NuHAG Numerical Harmonic Analysis Group

What are optimal Gabor families Criteria for Gabor frames and Riesz Sequences

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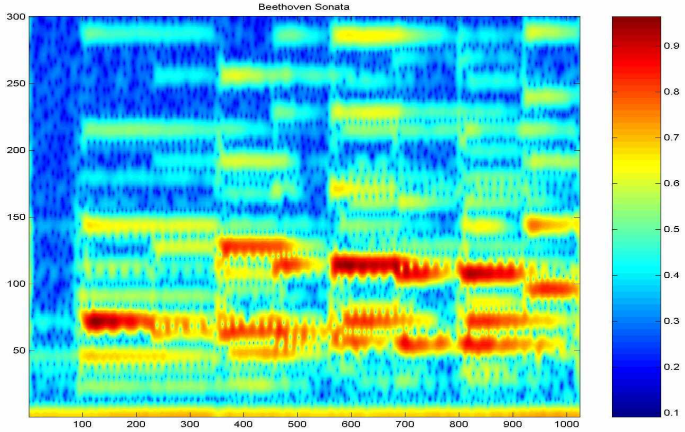
Gabor Analysis: Beethoven Piano Sonata



The META-message of this Talk: Images

Since you are going to hear a lot of theoretical and also applied material during this conference, and also because we have done in the last two decades a lot of numerical (MATLAB based) work, I will try to convey as much information as possible through images. NOTE however, that these images only support the intuition and should help to find appropriate formulations for the underlying theoretical questions. They also help us to formulate appropriate quality criteria!
NOTE also that we realize, using such images that there may be a difference between the visualized effects and the true, underlying numerical reality (think of phase factors, approximation errors) > new challenges for theory.

What are Gabor Multipliers?

Why are Gabor multipliers so interesting? *Because they allow us to mix visual impression (interpretation of the Gabor coefficients) with the chance to act (via linear operators) on signals!*

We will discuss **Gabor multipliers** and their continuous analogue, which could be called STFT (= short-time Fourier transform) multipliers, known in the literature by the name of **Anti-Wick symbolic calculus**, because every such multiplier on phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ gives us a linear operator with some desired properties. Let us first take a look at images, conveying the spirit of the task to be discussed.

The audio-engineer's work: Gabor multipliers

Digital realization of Gabor Multiplier

General Aspects of Gabor Analysis A

Gabor analysis can be described over general locally compact Abelian groups, and is this a branch of abstract harmonic analysis, making crucial use of the underlying Abelian group (of time- or space-) translation operators a family of functions on a group G, as well as the frequency multiplication operators.

This means that *Gabor multipliers* can be studied in a variety of concrete cases. The first test case are of course Gabor systems generated by a Gaussian atom, along the lattice $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, viewed as operators on $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$. But it makes also sense to consider Gabor multipliers on $\mathbb{R}^d$ (at least for $d = 1, 2, 3$), where they are used in practice to produce *slowly time-variant filters*.

It is also possible to build Gabor multipliers in the finite-discrete setting, where in fact eigenvectors and eigenvalues of the corresponding matrices are easily built.

General Aspects of Gabor Analysis A

The role of certain function spaces (e.g. *modulation spaces*) become clear, in order to describe boundedness of operators. In fact, a theory of *Banach Gelfand triples* has been developed meanwhile, which forms the backbone for Gabor analysis. First of all it is about the boundedness of analysis and synthesis operators, the choice of good windows, or the questions of continuous dependence (e.g. of dual atoms) on the ingredients, i.e. the window and the lattice used to build the Gabor family.

At the same time we have witnessed the improvement on the algorithmic side. While Gabor analysis was considered an interesting approach it was for a long time viewed as *computationally intensive and instable* task.

One can say: this problem has been overcome by now!

General Aspects of Gabor Analysis B

- **Linear Aspects**: dual frame = pseudo-inverse, Gabor Riesz basis: biorthogonal system
- **Algebraic Aspects**: lattice (Abelian Groups) act on a Hilbert space of signals via some *projective* representation, *commutation property* of Gabor frame operator, i.e. $[\pi(\lambda), S] == 0$, imply specific sparsity structure of S.
- **Functional Analytic features**: Convergence of double sums (sums over lattice) appear to be complicated (Bessel condition, unconditional convergence, etc.)
- Specific properties of the acting **Weyl-Heisenberg group** (!phase factors) or the validity of Poisson's formula for the symplectic Fourier transform (in contrast to wavelets);

Main Questions in Gabor Analysis I

Following the suggestion of Denis Gabor the focus was for a long time on the question: Given a (Gabor) window g, i.e. a smooth function which is well localized near the origin, what can one say about the pairs of lattice constants such that the family of TF-translates, usually denoted by $G(g, a, b)$ is a Gabor frame, i.e. allows to span all of the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ (in a stable way, i.e. using only $\ell^2(\mathbb{Z}^{2d})$-coefficients).
The study of this question is typically focussing on the realization of a pair of *frame inequalities* of the form: There exists $A, B > 0$ such that the family (g_λ) satisfies for all $f \in \mathbf{L}^2(\mathbb{R}^d)$

$$A\|f\|^2 \leq \sum_{\lambda \in \Lambda} |\langle f, g_\lambda \rangle|^2 \leq B\|f\|^2 \tag{1}$$

Main Questions in Gabor Analysis II

Of course this questions is known to be equivalent to the invertibility of the so-called *frame operator* S defined as

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda. \tag{2}$$

The fact that in the *regular case* Λ is a discrete subgroup of the additive group $G \times \hat{G}$ (resp. of *phase space*) implies that $S = S_{g,\Lambda}$ satisfies important *commutation relations*, i.e.

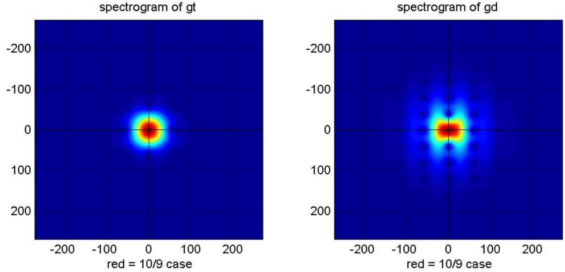
$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda. \tag{3}$$

This implies that the dual frame of a regular Gabor frame is generated using the *canonical dual* atom $\tilde{g} = S^{-1}(g)$, which inherits good TF-localization properties of g. (3) is also equivalent to the so-called *Janssen representation* of the Gabor frame operator.

Main Questions in Gabor Analysis

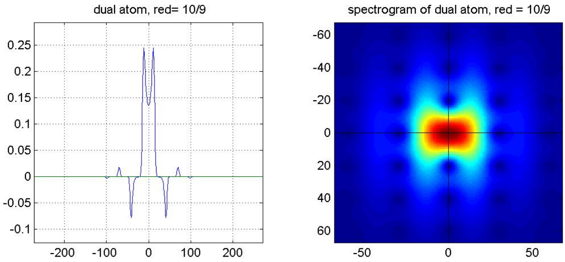
More recently this basic questions has been answered in a number of different ways, e.g. for totally positive functions Gröchenig and Stöckler, ([1]), generalizing the classical statement for the Gauss function due to Seip/Wallstein ([3]) and Lyubarskii ([2]) respectively, namely $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$ is OK if and only if $ab < 1$.
But it is also interesting to find out under which conditions alternative windows can be used. For example, in the 1D-case the so-called painless approach turned out to be useful, which works fine for compactly supported windows (also good for computations), with the extra condition that the frame-operator is just a simple (invertible) multiplication operator.
What is crucial in all these situations is the following:

Dual and tight Gabor atoms



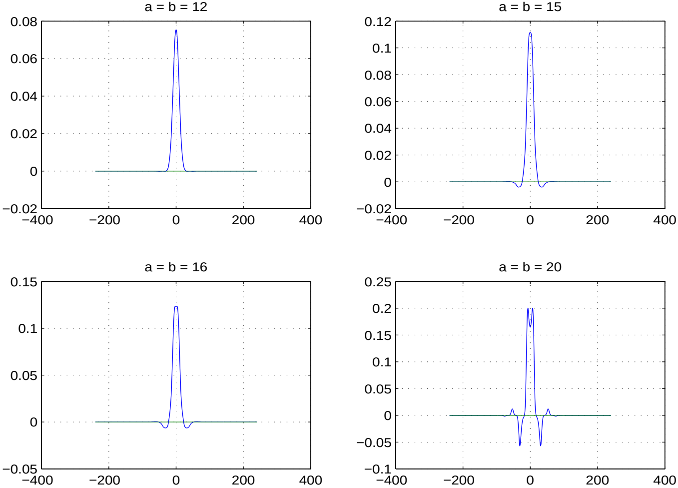
Dual and tight Gabor atoms IV

Showing the dual window at more critical situation:



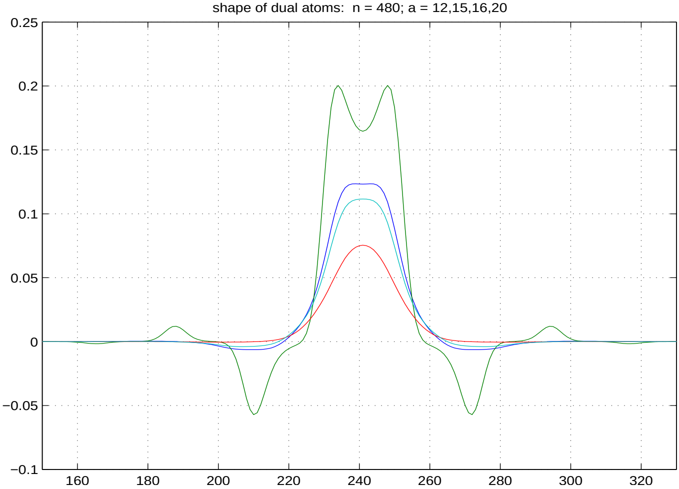
Between critical density and high redundancy

Showing the dual window at more critical situation:



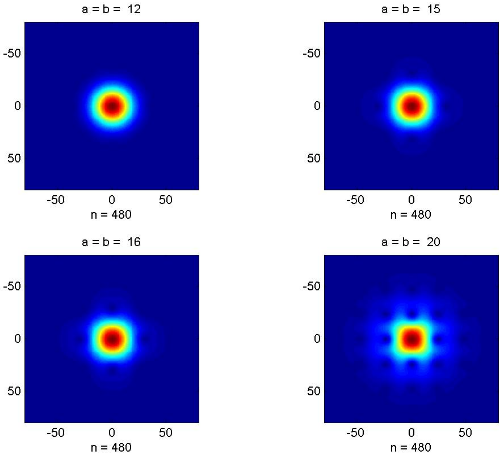
Between critical density and high redundancy

Showing the dual window at more critical situation:



Between critical density and high redundancy

Showing the dual window at more critical situation:



The general Janssen representation

The commutation relation $\pi(\lambda) \circ S_{g,\Lambda} = S_{g,\Lambda} \circ \pi(\lambda)$ is not only relevant for the fact, that the *dual* of a *Gabor frame* is again a Gabor frame, generated from what is called the *dual Gabor atom* (for the given atom g and lattice Λ), it has also other useful consequences, most of which directly related to the so-called Janssen representation of the Gabor frame operator.
Recall, that the spreading representation of an operator corresponds - in the case of finite groups - the representation of an $N \times N$ matrix as a superposition of N^2 TF-shift matrices. In fact, these matrices generate (up to the scaling factor $\sqrt(N)$) an orthonormal system within the $N \times N$-matrices, endowed with the natural Euclidean structure of $\mathbb{C}^{N^2}$ (Frobenius norm).

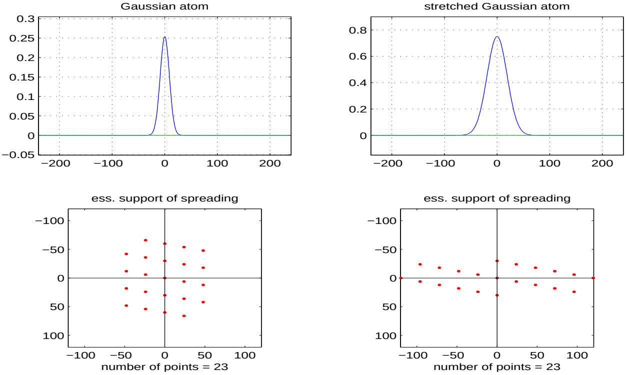
The general Janssen representation

Since this transition has many good properties analogous to the ordinary Fourier transform (e.g. periodization corresponds to sampling in the spreading domain!) it may not come as a complete surprise that the spreading operator $S : f \mapsto S(f) := \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ can be viewed as $S = \sum_{\lambda \in \Lambda} P_\lambda$ resp. as the $\Lambda-$ periodization of the rank one operator $P_o : f \mapsto \langle f, g_0 \rangle g_0$, which is nothing but the orthogonal projection onto the 1d-subspace generated by the atom (assuming from now on that $\|g_0\|_2 = 1$). Using that the spreading function of P_0 coincides with $V_{g_0} g_0$, the short-time Fourier transform of the atom g_0 with respect itself (also ambiguity) function, on ends up (using the symplectic Poisson formula) with

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ).$$

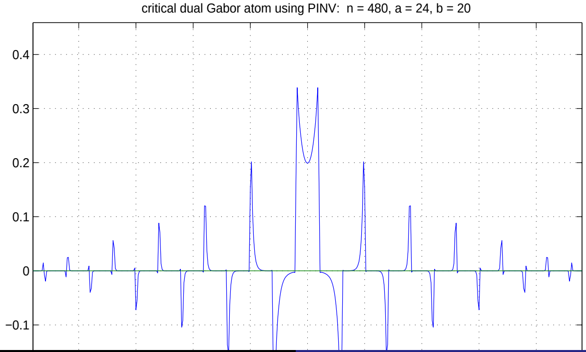
The general Janssen representation

In this plot we show how the concentration of the ambiguity function $V_g(g) = STFT(g,g)$ for g influences the essential support of the spreading operator of the Gabor frame operator. In each case the support is part of the (same) lattice, in our case a slanted lattice of redundancy $red = 1.5$.



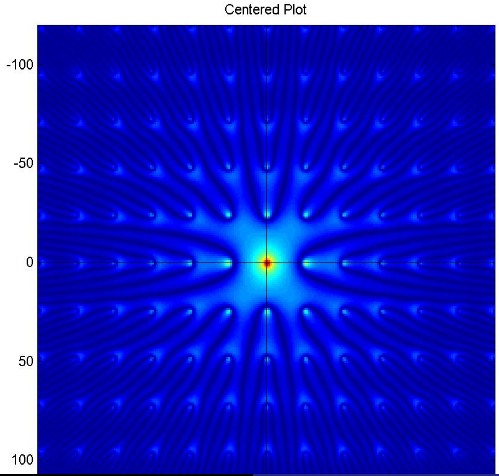
Need for Quality criteria for Gabor systems

Most authors would agree, that *well concentrated* Gabor systems, i.e. Gabor systems which allow perfect reconstruction of signals in a localized way are what one should go for. Just recall the Balian-Low principle, telling us that there is no stable dual for the critical case. In fact, one can compute (using PINV) the dual atom for the critical case and it looks like this:



The Balian-Low principle continued

Looking at the ambiguity function $V_{\tilde{g}}\tilde{g}$ of the "dual window" shows also (a nice pattern but) terrible TF-concentration.



QUESTIONS: Gabor's nightmare

First let us fix the rules of the game, resp. browse through different types of questions which one may want to ask. After all, the *very original* (natural) question posed by D. Gabor has turned into a (mathematical) nightmare! He was arguing essentially like this:

1. What is the best Gabor atom (in our terminology)? ANSWER: the Gauss-function, because it is a minimizer in the sense of the Heisenberg uncertainty relation (symmetry between time and frequency suggests $g_o(t) = e^{-\pi|t|^2}$!)
2. Why the integer lattice? ANSWER: by exclusion Considering lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$ one has in fact:
   - If $ab > 1$ the Gabor family is not even total in $\mathbf{L}^2(\mathbb{R})$.
   - If $ab < 1$ the Gabor family is linear dependent (not in the classical, but in a *practical* sense!)

Gabor's nightmare, continued

But still the conclusion: "Hence we should use $ab = 1$, resp. for reason of symmetry the lattice $a = 1 = b$" was not supported by evidence, **but rather by the hope**, that one might have at the boundary between too rich and too poor the appropriate mixture.

As we know now we do **not get a Riesz basis** (the linear independence which Gabor was hoping for is flowed, although somehow in a weak sense, one can represent the zero distribution in a non-trivial way using bounded $+/-1$ coefficients) **nor does one have a frame** (a stable set of generators)!

M. Bastiaans attempt to compute the dual window was courageous (and influential), but it resulted in a nasty, bounded function which does *not belong to* $\mathbf{L}^2(\mathbb{R})$.

Living with Redundancy: Banach frames

Since the lack of totality *cannot be overcome* it is reasonable to accept the need for redundancy in the representation (allowing for stable, localized signal representations), and even have uniqueness of representations pursuing the minimal norm representation. In fact, the *dual Gabor family* provides for every $f \in \mathbf{L}^2(\mathbb{R}^d)$ the minimal norm coefficients among all representations

$$f = \sum_{\lambda \in \Lambda} c_\lambda g_\lambda \quad \text{with} \quad (c_\lambda) \in \ell^2(\Lambda).$$

The good thing is that for windows $g \in \mathbf{S}_0(\mathbb{R}^d)$ also $\tilde{g} \in \mathbf{S}_0(\mathbb{R}^d)$ and hence there is a much larger range of spaces (modulation spaces) for which this particular method of expansion makes sense and allows to characterize membership in the corresponding (Fourier invariant) function spaces.

Quality criteria for pairs (g, Λ)

We have seen that one has to accept some redundancy. One can show that the dual atoms vary continuously in $\mathbf{S}_0(\mathbb{R}^d)$ with the lattice constants.
Since one has the perfect reconstruction from the full STFT (expansion of signals into coherent state in quantum physics) it is not surprising that for $(a, b) \to (0, 0)$ the shape of the dual atom is convergent to the original atom. So one has for high redundancy a dual atom which is almost like the Gabor atom, e.g. the Gauss function. Practically a redundancy of 9 is high enough for this purpose.
On the other hand one finds a kind of weak convergence towards the critical Bastiaan's dual function for $(a, b) \to (1, 1)$, which is not in $\mathbf{L}^2(\mathbb{R})$, and corresponding leakage of $\tilde{g}$ in the TF-plane.

Quality criteria for pairs (g, Λ)

Let us now come to a set of realistic question:

1. Given a Gabor atom g and a redundancy factor `red > 1`, what is the optimal (separable) lattice $a\mathbb{Z} \times b\mathbb{Z}$ such with $ab = 1/red$ in the sense of the **condition number of the corresponding frame operator** (resp. the analysis mapping); unfortunately this is computationally costly.
2. The condition number is of course indicative for the quality of the inverse Gabor frame operator $S^{-1} = S_{\tilde{g},\Lambda}$. For tight Gabor frames the condition number is clearly equal to the optimum $= 1$, but then one needs another quality criterion to distinguish between the localized tight atoms and the not-so localized ones. We propose to choose the $\mathbf{S}_0-$norm of $\tilde{g}$.
3. Since the $\mathbf{S}_0$-norm is computed using a fixed window (e.g. the Gaussian) a good alternative is the $\mathbf{L}^1$-norm of the ambiguity function $V_{\tilde{g}}\tilde{g}$.

Quality criteria for pairs (g, Λ), ctd.

Since Gabor families (especially tight Gabor families) are used to build to perform best approximation of slowly varying systems by Gabor multipliers we suggest that the stability of choosing the coefficients of these approximating Gabor multipliers is reasonable request. This stability deteriorates with increasing redundancy and promotes relatively low redundancy. It can be expressed through the Riesz bounds of the family $(P_\lambda)_{\lambda \in \Lambda}$, which can be computed easily (using properties of their Kohn-Nirenberg symbols). We call this the **projection condition number**.

Combined with the measures for the quality of Gabor system above we define a **compound condition number**, e.g. by taking the product of the condition number of the Gabor frame operator and the projection condition number.

Quality criteria for pairs (g, Λ), ctd.

Typical question using these criteria are:

- Given some atom and a family of lattices, e.g. the separable lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, which one is providing the minimum of the compound condition number?
- How much better are non-separable lattices, especially hexagonal lattices?
- Given a lattice, for example the critical lattice $\mathbb{Z} \times \mathbb{Z}$: what is the optimal compression factor?
- Is there a kind of new uncertainty telling us that the compound condition number can never be smaller than some number (say π, according to our experiments).
- Given a fixed (sufficiently dense lattice), and a parameterized family of possible atoms (e.g. generalized Gaussians), which one is the optimal one for this lattice?

Quality criteria for Gabor systems over $\mathbb{Z}_N$

There are many ways to control in one or the other way the quality of Gabor systems, both in the continuous setting (over $\mathbb{R}^d$) or in the discrete/finite setting.
We will CONCENTRATE in the rest of this talk on quality measures which can be computed in the finite setting, for *any concrete pair* (g, Λ) where g is a given Gabor atom (typically well concentrated in time and frequency near the origin) and an arbitrary lattice within $\mathbb{Z}_N \times \mathbb{Z}_N$.
In particular, we have now methods to do an exhaustive search over all such finite subgroups (thanks to Christoph Wiesmeyr, Radu Frunza and Nicki Holighaus, with some help from Laszlo Toth).

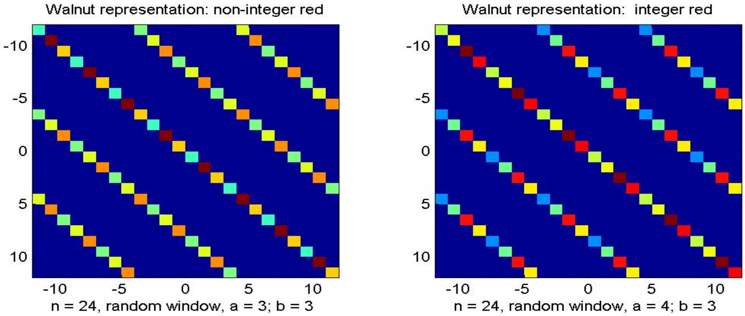
Quality criteria for Gabor systems over $\mathbb{Z}_N$

While there are meanwhile quite efficient algorithms for dual Gabor atoms respectively tight Gabor atoms in the separable case (see Peter Soendergaards LTFAT toolbox) some of the routines need a bit more work (ongoing) in the non-separable case. But in principle this is possible. But on the other hand the transition to the 2D-case is the next computational challenge.

The $\mathbf{S}_0$-norm can be computed using the full sum over the absolute STFT (over the discrete TF-plane), but with *some normalization* which allows then to compare also lattices and atoms which have generated with respect to different signal lengths (the idea being that they are just samples of the same, underlying, continuous function $g \in \mathbf{S}_0(\mathbb{R}^d)$).

The Walnut Representation

There are many sufficient conditions which guarantee that a Gabor family generated by a pair (g, Λ) is a Gabor frame, relying on the Walnut representation.



Double preconditioning methods

These results (also by Casazza and Christensen or Sigang Qiu) are mostly concerned with diagonal dominance of the Gabor frame matrix. This is typically the case if $1/b$ is relatively large and g is well concentrated in the TF-sense. In the "painless case" the whole frame matrix is just a diagonal matrix.
Since it was always one of the principles of Gabor to give the time and the frequency variable the same importance it is of course also interesting to consider the case where the frame matrix is diagonal dominant in the Fourier basis (which is often the case for small a). Combining these two ideas one comes up with the so-called double preconditioning method (developed jointly with Peter Balazs).

The Janssen Criterion

The methods described above are valid for the so-called separable case, i.e. for the case that $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$. Here the spreading function of S is supported on the *adjoint lattice* $\Lambda^\circ = \frac{1}{b}\mathbb{Z} \times \frac{1}{a}\mathbb{Z}$. For the non-separable case one does not have the simple *Walnut representation* anymore and it is better to directly work with the Janssen representation:

$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ)\pi(\lambda^\circ),$$

where Λ° is the adjoint group (a symplectic variant of the orthogonal group). Separating terms one finds that

$$Id - C_\Lambda^{-1} \cdot S_{g,\Lambda} = \sum_{\lambda^\circ \neq 0} V_g g(\lambda^\circ)\pi(\lambda^\circ),$$

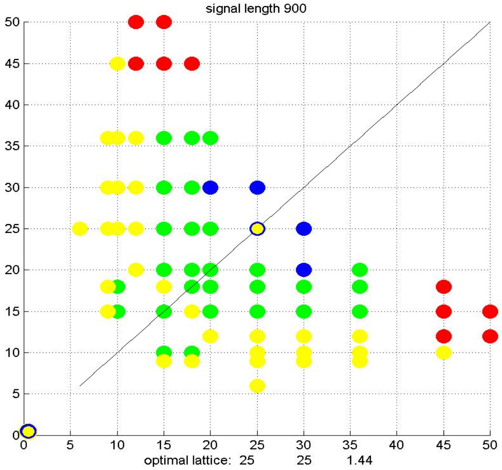
The Janssen Criterion

As a consequence we obtain as a sufficient condition that for an atom g with $\|g\|_2 = 1$ one has

$$\sum_{\lambda^\circ \neq 0} |V_g g(\lambda^\circ)| < 1$$

This implies that there is a good matching between the Gabor atom g and the lattice Λ if Λ° is adapted to the shape of $|V_g(g)|$, which is a general 2D-Gauss function for the case that g is some generalized Gauss-function (e.g. $V_g g$ is a stretched and rotated 2D Gaussian).

Illustrations of the Janssen Criterion



Selection of bibliographic items, see www.nuhag.eu

K. Gröchenig and J. Stöckler.
Gabor frames and totally positive functions.
preprint, 2011.

Y. I. Lyubarskii.
Frames in the Bargmann space of entire functions.
In *Entire and Subharmonic Functions*, volume 11 of *Adv. Sov. Math.*, pages 167–180. American Mathematical Society (AMS), Providence, RI, 1992.

K. Seip and R. Wallstén.
Density theorems for sampling and interpolation in the Bargmann-Fock space. II.
J. Reine Angew. Math., 429:107–113, 1992.