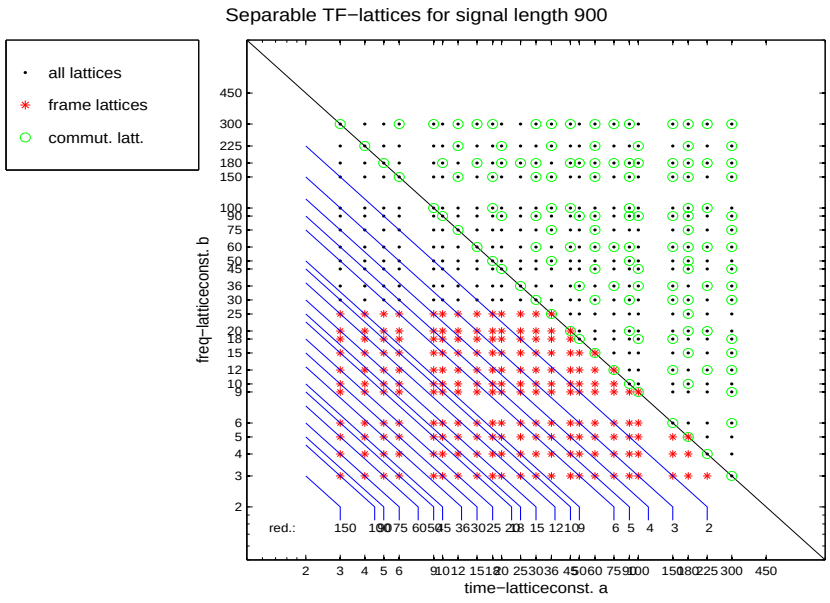
provides a plot, with blob-size 100, redundancy in the range from $red1 = 1$ to $red2 = 4$, and $maxex = 5$ (maximal excentricity).

An alternative way to display the properties of Gabor systems by identifying separable lattices in $\mathbb{Z}_n \times \mathbb{Z}_n$ with lattice constants (a, b) as points in the unit square, by associating to each pair (a, b) the point $(log(a), log(b))/log(n)$, where a, b have to be divisors of n. The plot below puts all pairs which satisfies these conditions, i.e. goes through all divisors of n ($alph = propdiv(n)$), which for $n = 900$ is

$$\begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 8 & 10 & 12 & 15 & 20 & 24 & 25 & 30 & 40 & 50 & 60 & 75 & 100 & 120 & 150 & 200 & 300. \end{pmatrix} \tag{1}$$

Red stars correspond to Gabor frames, while green rings mean Gabor atoms (verified numerically). The black lines represent pairs of equal *integer* redundancy, with the critical line $red = 1$ being on top. Obviously such a representation cannot show the quality of the Gabor family, i.e. the condition numbers arising in this context.

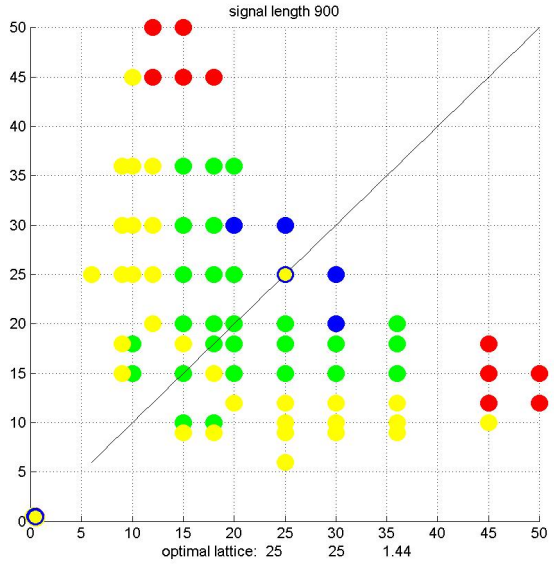
In our (the NuHAG toolbox) implementation clicking on one of those (red) stars displays the dual atom and the corresponding lattices.



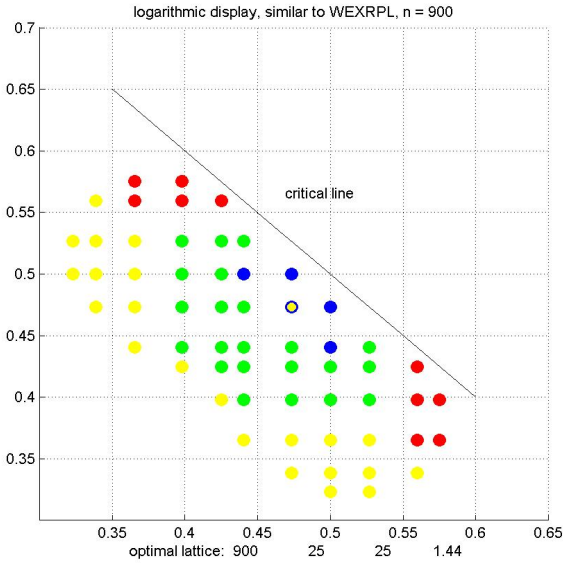
The stars or green rings however are based on a numerical verification of the invertibility of the Gabor frame-type operator resp. the Gramian matrix of the Gabor system.

The next plot is doing the representation it a slightly different manner, indicating the pairs for which the so-called *Janssen test* can be applied, and what kind of outcome is possible. The plot represents all the points/pairs of lattice constants for which the Janssen test provides a positive statement (i.e. implies the Gabor frame property). Green marking means, that the estimate obtained predicts good condition numbers, while the top five (in some cases four) are displayed in blue (the best one has an additional yellow dot). The yellow points means not so good condition number (say > 10) while the red points are marking the critical cases (the ones where the Janssen criterion is close to fail).

WE SHOULD CHECK HOW BAD THE CONDITION NUMBER IS ACTUALLY IN THIS CASE



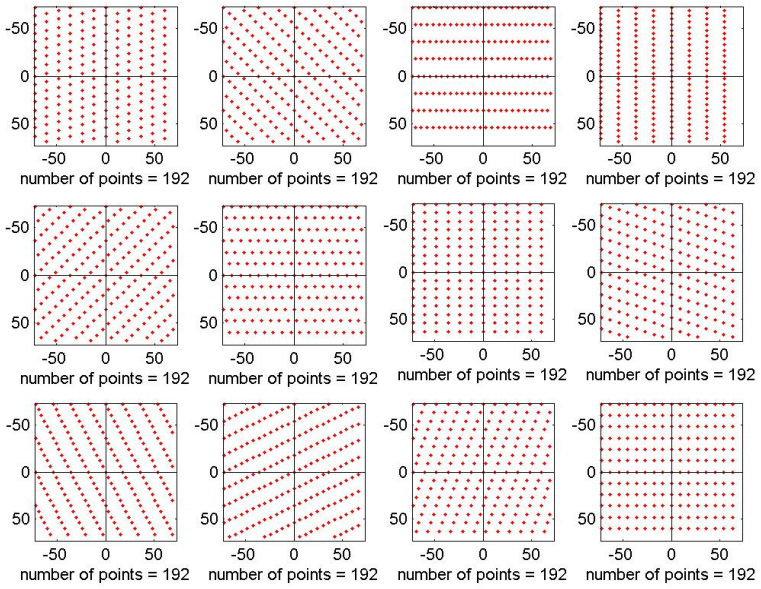
The same in the logarithmic display looks like this:



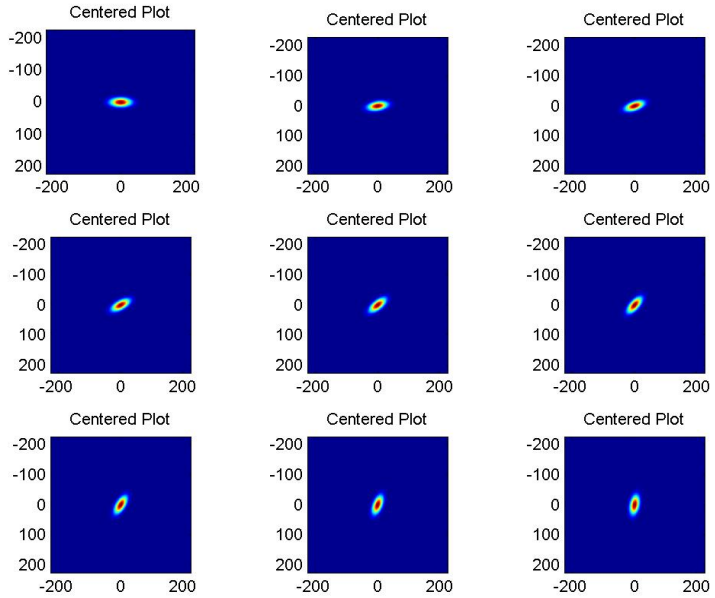
3.2 Convenient Territories of Separable Lattices

In addition to regular lattices we have systematically explored the possiblity of using non-separable lattices. In order to generate many such lattices we start from the given pairs of interesting lattice constants and apply systematically the automorphism named `SIDEDIGM` (with inverse `SIDE2MAT`) of the group $\mathbb{Z}_n \times \mathbb{Z}_n$. Since for every finite group the collection of finite subgroups of a given order is itself finite one must return to the original lattice after an appropriate (minimal) number of iterations of this automorphism, let us call it α for a moment. The observed length of a **cycle** of subgroups (they can be viewed as equivalent under α) has been occasionally in the order of 24, and sometimes more than 60, and even up to 120 for suitable triples (a, b, n) with $n \leq 1000$.

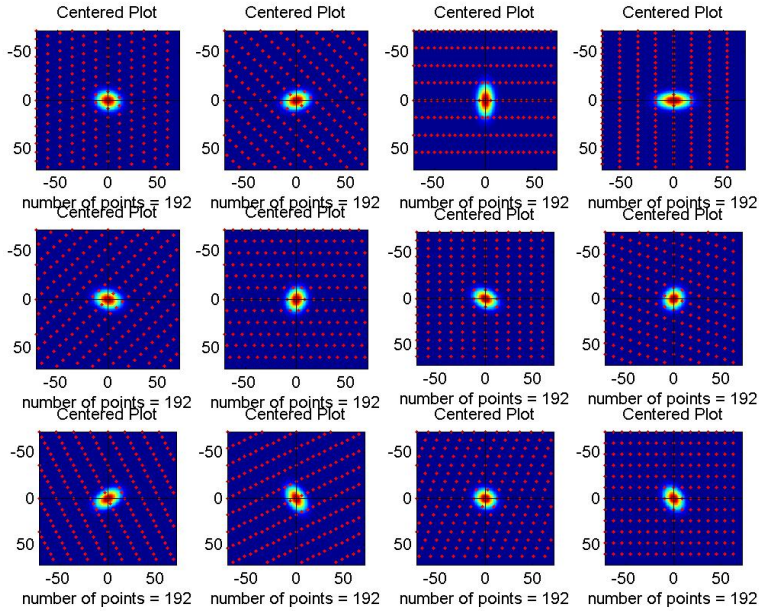
Typical examples of lattices with equal redundancy



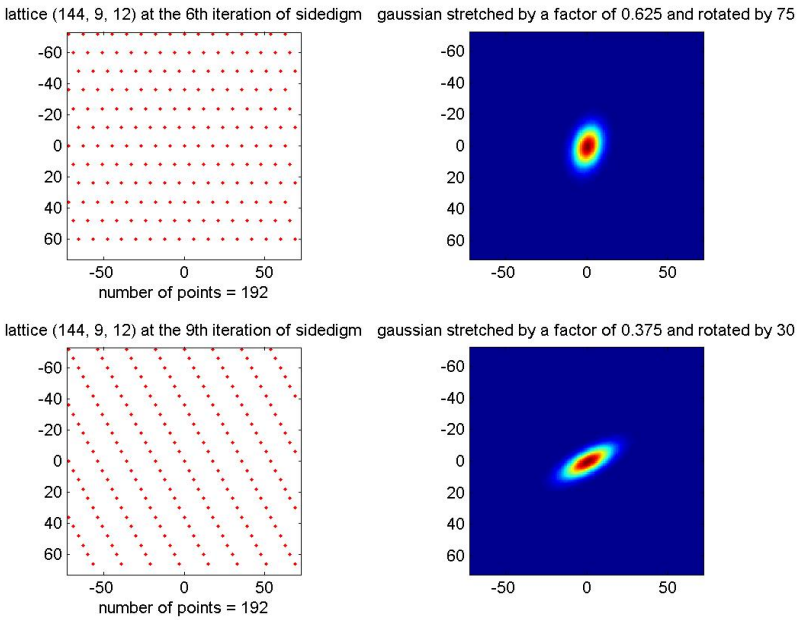
With this option can start to ask how *windows and lattices* are matching optimally. We have been running systematic test (hgfei + Radu Frunza) in order to find out, to which extent the optimal match of windows and lattice (Gabor atom g and TF-lattice Λ, to be more precise) coming from the idea of covering with ellipses corresponds to the numerical findings. In short/summary, one may claim/confirm that this is makes a valid.



Starting from an (appropriately) stretched discrete Gauss-function (imitated by taking $g.^{\cdot}2$ or similar exponents < 1) one can obtain (using Hermite rotations, i.e. discrete fractional Fourier transforms) comprehensive collections of atoms with different *orientation and eccentricity*.



or in a better closeup:


lattice (144, 9, 12) at the 6th iteration of sidedigm
gaussian stretched by a factor of 0.625 and rotated by 75
-60
-40
-20
0
20
40
60
-50
0
50
number of points = 192
lattice (144, 9, 12) at the 9th iteration of sidedigm
gaussian stretched by a factor of 0.375 and rotated by 30
-60
-40
-20
0
20
40
60
-50
0
50
number of points = 192

4 Consumer reports

When it comes to the practical use of Gabor systems it is not sufficient to know, that it may be better to use a well-localized Gabor system, and to apply an abstract theorem which provides a guarantee that also the dual system or the corresponding canonical tight system will be well localized in the TF-sense, but one has also to make use of one or the other algorithm. So it becomes an important practical question: Which algorithm, which version of the algorithm, which implementation should one use in order to do the computations most efficiently.

For this purpose we have started to produce - in connection with the Gabor landscape - a type of *consumer reports* which are supposed to help the applied scientist or the graduate student who is going to carry out some experiments in this field, which version should be used.

4.1 Consumer Report Principles

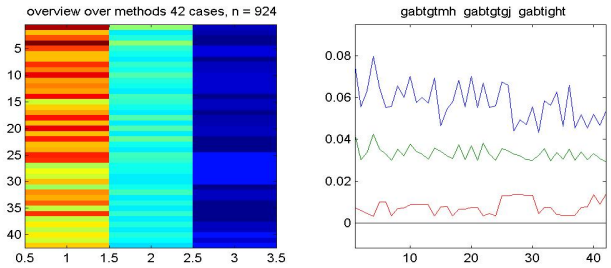
As it is common to consumer reports the recommendation depends on the application, and therefore the main purpose of our comments and listings below is to give the user some information that should help her/him to decide, which one to use. Obviously there are situations, where speed does not play such a big role, and generally applicable methods are preferred, while in other cases the repetitive application of specific method, which are only working in rather special situations, may be the best choice. Just think of the use of Zak-transform methods, which however work only in the case integer redundancy. Hence they are only of interest for redundancies of the form $red = 2, 3, 4$.

4.2 Methods for Tight Gabor atoms

One of the natural questions concerning Gabor families is the question, which *concrete algorithm* should be chosen to compute either a *dual* or the canonical *tight* Gabor window for a given situation.

We found that the NuHAG methods GABTGT**MH** (based on an implementation by **M**ario **H**ampejs) and GABTGT**GJ** (based on the methods described by **G**uido **J**anssen) perform well, but that for the signal size $n = 924$ the behaviour of the three methods always turns up the method GABTIGHT by Peter Sondergard (also called GABTGTPS.M with the standard input parameters `gd = gabtgtps(g,a,b)` instead of `gd = gabtight(g,a,n/b)` gives the output fastest.

In fact, as a later comparison shows (below) the algorithm in the LTFAT toolbox is showing no significant difference in the timing for the actual computation compared to the calculation of the canonical dual window. This is in sharp contrast the the second generation algorithms used within the NuHAG group until recently, where sometimes the mixing between atom and dual atom (iterated until stabilization of the mixing process occurred) was requiring a number of iterations using dual windows in order to approach the tight window properly.



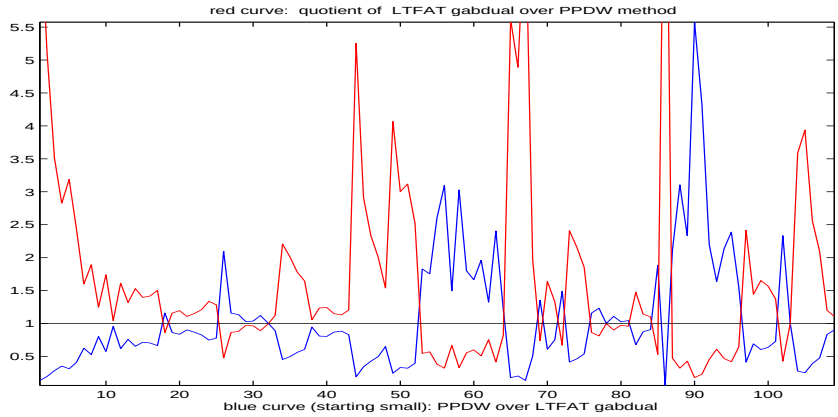
4.3 GABDUAL versus PPDW and GABDDD

Within the NuHAG toolbox we had a number of very efficient algorithms which are able to compute the dual window using iterative methods resp. matrix factorizations. Among them we have PPDW (Peter Prinz dual window [should be called gabdpp!) and GABDDD (historial name, based on sparse representation of the Gabor frame operator and the conjugate gradient method [should be called gabdcg]

GABDUAL is the file in the LTFAT toolbox, which is to be called differently:

```
gdpp = ppdw(g,a,b);
gdd  =  gabddd(g,a,b);
gdps =  gabdual(g,a,n/b);
```

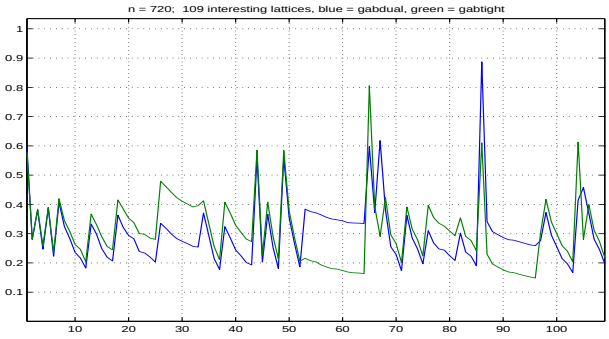
The following plot displays the timing (for $n = 720$) of the two most competitive methods, namely the method by Peter Prinz [14], but it works also for more general lattices, see [13].



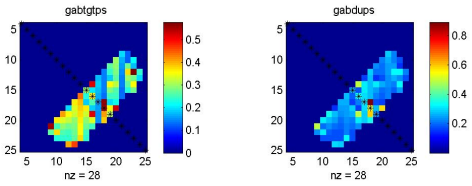
A comparison of the time for GABDUAL and GABTIGHT over the same family of lattices is provided by the follwoing graph (again these experiments are based on timings

for the ca. 110 most interesting lattices for $n = 900$. It shows clearly that the duration of computing the dual or tight Gabor window (using either GABDUAL or GABTIGHT from the LTFAT Toolbox) is increasing for the same cases. On the other hand the plot also shows that sometimes it may be even faster to compute the tight window instead of the dual window.

Consequently there is room for further improvement, e.g. by analyzing further, when one of these cases is clearly predictable, perhaps based on number theoretical properties of the triple (a, b, n). For example, Zak transform methods are likely to be extremely fast for the integer (= commutative) case, not only for the case of $1D$ Gabor signals. In addition one should find out, in which cases there is really a time to be gained by choosing one of the two algorithms. In other words, it would be enough to find out those pairs where one method is really slow and the other methods is good average, not so much the good ones compared against the extremely fast ones (if it takes time or computations to take such decisions).



Another method of display (using color code) is therefore of some interest. Again it can be used in principle to find patters in the speed or efficiency of one method against another one:



5 Statements Across various Signal Lengths

It is common to discuss Gabor Analysis in a concrete setting. For computations in the setting of finite Abelian groups G (which is the situation which can be implemented in MATLAB) this means that one thinks of a given Gabor atom and some lattice Λ within the phase space $G \times \widehat{G}$. Given these ingredients one wants to expand a signal on the same group into a Gaborian double-sum, or apply some Gabor multiplier to it.

However, in many cases one is interested in the continuous setting, i.e. the data are functions or distributions on $\mathbb{R}^d$ and the expansion is in terms of some lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.

Clearly in such situations one may ask, how a sequence of finite dimensional situations can be determined, such that the numbers computed on these approximating families (e.g. norms or pseudo-spectra of operators, or shape of dual atoms) can be used *to approximate the continuous limits*.

For such situations it is not only important to have fast algorithms for relatively large signal size (say n or L equals a few hundred or thousend samples), but in order to incorporate multi-grid or multi-resolution ideas one may favour iterative methods which allow to use the result of the coarse level as a starting point for the next level, and thus to control the overall computational costs better than just doing the whole computation from the beginning at the finest level necessary to reach the required precision (or order of approximation).

Hence such a setting requires to adjust and normalize all the constants and figures of merit in such a way that they will converge (to the appropriate concept for the continuous limit).

A good example is the normalization of the (discrete SO-norm). For the standard Gaussian function one can find that

$$\|g_0\|_{\boldsymbol{s}_0} = 2. \tag{2}$$

The corresponding "numerical verification is:

```
for jj = 1: 10; g = gaussnk(jj*100);  SOG(jj) = sonorm(g); end;
```

which requires a display in `format long` in order to even see that it is not exactly equal to the (correct) limit 2.

We can for example (as an easy test) try to find out the SO-norm of the dual window for the case $a = b$ (i.e. excentricity factor 1) and redundancy $red = 1/2$. For any natural number a one can imitate this be choosing $b = a$ and $n = 2 * a * b$:

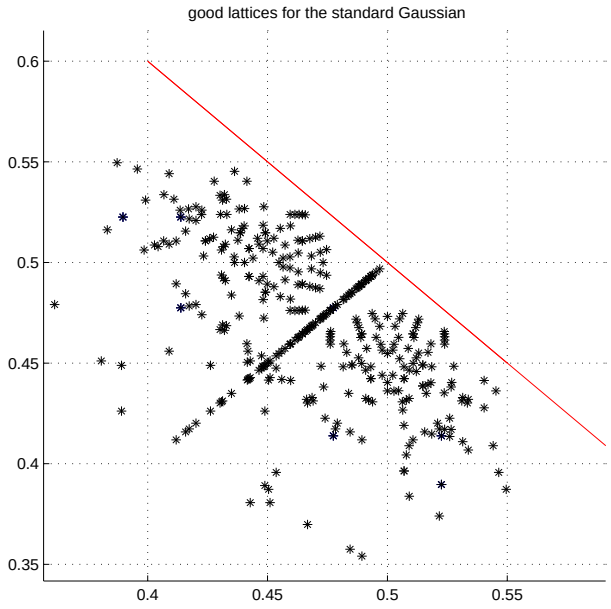
```
a1 = 12; b1= a1; n1 = 2*a1*b1 = 288;  g1 = gaussnk(n1);
a2 = 2*a1, a3 = 2*a2; b2 = a2;  b3 = a2; n2 = 2*a2*b2; n3 = 2*a3*b3;
gd1 = gabdups(g1,a1,b1); SOgdgd1 = stft(gd1,gd1); sum(abs(SOgdgd1(:)))/(n1*norm(gd1).^2
% or  >> sonorm(gd1)/norm(gd1):  ans =    2.151726725912067
gt1 = gabtgtps(g1,a1,b1); SOgtgt1 = stft(gt1,gt1);
sum(abs(SOgtgt1(:)))/(n1*norm(gt1).^2) = 2.076395300293107
gt2 = gabtgtps(g2,a2,b2);  SOgtgt2 = stft(gt2,gt2);
sum(abs(SOgtgt2(:)))/(n*norm(gt2).^2)  =  2.187420523088928
gt3 = gabtgtps(g3,a3,b3);  SOgtgt3 = stft(gt3,gt3);
sum(abs(SOgtgt3(:)))/(nn*norm(gt3).^2) =    2.187424107738490
```

```
sonorm(gd2)/norm(gd2) =  2.389211780111419
sonorm(gd3)/norm(gd3) =  2.389235714447005  etc.
```

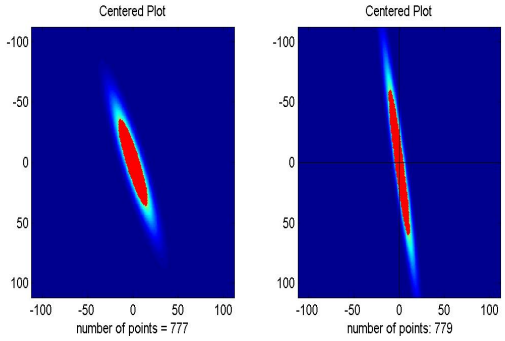
The log/log-representation allows to make statements about "good Gabor families" also for different values of n. This is an *absolutely new phenomenon* ad highly interesting, and has in turn to be investigated. For example, a systematic search for good lattice pairs across the family of signal length with rich divisor structure (say the first 50 of them, up to $n = 1040$):

$$\begin{pmatrix}
180 & 420 & 560 & 660 & 792 & 882 & 990 & 1092 & 1170 & 1280 & 1368 & 1452 & 1530 & 1596 & 1692 \\
240 & 432 & 576 & 672 & 800 & 900 & 1008 & 1100 & 1176 & 1296 & 1380 & 1456 & 1536 & 1600 & 1700 \\
252 & 450 & 588 & 684 & 810 & 912 & 1020 & 1104 & 1188 & 1300 & 1386 & 1470 & 1540 & 1620 & 1710 \\
288 & 468 & 600 & 700 & 816 & 924 & 1040 & 1116 & 1200 & 1320 & 1392 & 1476 & 1548 & 1632 & 1716 \\
300 & 480 & 612 & 720 & 828 & 936 & 1044 & 1120 & 1224 & 1332 & 1400 & 1488 & 1560 & 1638 & 1728 \\
336 & 504 & 624 & 756 & 840 & 960 & 1050 & 1134 & 1232 & 1344 & 1404 & 1500 & 1568 & 1650 & 1740 \\
360 & 528 & 630 & 768 & 864 & 972 & 1056 & 1140 & 1248 & 1350 & 1428 & 1512 & 1575 & 1656 & 1760 \\
396 & 540 & 648 & 780 & 880 & 980 & 1080 & 1152 & 1260 & 1360 & 1440 & 1520 & 1584 & 1680 & 1764
\end{pmatrix} \tag{3}$$

This image shows also a strong concentration on the axis $a = b$, which seems to be a good choice for many signal sizes (however this is not a surprise, since these investigations have been carried out for the fixed Gaussian window, which can be considered to be a radial symmetric object in phase space).



We are expecting that the density of points has to be adapted to the covering properties of the corresponding ellipses describing the *half-width maximum*



but it has to be still decided, whether it is better to use the auto-ambiguity function $V_g(g)$ or the STFT with Gaussian window for this purpose.

6 Bringing Flexibility to Gabor Analysis

Although there are still papers considering interesting questions for the by now *classical setting*, i.e. for atoms in $\boldsymbol{L}^2(\mathbb{R})$ with a TF-lattice of the form $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, it is by now common to either work with a fixed Gabor atom g (or a finite family of such atoms, used at each position, in the case of *multi-window Gabor families*) and a general lattice $\Lambda = \boldsymbol{A} * \mathbb{Z}^{2d}$, for some non-singular $d \times d$-matrix $\boldsymbol{A}$.

On the other hand theoretical studies (and to a modest amount numerical simulations) have been carried out for *irregular Gabor families*, typically families of Gabor atoms, which are obtained by applying TF-shifts from a *well-spread* random set of points in the TF-plane, to a given Gabor atom.

We believe that it will be important to allow various kinds of semi-structured and intermediate type of Gabor expansions, more suitable for new applications, e.g. in the theory of pseudo-differential operators.

6.1 Musical Gabor Families

The idea of "*musical Gabor families* is related to the idea that in a real-life musical performance (we think still of classical music) there is a basic rhythm, and usually a fixed scale of tunes that are in use during the given piece of music, but the ideal musical scaling is not just an arithmetic progression in the strict mathematical sense (and even the invertible version of the constant-Q-transform developed recently at NuHAG, see [10, 18]

6.2 Quilted Gabor Frames

[1]

6.3 Hamiltonian Gabor Frames

The idea of *Hamiltonian Gabor frames* is based on the consideration that it may be quite important to adapt the building blocks used in a Gabor expansion to the signal or distribution to be expanded (in the spirit of adaptive signal processing), or to the pseudo-differential operator under discussion. In this way one has a continuous family of (slowly-varying in terms of shape) atoms over phase space, or what is called a *continuous frame*. It is then natural to adapt the discretization strategies to the given rate and kind of change occurring slowly. Although the general principle outlined in [7] is applicable in this situation the effective realization of this general principle has not been carried out, not even for relatively specialized situations. In particular quantitative estimates allowing a concrete choice of the discretizations have to be developed, both in a concrete way

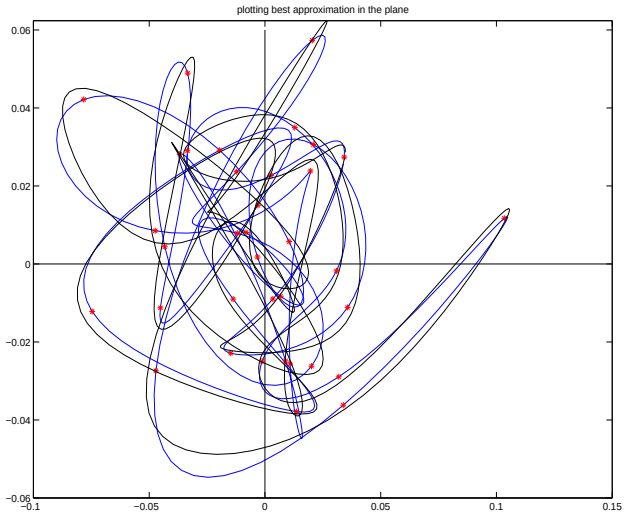
See also the ongoing NuHAG project by Maurice de Gosson and Franz Luef on Hamiltonian Deformations of Gabor Frames

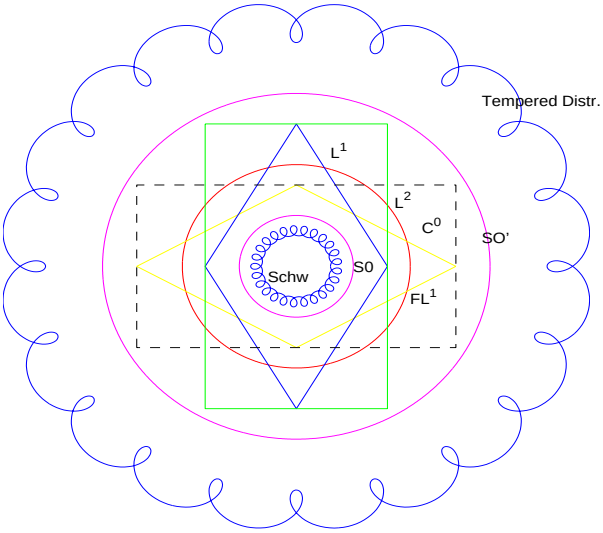
7 Gabor Multipliers and Best Approximation

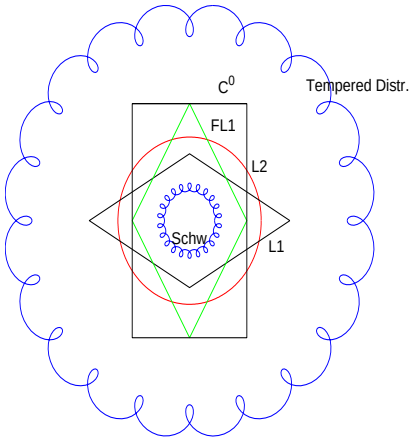
```
>> depfunnhg('gmappmh');
----------------------------------------
files depending on the file gmappmh.m  are:
    'C:\ml5\nuhagml\marioml\gmappmh.m'
    'C:\ml5\gabml\wks01\perrc.m'
    'C:\ml5\select\col2dig.m'
    'C:\ml5\select\dig2col.m'
    'C:\ml5\select\oneover.m'
----------------------------------------
>> GM1 = gabmulmh(W(1:b:n, 1:a:n),gt);
>> compnorm(GM1,gmappmh(GM1,gt,a,b));
quotient of norms: norm(x)/norm(y) = 1
difference of normalized versions  = 2.6291e-016
```

8 Function Spaces

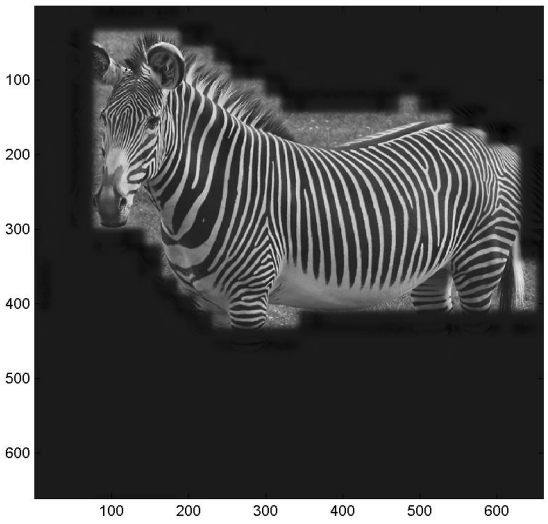
For a proper description of the transition between continuous and discrete settings and the control of operators arising in Gabor analysis modulation space turned out to be the right setting (see [8] for a good summary), but also [3, 6] for concrete settings. In particular the theory of Gabor multipliers is making use of the Banach Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

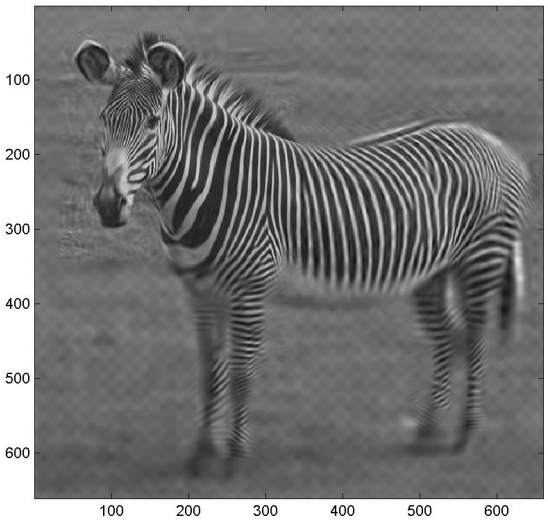



Tempered Distr.
L1
L2
C0
SO'
Schw
S0
FL1


C0
Tempered Distr.
FL1
L2
Schw
L1

8.1 2D Gabor transform, filtering



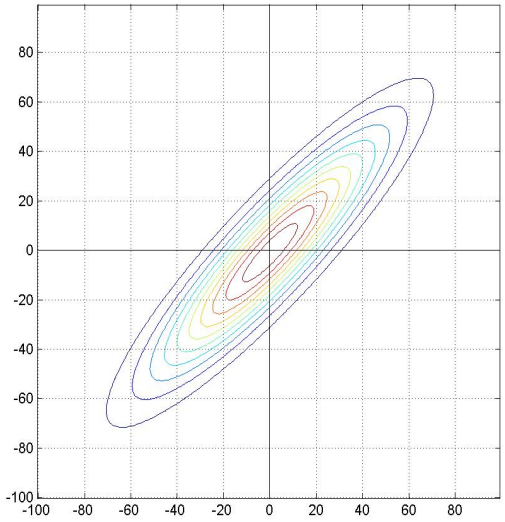


Just for the purpose of test let us recall, that a command like this one

```
\begin{center}
 \includegraphics [width=10cm,height=8cm] {\PDFS test111.jpg}
\end{center}
```

gives the following plot, showing the contourlines of the spectrogram (with Gaussian window) of a generalized Gauss function, obtained via

```
g5 = hermrot(g.^.05,45,HERM);
```



A *regular Gabor frame* is obtained by restricting the family of TF-shifts on a given atom to some lattice Λ, i.e. some discrete subgroup of the TF-plane (abstractly speaking, a subgroup of the Abelian group $G \times \widehat{G}$).

It is now interesting in general to find out which windows and which lattices $\Lambda \triangleleft G \times \widehat{G}$ provide good Gabor frames. Due to the Wexler-Raz principle (also going back to M. Rieffel's work on projective modules from 1988, [16]) the question is equivalent to establishing the Riesz basis property for the adjoint orbit, i.e. for the family (g_{λ°), with $\lambda^\circ \in \Lambda^\circ$.

For the case $G = \mathbb{Z}_N$ we are mostly interested in Gabor atoms (i.e. building blocks, or "windows" in the terminology of short-time Fourier transforms) which are obtained from the classical Gauss-function, and stretched resp. compressed versions (realized by real exponents) and fractional FT's of it (obtained as rotations in the TF-plane, using discrete Hermite-functions).

We hope to discuss/answer (experimentally resp. theoretically) questions of the following form:

1. Given a lattice, what is the best *generalized Gaussian* for this family (and how do condition number and redundancy relate to each other); CONJECTURE: for reasonable lattices it is always possible to find a pairing for that given redundancy (e.g. $red = 1.2$) such that a uniform quality of all the optimally adapted families can be guaranteed...!

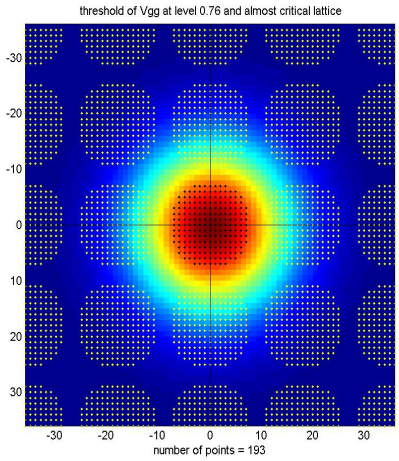
2. Given a Gabor atom from this family and a redundancy, what is the optimal lattice: Radu Frunza (Jacobs University) has done systematic experiments in this direction during his summer internships at NuHAG in 2010 and 2011;

3. x

Citations: The most important facts about the invertibility of the Gabor frame matrix is the use of Janssen's representation, which tells us that the Gabor frame operator corresponding to a pair (g, Λ), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ is an absolutely convergent sum of the form

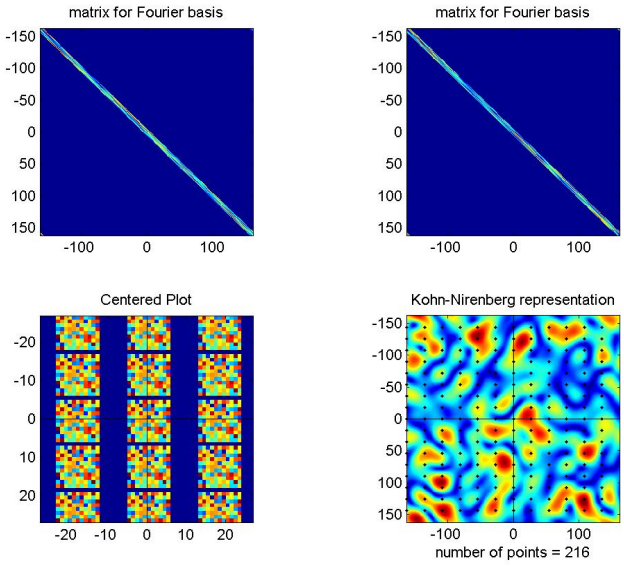
8.2 sufficient conditions for Gabor frames

$n = 360, a = b = 18; red = 10/9;$



8.3 Pilot tone arrangements

$n = 324.$



9 MORE LEFT OVER MATERIAL

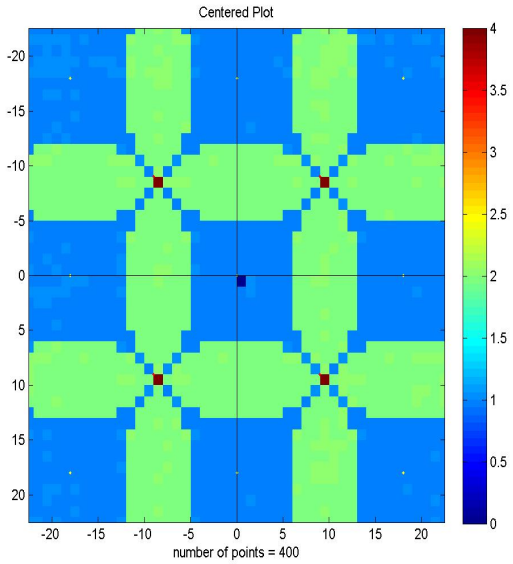
9.1 Maurice-Darian paper

: CHANGE OF TITLE TO: MULTIVARIATE GABOR FRAMES WITH GAUSSIAN WINDOWS

Abstract (another suggestion): It is the purpose of the paper to stimulate a more systematic investigation of multi-dimensional Gabor frames with respect to general lattices. In particular, we are interested in families generated from generalized Gaussians, which are also known as squeezed coherent states in quantum mechanics, resp. the action of non-separable groups. A number of concrete formulas and the fact that the metaplectic group is acting transitively on this family allow to derive sufficient (and sometimes necessary) conditions for many cases should allow users to explore their usefulness in practice.[1] Alternatively a sufficient condition based on the Janssen representation of the frame operator is formulated, which is easy to apply and provides reasonable estimates for partically interesting cases. At the end we state a conjecture relating the Gabor frame property of a pair (g, Λ) to the symplectic capacity of a certain ellipsoid associated to this pair.

Apply the *Janssen criterion* (sufficient condition based on the properties of the Janssen representation of a regular Gabor frame operator) in order to show

With the corollary, that in the case it is applied we also have a control on the frame operator, namely that it is even invertible on SO(Rd), with a control on the operator norm !!! (geometric series..!) which guarantees a certain level



9.2 Hexagonal lattices

The generating matrix is this one,

$$\begin{pmatrix} -0.7598 & -0.3799 \\ -0.0000 & 0.6580 \end{pmatrix} \tag{4}$$

[1]Part of the motivation for this paper was work done by the second named author within the framework of the UnlocX EU-project, DEDICATION!!!

coming from the command

```
[H2,H2adj,red] = showadjred(hx1,sqrt(.5),1);
>> sqrt(.5) : ans = 0.707106781186548 >> help gausumsq GAUSUMSQ.M hgfei modified
from gausum.m Dec.2011 ordinary gausum for stft(g,g)!
   Usage: gsq = gausumsq(A1,dilf,show);
```

The Λ° has index 4 within Λ, in particular

10 REFERENCES

[5, 8, 15]

Literatur

[1] M. Dörfler. Quilted Gabor frames - A new concept for adaptive time-frequency representation. *Advances in Applied Mathematics*, 47(4):668 – 687, Oct. 2011.

[2] H. G. Feichtinger and N. Kaiblinger. Varying the time-frequency lattice of Gabor frames. *Trans. Amer. Math. Soc.*, 356(5):2001–2023, 2004.

[3] H. G. Feichtinger and W. Kozek. Quantization of TF lattice-invariant operators on elementary LCA groups. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms. Theory and Applications.*, Applied and Numerical Harmonic Analysis, pages 233–266, 452–488, Boston, MA, 1998. Birkhäuser Boston.

[4] H. G. Feichtinger, W. Kozek, and F. Luef. Gabor Analysis over finite Abelian groups. *Appl. Comput. Harmon. Anal.*, 26(2):230–248, 2009.

[5] H. G. Feichtinger and T. Strohmer. *Gabor Analysis and Algorithms. Theory and Applications.* Birkhäuser, Boston, 1998.

[6] H. G. Feichtinger and G. Zimmermann. A Banach space of test functions for Gabor analysis. In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.

[7] M. Fornasier and H. Rauhut. Continuous frames, function spaces, and the discretization problem. *J. Fourier Anal. Appl.*, 11(3):245–287, 2005.

[8] K. Gröchenig. *Foundations of Time-Frequency Analysis.* Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

[9] K. Gröchenig and J. Stöckler. Gabor frames and totally positive functions, 2011.

[10] A. Holzapfel, G. Velasco, N. Holighaus, M. Dörfler, and A. Flexer. Advantages of nonstationary Gabor transforms in beat tracking. In *Proceedings of MIRUM11.*, November 2011.

[11] A. J. E. M. Janssen. Zak transforms with few zeros and the tie. In H. G. Feichtinger and T. Strohmer, editors, *Advances in Gabor Analysis*, Applied and Numerical Harmonic Analysis, pages 31–70. Birkhäuser, Basel, 2003.

[12] Y. I. Lyubarskii. Frames in the Bargmann space of entire functions. In *Entire and Subharmonic Functions*, volume 11 of *Adv. Sov. Math.*, pages 167–180. American Mathematical Society (AMS), Providence, RI, 1992.

[13] P. Prinz. Calculating the dual Gabor window for general sampling sets. *IEEE Trans. Signal Process.*, 44(8):2078–2082, 1996.

[14] P. Prinz. Theory and Algorithms for Discrete 1-Dimensional Gabor Frames. Master's thesis, University of Vienna, 1996.

[15] H. Reiter and J. D. Stegeman. *Classical Harmonic Analysis and Locally Compact Groups. 2nd ed.* Clarendon Press, Oxford, 2000.

[16] M. A. Rieffel. Projective modules over higher-dimensional noncommutative tori. *Canad. J. Math.*, 40(2):257–338, 1988.

[17] K. Seip and R. Wallstén. Density theorems for sampling and interpolation in the Bargmann-Fock space. II. *J. Reine Angew. Math.*, 429:107–113, 1992.

[18] G. A. Velasco, N. Holighaus, M. Dörfler, and T. Grill. Constructing an invertible constant-Q transform with non-stationary Gabor frames, 2011.