NuHAG Numerical Harmonic Analysis Group

Postmodern Harmonic Analysis

[How far Soft Analysis can get you!]

Hans G. Feichtinger
www.nuhag.eu

August 23rd, 2012
ABEL SYMPOSIUM, OSLO

Books on the market: Harmonic Analysis

- Juergen Jost: **Postmodern Analysis**. 2nd ed. [16]
- The **"old testament"** by Antoni Zygmund: Trigonometric series. 2nd ed. Vols. I, II Camb. Univ. Press, (1959, [24])
- Edwin Hewitt and Kenneth A. Ross: **Abstract Harmonic Analysis**. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups [15]
- Hans Reiter [and Jan Stegeman]: is entitled "**Classical Harmonic Analysis** and Locally Compact Groups" (2nd ed., 2000, [20], first ed. 1968).
- L. Grafakos: **Classical and Modern Fourier Analysis** [8] has later been split into two books.
- G. Folland: A Course in Abstract Harmonic Analysis [7]

Books on the market

More recent books are:

- H. J. Weaver: Applications of Discrete and Continuous Fourier Analysis (1983, [22]).
- R. Shakarchi and Elias M. Stein: Fourier Analysis: An Introduction Princeton University Press, Princeton Lectures in Analysis, (2003, [21]).
- Robert Marks: Handbook of Fourier Analysis and its Applications Oxford University Press, (2009 , [19]).
- M.W. Wong: Discrete Fourier analysis. Pseudo-Differential Operators. Theory and Applications 5. Basel: Birkhäuser (2011, [23]).

More recent books providing all necessary details from measure theory are (among many others);

Books on the market

- John J. Benedetto and Wojciech Czaja: **Integration and Modern Analysis** Birkhäuser, (2009, p.576, [1]);
- or the upcoming book by Chris Heil: Introduction to Harmonic Analysis,
- ——————————- books on frames, bases
- O. Christensen: Frames and Riesz Bases ([2]);
- C. Heil: A Basis Theory Primer. Expanded ed., ([14]);
- ——————————————————- or TF analysis:
- G. Folland: Harmonic Analysis on Phase space ([6]);
- K. Gröchenig "Foundations of TF-analysis" ([13]);

Fourier Analysis “in the books”

Let us take a look back into the historical development of FOURIER ANALYSIS:
First J.B. Fourier’s claim that *every* periodic function can be *expanded* into a Fourier series was challenging the mathematical community to develop proper notions of **functions** and **integrals** (Riemann, Lebesgue) and different types of convergence. The need to properly describe measurability and exceptional sets has certainly greatly influenced the **theory of sets**.
The last century saw the development from **classical Fourier analysis** (over Euclidean spaces $\mathbb{R}^d$) to what has been sometime called **abstract Harmonic Analysis** (over LCA groups), but also the invention of the theory of **(tempered) distributions** as a tool to extend the domain of definition to include point masses and polynomials.

General Achievements in the Last Century

Recall that we have come a long way from Fourier to Modern Harmonic Analysis:

- in the late 19th century concepts of pointwise convergence had its first high period (De-la-Vallee Poussin, Dirichlet);
- Set theory and integration theory have been founded, then topology and functional analysis have been developed;
- Banach algebras and in particular Gelfand's theory provide proper foundations of Fourier analysis over LCA groups;
- the existence of a Haar measure and Pontrjagin's theorem, as well as Plancherel's theorem have been established;
- L. Schwartz established his theory of tempered distributions, extended by Bruhat to LCA groups;
- Cooley-Tukey invent the FFT around 1965;

Fourier Analysis “in the books” II

If we take a look into published books on Fourier Analysis one finds a majority of books following the **historical path**:

- First starting with Fourier series and Hilbert spaces
- Then the Fourier transform, Riemann-Lebesgue theorem
- Fourier inversion, uniqueness
- Plancherel’s theorem, convolution theorem
- $\mathbf{L}^p$-spaces, Hausdorff-Young theorem
- Poisson’s formula, Sampling, Shannon’s theorem
- Fast Fourier Transforms (FFT)
- perhaps some wavelets or other recent stuff
- ideally indications about the extension to distributions

Fourier Analysis "in the books" III

A side aspect or rather an *enabling tool* in performing this task, is the availability of suitable function spaces (in fact Banach spaces of functions or distributions) aside from the classical $\mathbf{L}^p$-spaces, such as

- Wiener amalgam spaces
- Modulation spaces
- Sobolev spaces
- Besov-Triebel-Lizorkin spaces
- and others ....

What are the Proposed Ingredients?

While most books still follow the historical path, starting form Lebesgue integration theory, going to $\mathbf{L}^p$-spaces and finish by mentioning recent developments like FFT, wavelets or frames I think we should **reconsider the catalogue** of important concepts and how we want to introduce them.
It is just like linear algebra, where the rapid development of numerical linear algebra is having its impact (to some extent) on the content of modern linear algebra books, which try to accommodate the balance between concrete and often simple examples, to abstract viewpoints and then not-so-trivial applications.

The New Catalogue of Terms I

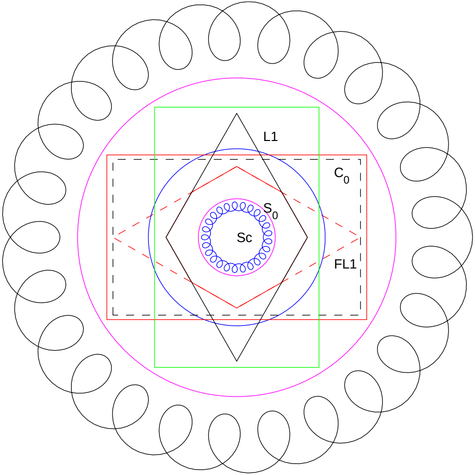
Overall I may suggest a more functional analytic approach to Fourier/Harmonic Analysis. By this I mean to *acknowledge* the fact of life that linear spaces of signals or operators are inherently *infinite dimensional*, so that the idea of working with a fixed basis is not a good idea, but also definitely requires to have infinite series, and thus *completeness* must play an important role.
So in principle we need **Banach spaces and their duals**, in many cases it will be convenient to work in a **Hilbert space**. In the case of dual elements we also have to allow considerations of the w^*-convergence of functionals.
OVERALL: nothing exotic so far!

The New Catalogue of Terms II

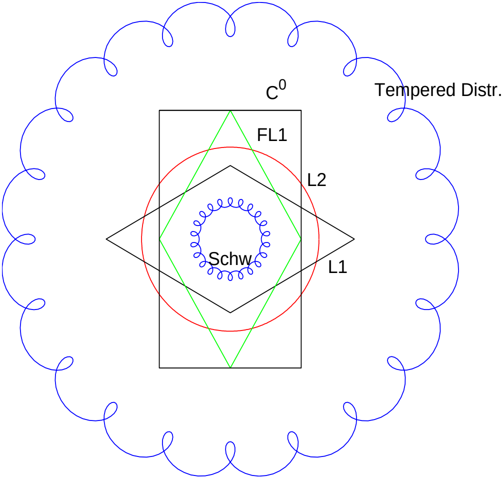
Concepts and terms which are perhaps less familiar are, some of which already are interesting in a linear algebra setting:

1. *Banach algebras* and **Banach modules**;
2. approximate units (e.g. Dirac sequences for convolution);
3. the four spaces associated with a linear mapping (G. Strang);
4. the SVD (singular value decomposition);
5. **Banach frames** and Riesz (projection) bases as extensions of generating and lin. indep. sets;
6. **Banach Gelfand triples**, retracts in this category;
7. unitary Banach Gelfand triple isomorphism;

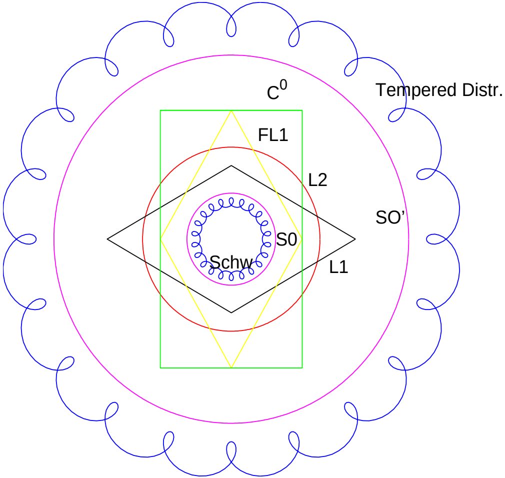
The classical function spaces I



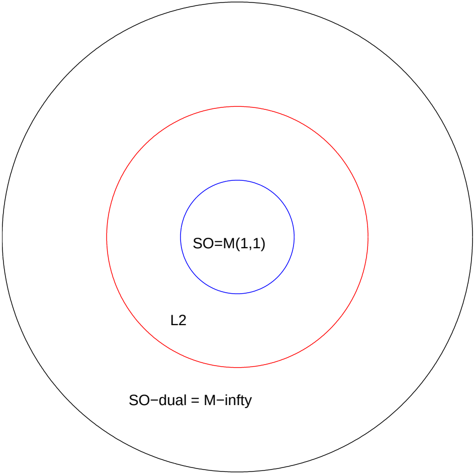
The classical function spaces II



The enriched family: two more spaces



The Banach Gelfand Triple



Pro and Contra Standard Approach

Trying to *change standards* or even claim that there are alternative options to the way how *things are to be done properly* within a given community are always a bit controversial.
Hence I would like to carefully look into the arguments for and against that traditional way, starting from Lebesgue integration, and check its validity.
I am not! concerned about our PhD students who need measure theory and tempered distribution anyway, but the vast majority of applied scientist who *deserve a properly formulated mathematical foundation* of what they are doing.

Applications of Fourier Analysis in Modern Life

Although it is not the usual way to justify the teaching of mathematical topics it is not unreasonable to reconsider those areas where Fourier Analysis and in particular the use of the FFT has a significant impact on our daily life!
Let us just mention a view cases:

- digital signal and image processing
- mobile communication
- medical imaging (e.g. tomography)
- why are WAV files stored using the sampling rate 44100/sec?
- how is MP3 able to compress musical information?

But what is the relationship to the Fourier Transform we are teaching in our analysis classes?

Fourier Analysis: Technical requirements I

In order to run this program properly one has to explain the necessary terms first. The Fourier transform appears a priori to be defined as an **integral transform**, namely via the equation

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega. \tag{2}$$

We all know about the short-comings of the (nice and simple) Riemann integral and thus vote for *Lebesgue integration*.
BUT it cannot justify the inversion formula!

Fourier Analysis: Technical requirements II

There appear to be many further reasons for assuming that the theory of Lebesgue integration learned properly is a requirement for a deeper understanding of Fourier analysis:

- The Banach space $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is not only the "proper domain" of the (integral) FT, but also for **convolution**!
-

$$f * g(x) := \int_{\mathbb{R}^d} g(x-y)f(y)dy = \int_{\mathbb{R}^d} T_y g(x)f(y)dy \quad (3)$$

- the convolution theorem $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
- checking associativity of convolution requires Fubini;

Fourier Analysis: Technical requirements III

Still there are cases where Lebesgue integration is *not be able to provide good answers*:

- The Fourier inversion problems requires to introduce summability kernels and take limits of functions (in suitable function spaces);
- There are discrete measures having well defined Fourier (Stieltjes) transforms, but the corresponding theory of *transformable measures* is equally asymmetric! (developed by Argabright and Gil de Lamadrid)
- Pure frequencies (characters) are not in the domain of classical Fourier transforms (also Dirac measures);
- For PDE applications the Schwartz theory of tempered distributions is the right thing anyway!

Fourier Analysis: Technical requirements IV

So what do we need in order to properly define **generalized functions** and in particular **tempered distributions**.

- To define them we *need suitable spaces of test functions*, typically infinitely differentiable functions, with compact support, are sufficiently rapid decay;
- in the case of the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ one takes only functions which together with all their (partial) derivates decay faster than any polynomial;
- in order to properly define the dual space one has to at least indirectly put a convergence on such spaces of test functions in order to select the *continuous linear functionals*;
- the useful test functions spaces (such as $\mathcal{S}(\mathbb{R}^d)$) fulfill extra properties, such as Fourier invariance.

Fourier Analysis for Applied Scientists

If we look into this program and the potential role that Fourier Analysis (and more generally Harmonic Analysis) should play the teaching of Fourier Analysis we may ask ourselves:

- are we teaching all of these concepts to the majority of our master students (or teacher students)?
- is there a chance that applied scientist such as engineers of computer scientists can learn Fourier analysis in this way;
- are there ways of teaching the essence of Fourier analysis in a different way, without making wrong claims, such as: THE DIRAC FUNCTION IS ZERO EVERYWHERE EXCEPT AT ZERO, BUT IT IS SO IMMENSELY HUGE, SOMEHOW LARGER THAN PLUS INFINITY, SUCH THAT ITS INTEGRAL SATISFIES $\int_{\mathbb{R}} \delta_0(t)dt = 1$.
  So should we now forget about Lebesgue's integral??

Assessment of needs: bottom up approach

Let us try a bottom up approach, analyzing the emphasize put on those items that take an important role in typical engineering books (as opposed to the mathematical literature):

- The basic terms are filters, time-invariant systems, etc.
- Characterization of linear time-invariant systems as *convolution operators* by means of the impulse response function;
- Characterization of LTIs using the *transfer function*
- the convolution theorem connecting impulse response and transfer function

Assessment of needs: bottom up approach: II

Asking critically what is needed.

- is Lebesgue integration crucial, or even more important than the concept of "generalized functions"
- is it OK to leave engineers with the all-important "sifting property" of the Dirac δ-distribution/function?
- is there a way to teach distribution theory in a natural context? (i.e. without topological vector spaces)
- What is the role of the FFT? Just the computational little brother of the true Fourier transform, based on Lebesgue integration?

Assessment of needs: bottom up approach: III

Given the realistic situation that a typical, even mathematically oriented engineer will

- never be able to follow in all details, but
- nowadays students learn about linear algebra in a more applied spirit, using MATLAB^{TM} in their course, and can carry out numerical experiments;
- that a certain understanding of generalized functions or distributions is unavoidable, if the computation of divergent integrals resulting in Dirac's or other computational tricks (which do not even have a precise meaning to those who have digested more theory);

Comparing with the Number Systems

Preparing for the suggestion to make use of so-called *Banach Gelfand Triples* let us recall what the situation is when we **use different number system**. Once we have learned to see the advantages of the different types of numbers, taken from the fields

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

it is easy to work with them, following properly given rules.

- Inversion is only "simple" within $\mathbb{Q}$!
- Taking square roots (and other things) is only possible within the (positive part of) $\mathbb{R}$;
- Once $\mathbb{C}$ is well-defined as pairs of reals, with properly defined addition and multiplication, the field properties of $\mathbb{C}$ follow without reflection at each step *how e.g. inversion is carried out within* $\mathbb{R}$!

Comparing with the Number Systems, II

So in teaching analysis one has to think about the choice of spending a lot of time on the establishment of the existence and basic properties of the real number system, or rather assume from the beginning that *there exists some totally ordered field, containing the rational numbers as a dense subfield*, and go deeper into analysis:
Better spend time on concepts of differentiability of functions, Taylor expansions of differentiable functions, or other properties which will the students enable to properly solve differential equations or develop realistic models for application areas.
Of course we are *not advocating* here to leave the students without an awareness of what has been shown and what is taken as granted, what could be proved in detail, and what is actually carried out step by step.

Case Study I: Characterization of TILS

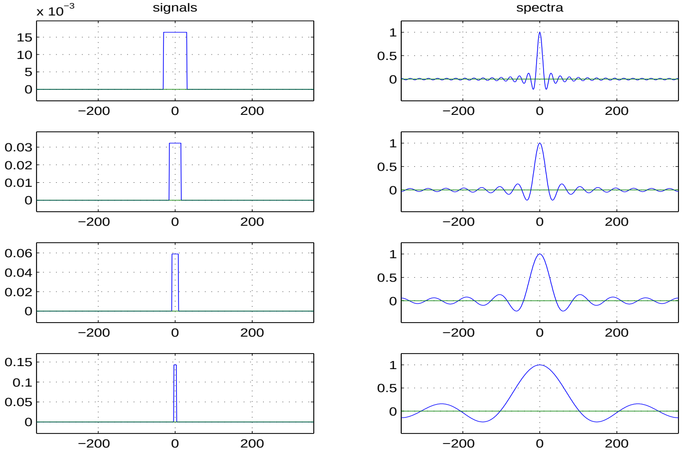
It is one of the first mathematical claims taught to electrical engineers students that every translation invariant linear system T can be viewed as a convolution operator. The standard rule (often with some hand-waving) is the following:

If one treats the case of discrete signals, say over the group $\mathbb{Z}$ of integers, then the *impulse response* is just the output arising if the unit-vector $\mathbf{e}_0$ is used as input vector. Clearly unit vectors at $n \in \mathbb{Z}$ can be obtained from $\mathbf{e}_0$ by n units (left or right) thus allowing to obtain a mathematically correct characterization of T

But what kind of problems are put under the rug if we have continuous variables, say we are interested in TILs over the real line $\mathbb{R}$? Should/can we simply replace $\mathbf{e}_0$ by δ_0?

Case Study I: Characterization of TILS: II

The usual way , Dirac sequences of box functions, vague arguments . . . ;



Applications of Fourier Analysis in Modern Life

Although it is not the usual way to justify the teaching of mathematical topics it is not unreasonable to reconsider those areas where Fourier Analysis and in particular the use of the FFT has a significant impact on our daily life!
Let us just mention a view cases:

- digital signal and image processing
- mobile communication
- medical imaging (e.g. tomography)
- why are WAV files stored using the sampling rate 44100/sec?
- how is MP3 able to compress musical information?

But what is the relationship to the Fourier Transform we are teaching in our analysis classes?

Case Study I: Characterization of TILS: III

Not too long ago an obviously mathematically trained engineer, Irwin Sandberg, was observing that there is a "scandal". Between 1998 and 2004 he published a series of papers, e.g. with the title:

Continuous multidimensional systems and the impulse response scandal.

So far the impact of his observations was quite modest, although he demonstrates among others that there are bounded linear mappings from the space $\mathbf{C}_b(\mathbb{R}^d)$ (endowed with the sup-norm) into itself, which are translation invariant but *cannot be represented* in the usual/expected way!
It seems that the community prefers to ignore a pedantic member rather than doing ground work on the basics!

Case Study I: Characterization of TILS: IV

But it is easy to correct the problem by replacing $\mathbf{C}_b(\mathbb{R}^d)$ by $\mathbf{C}_0(\mathbb{R}^d)$, the space of continuous, complex-valued functions vanishing at infinity (mostly because this space is separable!). Recall that by definition (justified by the Riesz-Representation Theorem) the *dual space* to $\left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ can be called the space of bounded (regular) Borel measures, denoted by $\mathbf{M}(\mathbb{R}^d)$.

Theorem

There is an isometric isomorphism between the Banach space of bounded linear mappings from $\left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ into itself which commutes with all translation operators $T_x, x \in \mathbb{R}^d$, and the space $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$ (with the norm as dual space).
Given $\mu \in \mathbf{M}(\mathbb{R}^d)$ one defines the convolution operator

$$C_\mu f : f \mapsto \mu * f(x) = \mu(T_x \check{f}), \quad \check{f}(x) = f(-x).$$

Case Study I: Characterization of TILS: V

This results requires a couple of auxiliary observations, such as the (norm) density of compactly supported measures within $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$ in order to ensure that C_μ is not only *uniformly continuous* but in fact even in $\mathbf{C}_0(\mathbb{R}^d)$, for every input signal $f \in \mathbf{C}_0(\mathbb{R}^d)$.

On the other hand all one needs is the existence of sufficiently many functions in $\mathbf{C}_0(\mathbb{R}^d)$, i.e. the density of functions with compact support within $\left(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$. This fact is easily established using the local compactness of the additive group $\mathbb{R}^d$, and thus the proof of the theorem is valid in the **context of general locally compact Abelian groups G !**

Note that it also does not depend on the Haar measure on G.

Case Study I: Characterization of TILS: VI

There is a number of consequences from this observation, starting from the observation that the simple translation operators T_x describe of course translation invariant operators on $\mathbf{C}_0(G)$, and correspond exactly to the bounded measure $\delta_x : f \mapsto f(x)$.

Using uniform partitions of unity one can show that finite discrete measures are dense (in the w^*-topology) in $\mathbf{C}_0'(\mathbb{R}^d) = \mathbf{M}(\mathbb{R}^d)$, or equivalently any TILS can be approximated in the strong operator topology by finite linear combinations of translation operators: For $\varepsilon > 0$ and any finite subset $F \subset \mathbf{C}_0$ one has for a suitable sequence (a_n) in $\mathbb{C}$ and a finite sequence (x_n) in $\mathbb{R}^d$ such that:

$$\| T(f) - \sum_{k=1}^{l} a_n T_{x_n} f \|_\infty < \varepsilon \quad \forall f \in F.$$

Case Study I: Characterization of TILS: VII

The most important consequence of the identification of TILS and bounded measures is the fact that one can transfer the Banach algebra properties of the family of all TILS to the bounded measures. *By definition* we define composition of bounded measures (to be called **convolution!**) via the **composition of the corresponding operators**. Obviously convolution is thus an associative and bilinear operation on MbG.
Since the translation operators on a LCA group commute with each other and since they can be used to approximate generals TILS on find out that this convolution is also commutative.
Finally one shows that any $\mu \in \mathbf{C}_0'(G)$ extends to all of $\mathbf{C}_b(G)$ in a unique way, hence $\hat{\mu} = \mu(\chi_s)$ (with $\chi \in \widehat{G}$) is well defined and one shows the convolution theorem: $\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_1}$.
Again all this can be done without any integration theory!

Case Study I: Characterization of TILS: VIIa

Here let us introduce some terminology about Banach modules, which may be less familiar to the "general public":

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a *Banach module* over a Banach algebra $(\mathbf{A}, \cdot, \|\cdot\|_{\mathbf{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\mathbf{A} \times \mathbf{B}$ into $\mathbf{B}$ with

$$\|a \bullet b\|_{\mathbf{B}} \leq \|a\|_{\mathbf{A}} \|b\|_{\mathbf{B}} \quad \forall\, a \in \mathbf{A}, b \in \mathbf{B}$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \mathbf{A}, b \in \mathbf{B}.$$

Case Study I: Characterization of TILS: VIIb

Following [18] (Katznelson) we define:

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ of locally integrable functions is called a *homogeneous Banach space* on $\mathbb{R}^d$ if it satisfies

1. $\|T_x f\|_{\mathbf{B}} = \|f\|_{\mathbf{B}} \quad \forall f \in B, x \in \mathbb{R}^d$;
2. $\|T_x f - f\|_{\mathbf{B}} \to 0$ for $x \to 0, \quad \forall f \in \mathbf{B}$.

Important examples are of course the spaces $\mathbf{B} = \mathbf{L}^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ (which obviously are not sitting inside of $\mathbf{L}^1(\mathbb{R}^d)$). It can be shown in an elementary way (just using a kind of vector-valued Riemann-integrals) that any such space is a Banach convolution module over $\mathbf{M}(\mathbb{R}^d)$:

$$\|\mu * f\|_{\mathbf{B}} \leq \|\mu\|_{\mathbf{M}} \|f\|_{\mathbf{B}} \quad \forall \mu \in \mathbf{M}(\mathbb{R}^d), f \in \mathbf{B}. \tag{4}$$

Case Study I: Characterization of TILS: VIII

In order not to play down the role of the *Haar measures* on any LCA group let us remind that it is a why which enables the **identification of ordinary functions with bounded measures**! In fact, one can define for $k \in \mathbf{C}_c(\mathbb{R}^d)$ (continuous with compact support) the measure

$$\mu_k : h \mapsto \int_{\mathbb{R}^d} h(x)k(x)dx, \; h \in \mathbf{C}_0(\mathbb{R}^d),$$

With some nice tricks one then has to show that

$$\|\mu_k\|_{\mathbf{M}} = \int_{\mathbb{R}^d} |k(x)|dx,$$

i.e. the Haar measure (a linear functional) applied to $|k|$!, and from there one can define $\mathbf{L}^1(\mathbb{R}^d)$ as the closure of all those measures in $(\mathbf{M}(\mathbb{R}^d), \| \cdot \|_{\mathbf{M}})$ and and closed ideal in $\mathbf{M}(\mathbb{R}^d)$.

Summary so far

Summarizing so far we can claim that it is possible to introduce - even in the full generality of locally compact Abelian groups and without the use of the Haar measure basic facts about

- the commutative Banach **convolution algebra** $\mathbf{M}(G)$;
- the **Fourier (Stieltjes) transform** satisfying the convolution theorem;
- the extension of the group action on homogeneous Banach spaces to $\mathbf{M}(G)$, which is thus taking the role of the *group algebra* over finite Abelian groups.
- the technical tool to achieve this are fine partitions of unity!

This also shows that one should not let oneself be *distracted* from certain technical questions which arise from measure theoretical considerations!

Recalling some Basic Questions

Most often the "naturalness" of the assumption of Lebesgue integrability, i.e. the restriction to $\mathbf{L}^1(\mathbb{R}^d)$ is coming from considerations concerning the **existence** (!!??!!) of the Fourier transform or some convolution. But *WHAT DO WE MEAN*, when making one of the following claims?

- The function has a Fourier transform?
- The convolution $f * g$ of two function f, g exists?
- The convolution operator $f \mapsto f * g$ is well defined on a given Banach space of functions $\left(\mathbf{B}, \|\cdot\|_{\mathbf{B}}\right)$;
- what is the validity range for the convolution theorem

$$\widehat{\sigma_1 * \sigma_2} = \widehat{\sigma_1} * \widehat{\sigma_2}$$

Recalling some Basic Questions, II

Let us consider the first question. We have many options

- The Fourier transform is well defined pointwise as a Lebesgue integral. *This is of course just a slightly reformulation of the assumption* $f \in \mathbf{L}^1(\mathbb{R}^d)$*!* (and thus not very interesting);
- But isn't then the Fourier transform of the SINC function: $SINC(t) = sin(\pi t)/\pi t, t \in \mathbb{R} \setminus \{0\}$ not well defined, *although one could take the Fourier integral in the improper Riemannian sense. Should we prohibit this option, but how far can we go in making exceptions?*
- certainly *SINC* has an (inverse) Fourier transform (the box-function) in the $\mathbf{L}^2$-sense.
- in such situations: should we allow a combination of summability methods, but require pointwise convergence?

Recalling some Basic Questions, III

What about the well-definedness of linear operators for bilinear operations such as convolution? When deserves a function (or distribution) to be called a **convolution kernel** for a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ (such as $\mathbf{L}^2(\mathbb{R}^d)$ or $\mathbf{L}^p(\mathbb{R}^d)$)?
Again there are various options:
a) the *strict one*: For every $h \in \mathbf{B}$ the Lebesgue integral $\int_{\mathbb{R}^d} h(x-y)f(y)dy$ exists almost everywhere and defines a new element to be called $f * h \in \mathbf{B}$;
b) The integral is well-defined for test functions $h \in \mathbf{B}$, but

$$\|f * h\|_{\mathbf{B}} \leq C_f \|h\|_{\mathbf{B}}$$

for all such h, assuming that they are dense in $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$;

Recalling some Basic Questions, IIIb

In many situations there will be now difference, e.g. if both $f, g \in \mathbf{L}^2(\mathbb{R}^d)$ then the Cauchy-Schwarz inequality guarantees the existence of convolution integrals and

$$|f * g(x)| \leq \|f\|_2 \|g\|_2 \quad \forall f, g \in \mathbf{L}^2$$

But what about functions $g \notin \mathbf{L}^2(\mathbb{R}^d)$ which happen to have a bounded Fourier transform, and hence define a bounded Fourier multiplier thanks to Plancherel's Fourier transform. Chirp functions of the form $t \mapsto e^{i\alpha t^2}$ are typical examples, because their FT is another chirp.

In other words, the pointwise concept is imposing restrictions which are not suitable from a functional analytic viewpoint. This was also the cause of troubles in work on tempered $\mathbf{L}^p$-functions by K. McKennon in the 70th ([9]).

The Proper Setting from a TF-viewpoint I

Surprisingly many classical questions can be given a more elegant interpretation using function spaces which would be mostly considered as suitable and relevant for TF applications (only), such as the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ (resp. $\mathbf{S}_0(G)$) and its dual.
Originally ([3]) it got its symbol because it is the smallest (therefore the index zero!) in a class of so-called Segal algebras as studied by Hans Reiter ([20]).
It is defined as follows: A bounded, continuous and integrable function f belongs $\mathbf{S}_0(\mathbb{R}^d)$ if (and only if) it has a (Riemann-) integrable Short-time Fourier transform $V_{g_0}f$, defined as

$$\|f\|_{\mathbf{S}_0} := \|V_{g_0}f\|_{\mathbf{L}^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}$$

The assumption is so strong that the same space arises even if one just assumes a priori that f is a tempered distribution.

The Proper Setting from a TF-viewpoint II

It is not difficult to show that the space is the same if g_0 is replaced by any other non-zero function from the Schwartz space, or even from $\mathbf{S}_0(\mathbb{R}^d)$ itself. In such a way the space can be defined for general LCA groups (using e.g. smooth and compactly supported functions instead of g_0, or a some $g \in \mathbf{L}^1(G)$ with compactly supported Fourier transform, i.e. a band-limited $\mathbf{L}^1$-function).
The advantage of the choice $g = g_0$ is the Fourier invariance of g_0, which in turn implies that $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under the Fourier transform!
Depending on the viewpoint, one can say that it is one of the few Fourier invariant Banach spaces (aside from $\mathbf{L}^2(\mathbb{R}^d)$), sitting inside of $\mathbf{L}^2(\mathbb{R}^d)$, or otherwise, that the Fourier transform is not only isometric on $\mathbf{S}_0(\mathbb{R}^d)$, but also isometric in the $\mathbf{L}^2$-norm and therefore extends to a unitary transformation for $\mathbf{L}^2(\mathbb{R}^d)$, viewed as the completion of $\mathbf{S}_0(\mathbb{R}^d)$ with respect to $\| \cdot \|_2$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\mathbf{M}^1 = \mathbf{S}_0(\mathbb{R}^d)$

Lemma

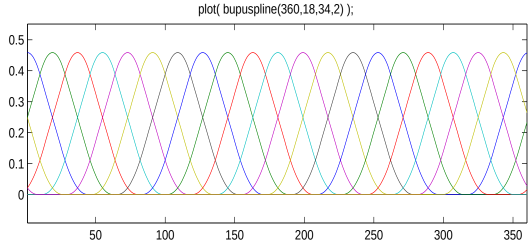
Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, then the following holds:

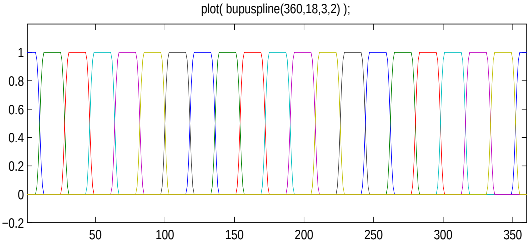
(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\big(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\big)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).
There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) = \mathbf{M}^0_{1,1}(\mathbb{R}^d)$.

$S_0(\mathbb{R}^d)$ as prototypical Modulation Space: $M_0^{1,1}(\mathbb{R}^d)$





Basic properties of $\mathbf{M}^\infty(\mathbb{R}^d) = \mathbf{S}_0'(\mathbb{R}^d)$

- $\mathbf{S}_0'(\mathbb{R}^d)$ is the *largest* (Fourier invariant) Banach space of distributions which is isometrically invariant under time-frequency shifts.
- As an Wiener amalgam space one has $\mathbf{S}_0'(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)' = \mathbf{W}(\mathcal{F}\mathbf{L}^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, resp. $\mathbf{M}^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in \mathbf{S}_0(\mathbb{R}^d)$).
- Consequently norm convergence in $\mathbf{S}_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while atomic characterizations of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT. – Think of Hifi recordings! CD player: up to 44100 Hz!

BANACH GELFAND TRIPLES: a new category I

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}'_1)$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}'_2)$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}'_1$ and $\mathbf{B}'_2$.

BANACH GELFAND TRIPLES: a new category II

It is a nice consequence of the abstract thinking of category theory that it allows to express universal facts and terms (like isomorphism, retract, embedding, epimorphism, etc.) automatically, once the objects (here triples of Banach spaces with their 4 topologies) and morphisms are defined. Categorial expression of properties through diagrams.

It also allows to think in terms if levels of complexity of categories (such as vector spaces, normed spaces, Banach Gelfand triples, etc.) and ask systematic questions (see Banach's theorem).

This will enable us in a simple way to transfer concepts (such as frames, etc.) from say linear algebra to Banach Gelfand triples using appropriate diagrams.

Banach Gelfand Triples associated with ONBs

In principle every CONB (= complete orthonormal basis) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish a BGTr, i.e. an isomorphism to $(\ell^1, \ell^2, \ell^\infty)$. by choosing as **B** the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{U})$, the space of absolutely continuous Fourier series.
It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{U}) = \mathbf{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.
For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the "ax+b"-group) one finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler's formula) to cos and sin functions in order to build a Wilson ONB for $\mathbf{L}^2(\mathbb{R}^d)$.
In this way another BGT-isomorphism between $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{5}$$

is valid for $(f,g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.
This is a typical case, where we can see, that the w^*-continuity plays a role, and where the fact that $\delta_x \in \mathbf{S}_0'(\mathbb{R}^d)$ as well as $\chi_s \in \mathbf{S}_0'(\mathbb{R}^d)$ are important.
In the STFT-domain the w^*-convergence has a particular meaning: a sequence σ_n is w^*-convergent to σ_0 if $V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda)$ uniformly over compact subsets of the TF-plane (for one or any non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$).

Test functions and finite dimensional Approximations

The main role of the space of test functions $\mathbf{S}_0(G)$ or $\mathbf{S}_0(\mathbb{R}^d)$ is that it is easily possible to relate it to (long) finite sequence, typically via sampling and periodization. This is a procedure which is Fourier invariant and compatible with the FFT.
In fact this can be used to approximately factorize the integral transform $f \mapsto \hat{f}$ through the FFT (which maps exactly discrete, periodic signals into periodic, discrete signals, thanks to Poisson's formula) which is perfectly valid for functions in $\mathbf{S}_0(G)$.
The corresponding formal approximation result has been rigorously established in a paper by Norbert Kaiblinger ([17]).

Regularizing operators: $\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$ versus $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$

The *bold message* concerning the possible use of $\mathbf{S}_0(G)$ and $\mathbf{S}_0'(G)$ would be to say that it is an *excellent replacement* for the Schwartz-Bruhat space over LCA groups for every application which does not involve PDE and differentiation (while modulation spaces are suitable for the setting of pseudo-differential operators).

Since all the $\mathbf{L}^p$-space are in between $\mathbf{S}_0$ and $\mathbf{S}_0'$, i.e. satisfy $\mathbf{S}_0 \subset \mathbf{L}^p \subset \mathbf{S}_0'$ with continuous embeddings, every linear operator from $\mathbf{L}^p$ to $\mathbf{L}^q$ $(1 \leq p, q, \leq \infty)$ can also be viewed as element of $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$.
In the opposite direction it is fair to call the elements of $\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$ the *regularizing operators*, as they map distributions into test functions.

Convolution-Product Operators as Regularizers

There are many ways to find regularizing operators, even such operators which in the sense of the strong operator topology of must function spaces, in particular of $\mathbf{S}_0(G)$ or $\mathbf{L}^2(G)$ would approximate the identity operator.
The idea is that one take a combination of smoothing and localization (via pointwise multiplication) to turn a "nasty" object into a nice test function. Therefore one uses product-convolution or convolution product operators by $\mathbf{S}_0$-kernels. It is good to know

$$(\mathbf{S}_0' * \mathbf{S}_0) \cdot \mathbf{S}_0 \subset \mathbf{S}_0$$

$$(\mathbf{S}_0' \cdot \mathbf{S}_0) * \mathbf{S}_0 \subset \mathbf{S}_0$$

Kernel Theorem for general operators in $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$

Theorem

If K is a bounded operator from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathbf{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \mathbf{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)g(x)dxdy$$

Kernel Theorem: Composition of operators

As in the analogue case of matrix multiplication one has to be careful with the composition of linear mappings and corresponding composition of kernels using the continuous analogue of matrix multiplication.
But the setting of BGTs tells us how to do it: approximate first the individual kernels by regularizing kernels and then compose those. Under suitable assumptions it will be allowed to take the limit. When one applies this trick to the FT and its inverse, with $KF(x,y) = exp(2\pi ixy)$ and its conjugate respectively we come up with the engineering formula:

$$\int_{\mathbb{R}^d} e^{2\pi i x\cdot y}\, dy = \delta_0(x).$$

Kernel Theorem II: Hilbert Schmidt Operators

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels. The complete picture can be best expressed by a unitary Gelfand triple isomorphism. First the innermost shell:

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\mathbf{L}^2(\mathbb{R}^{2d})$ on the kernels. An operator T has a kernel in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ if and only if the T maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$, boundedly, but continuously also from w^*-topology into the norm topology of $\mathbf{S}_0(\mathbb{R}^d)$.

Kernel Theorem III

Remark: Note that for such regularizing kernels in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ the usual identification. Recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

Note that $\delta_y \in \mathbf{S}_0'(\mathbb{R}^d)$ implies that $K(\delta_y) \in \mathbf{S}_0(\mathbb{R}^d)$ by the regularizing properties of K, hence pointwise evaluation is OK. The kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ into the Gelfand triple of operator spaces

$$(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')).$$

AN IMPORTANT TECHNICAL warning!!

How should we realize these various BGT-mappings?
Recall: How can we check numerically that $e^{2\pi i} = 1$??
Note: we can only do our computations (e.g. multiplication, division etc.) properly in the rational domain $\mathbb{Q}$, we get to $\mathbb{R}$ by approximation, and then to the complex numbers applying "the correct rules" (for pairs of real numbers).
In the BGT context it means: All the (partial) Fourier transforms, integrals etc. only have to be meaningful at the $\mathbf{S}_0$-level (using simply Riemanian integrals!), typically isometric in the $\mathbf{L}^2$-sense, and extend by duality considerations to $\mathbf{S}_0'$ when necessary, using w^*-continuity!
The Fourier transform is a good example (think of Fourier inversion and summability methods), similar arguments apply to the transition from the integral kernel of a linear mapping to its Kohn-Nirenberg symbol., for example.

The Spreading Representation

The kernel theorem corresponds of course to the fact that every linear mapping T from $\mathbb{C}^n$ to $\mathbb{C}^n$ can be represented by a uniquely determined matrix $\mathbf{A}$, whose columns $\mathbf{a}_k$ are the images $T(\vec{e}_k)$. When we identify $\mathbb{C}^N$ with $\ell^2(\mathbf{Z}_N)$ (as it is suitable when interpreting the FFT as a unitary mapping on $\mathbb{C}^N$) there is another way to represent every linear mapping: we have exactly N cyclic shift operators and (via the FFT) the same number of frequency shifts, so we have exactly N^2 TF-shifts on $\ell^2(\mathbf{Z}_N)$. They even form an orthonormal system with respect to the Frobenius norm, coming from the scalar product

$$\langle \mathbf{A}, \mathbf{B} \rangle_{Frob} := \sum_{k,j} a_{k,j} \bar{b}_{k,j} = trace(A * B')$$

This relationship is called the spreading representation of the linear mapping T resp. of the matrix $\mathbf{A}$. It can be thought as a kind of operator version of the Fourier transform.

The unitary spreading BGT-isomorphism

Theorem

There is a natural (unitary) Banach Gelfand triple isomorphism, called the spreading mapping, which assigns to operators T from $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ the function or distribution $\eta(T) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. It is uniquely determined by the fact that $T = \pi(\lambda) = M_\omega T_t$ corresponds to $\delta_{t,\omega}$.

Via the symplectic Fourier transform, which is of course another unitary BGT-automorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$ we arrive at the Kohn-Nirenberg calculus for pseudo-differential operators. In other words, the mapping $T \mapsto \sigma_T = \mathcal{F}_{symp}\eta(T)$ is another unitary BGT isomorphism (onto $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$, again).

Consequences of the Spreading Representation

The analogy between the ordinary Fourier transform for functions (and distributions) with the spreading representation of operators (from nice to most general within our context) has interesting consequences.
We know that Λ-periodic distributions are exactly the ones having a Fourier transform supported on the orthogonal lattice $\Lambda^\perp$, and periodizing an $\mathbf{L}^1$-function corresponds to sampling its FT.
For operators this means: an operator T commutes with all operators $\pi(\Lambda)$, for some $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, if and only if $\mathrm{supp}(\eta(T)) \subset \Lambda^\circ$, the adjoint lattice. The Gabor frame operator is the Λ-periodization of $P_g : f \mapsto \langle f, g\rangle g$, $\eta(S)$ is obtained by multiplying $\eta(P_g) = V_g(g)$ by $⊔⊔_{\Lambda^\circ} = \sum_{\lambda^\circ \in \Lambda^\circ} \delta_{\mu^\circ}$.

Consequences of the Spreading Representation 2

This observation is essentially explaining the Janssen representation of the Gabor frame operator (see [5]).
Another analogy is the understanding that there is a class of so-called underspread operators, which are well suited to model slowly varying communication channels (e.g. between the basis station and your mobile phone, while you are sitting in the - fast moving - train).
These operators have a known and very limited support of their spreading distributions (maximal time- and Doppler shift on the basis of physical considerations), which can be used to "sample" the operator (pilot tones, channel identification) and subsequently decode (invert) it (approximately).

Classical Results in Banach Space Theory

Wiener's inversion theorem:

Theorem

Assume that $h \in \mathbf{A}(\mathbb{U})$ is free of zeros, i.e. that $h(t) \neq 0$ for all $t \in \mathbb{U}$. Then the function $g(t) := 1/h(t)$ belongs to $\mathbf{A}(\mathbb{U})$ as well.

The proof of this theorem is one of the nice applications of a spectral calculus with methods from Banach algebra theory.
This result can be reinterpreted in our context as a results which states:
Assume that the pointwise multiplication operator $f \mapsto h \cdot f$ is invertible as an operator on $\left(\mathbf{L}^2(\mathbb{U}), \|\cdot\|_2\right)$, and also a BGT-morphism on $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})$ (equivalent to the assumption $h \in \mathbf{A}(\mathbb{U})$!), then it is also continuously invertible as BGT-morphism.

SOGTr-results in Banach Triple terminology

In the setting of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ a quite similar results is due to Gröchenig and coauthors:

Theorem

Assume that for some $g \in \mathbf{S}_0$ the Gabor frame operator $S: f \mapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ is invertible at the Hilbert space level, then S defines automatically an automorphism of the BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$. Equivalently, when $g \in \mathbf{S}_0$ generates a Gabor frame (g_λ), then the dual frame (of the form $(\widetilde{g}_\lambda)$) is also generated by the element $\widetilde{g} = S^{-1}(g) \in \mathbf{S}_0$.

The first version of this result has been based on matrix-valued versions of Wiener's inversion theorem, while the final result (Gröchenig and Leinert, see [12]) makes use of the concept of *symmetry* in Banach algebras and Hulanicki's Lemma.

Classical Results in Banach Space Theory 2

Theorem (Theorem by S. Banach)

Assume that a linear mapping between two Banach spaces is continuous, and invertible as a mapping between sets, then it is automatically an isomorphism of Banach spaces, i.e. the inverse mapping is automatically linear and continuous.

So we have invertibility only in a more comprehensive category, and want to conclude invertibility in the given smaller (or richer) category of objects.

Other relations to the finite-dimensional case

The paper [11]: Gabor frames without inequalities Int. Math. Res. Not. IMRN, No.23, (2007) contains another collection of statements, showing the strong analogy between a finite-dimensional setting and the setting of Banach Gelfand triples: The main result (Theorem 3.1) of that paper shows, that the Gabor frame condition (which at first sight looks just like a two-sided norm condition) is in fact equivalent to injectivity of the analysis mapping (however at the "outer level", i.e. from $\mathbf{S}_0'(\mathbb{R}^d)$ into $\ell^\infty(\mathbb{Z}^d)$), while it is also equivalent to surjectivity of the synthesis mapping, but this time from $\ell^1(\mathbb{Z}^d)$ onto $\mathbf{S}_0(\mathbb{R}^d)$.

Anti-Wick Operators and BGTRs

Another example where the BGTR $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ are useful is in the context of STFT-multipliers resp. Anti-Wick operators.
These are pointwise multipliers on the STFT-side. Because of the smoothness of a STFT it makes sense to multiply the STFT even by a distribution. In fact, mostly the global behavior of a distribution plays a role (all Sobolev-spaces are globally ℓ^2);

Theorem

Let g, γ be a pair of functions in $\mathbf{S}_0(\mathbb{R}^d)$, e.g. $g = \gamma$. then the mapping from upper symbols (i.e. multipliers over phase space) σ to the Anti-Wick operator $T := V_\gamma^* M_\sigma V_g$, is a Banach Gelfand triple mapping from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ to $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$.

Anti-Wick Operators and Gabor Multipliers

Under certain explicitly formulated conditions, which of course are satisfied for the choice $g = g_0$ (the Gauss function) we can claim even more:

Theorem

For any fixed lattice Λ within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ the mapping from upper symbols $(c_\lambda) \in (\ell^1, \ell^2, \ell^\infty)$ to the Gabor multipliers is an isomorphism between $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ and its image within $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$.

In fact, at the Kohn-Nirenberg level this is comparable to the claim that spline functions are in $\mathbf{L}^p(\mathbb{R}^d)$ if and only if their coefficients are in the corresponding ℓ^p-space, for $1 \leq p \leq \infty$.
Similar to Shannon's sampling one can show
recovery of channels (Gabor multipliers) from lower symbols
(cf. PhD of Elmar Pauwels, Vienna 2011).

Frames in Hilbert Spaces: Classical Approach

Definition

A family $(f_i)_{i \in I}$ in a Hilbert space $\mathcal{H}$ is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \quad (6)$$

It is well known that condition (6) is satisfied if and only if the so-called frame operator is invertible, which is given by

Definition

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for} \quad f \in \mathcal{H},$$

Frames in Hilbert Spaces: Classical Approach II

The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\widetilde{f_i})_{i \in I}$, defined by $\widetilde{f_i} := S^{-1}(f_i)$ has the property that one has for $f \in \mathcal{H}$:

Definition

$$f = \sum_{i \in I} \langle f, \widetilde{f_i} \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \widetilde{f_i} \tag{7}$$

Moreover, applying S^{-1} to this equation one finds that the family $(\widetilde{f_i})_{i \in I}$ is in fact a frame, whose frame operator is just S^{-1}, and consequently the "second dual frame" is just the original one.

Frames in Hilbert Spaces: Approach III

Since S is *positive definite* in this case we can also get to a more symmetric expression by defining $h_i = S^{-1/2} f_i$. In this case one has

$$f = \sum_{i \in I} \langle f, h_i \rangle h_i \quad \text{for all } f \in \mathcal{H}. \tag{8}$$

The family $(h_i)_{i \in I}$ defined in this way is called the *canonical tight frame* associated to the given family $(g_i)_{i \in I}$. It is in some sense the closest tight frame to the given family $(f_i)_{i \in I}$.

LINEAR ALGEBRA: Gilbert Strang's FOUR SPACES

Let us recall the *standard linear algebra situation*. Given some $m \times n$ -matrix $\mathbf{A}$ we view it as a collection of *column* resp. as a collection of *row vectors*. We have:

$$\textbf{row-rank}(\mathbf{A}) = \textbf{column-rank}(\mathbf{A})$$

Each *homogeneous linear system of equations* can be expressed in the form of scalar products[1] we find that

$$\textbf{Null}(\mathbf{A}) = \quad \textbf{Rowspace}(\mathbf{A})^{\perp}$$

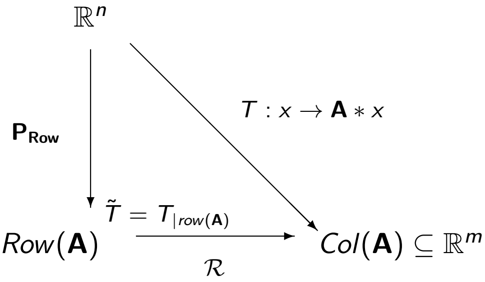
and of course (by reasons of symmetry) for $\mathbf{A}' := conj(A^t)$:

$$\textbf{Null}(\mathbf{A}') = \quad \textbf{Colspace}(\mathbf{A})^{\perp}$$

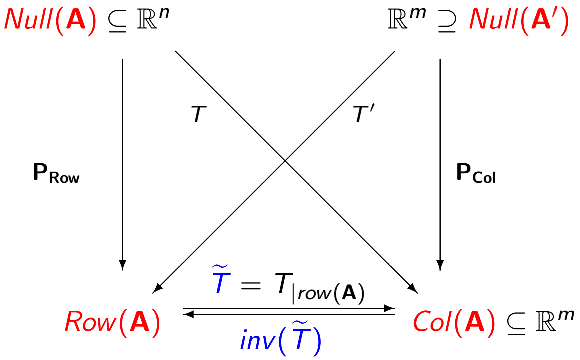
[1] Think of $3x + 4y + 5z = 0$ is just another way to say that the vector $\mathbf{x} = [x, y, z]$ satisfies $\langle \mathbf{x}, [3, 4, 5] \rangle = 0$.

Geometric interpretation of matrix multiplication

Since *clearly* the restriction of the linear mapping $x \mapsto \mathbf{A} * x$



Geometric interpretation of matrix multiplication



$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Four spaces and the SVD

The **SVD** (the so-called Singular Value Decomposition) of a matrix, described in the MATLAB helpful as a way to write A as

$$A = U * S * V'$$

, where the columns of U form an ON-Basis in $\mathbb{R}^m$ and the columns of V form an ON-basis for $\mathbb{R}^n$, and S is a (rectangular) diagonal matrix containing the non-negative *singular values* (σ_k) of A. We have $\sigma_1 \geq \sigma_2 \ldots \sigma_r > 0$, for $r = rank(A)$, while $\sigma_s = 0$ for $s > r$. In standard description we have for A and $pinv(A) = A^+$:

$$A * x = \sum_{k=1}^{r} \sigma_k \langle x, v_k \rangle u_k, \quad A^+ * y = \sum_{k=1}^{r} \frac{1}{\sigma_k} \langle y, u_k \rangle v_k.$$

Generally known facts in this situation

The Four Spaces are well known from LINEAR ALGEBRA, e.g. in the **dimension formulas**:

ROW-Rank of **A** *equals COLUMN-Rank of* **A***.*

The defect (i.e. the dimension of the Null-space of **A***) plus the dimension of the range space of* **A** *(i.e. the column space of* **A***) equals the dimension of the domain space* $\mathbb{R}^n$*.* Or in terms of linear, homogeneous equations: The dimension of set of all solution to the homogeneous linear equations equals the number of variables minus the dimension of the column space of **A**.
The SVD also shows, that the *isomorphism between the Row-space and the Column-space* can be described by a diagonal matrix, if suitable orthonormal basis for these spaces are used.

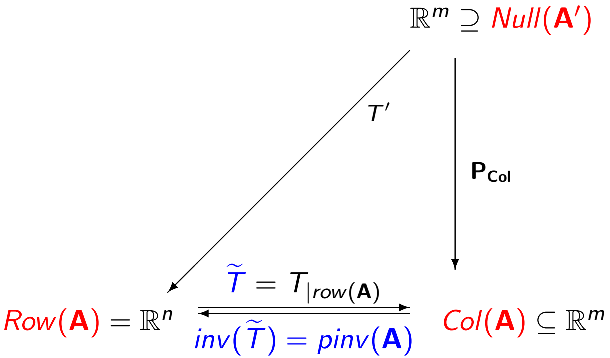
Consequences of the SVD

We can describe the quality of the isomorphism $\widetilde{T}$ by looking at its condition number, which is σ_1/σ_r, the so-called **Kato-condition number** of T.
It is not surprising that for **normal matrices** with $A' * A = A * A'$ one can even have diagonalization, i.e. one can choose $U = V$, because

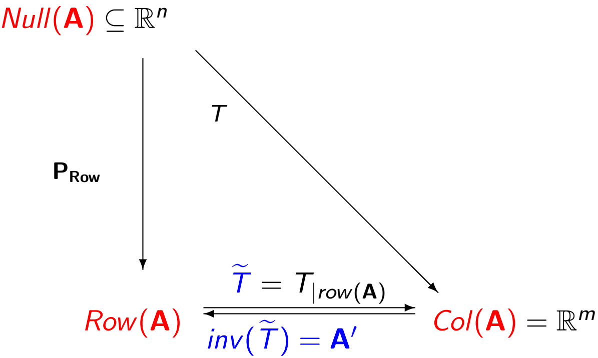
$$Null(A) =_{always} \; Null(A' * A) = Null(A * A') = Null(A').$$

The most interesting cases appear if a matrix has maximal rank, i.e. if $rank(\mathbf{A}) = min(m, n)$, or equivalently if one of the two Null-spaces is trivial. Then we have either linear independent columns of $\mathbf{A}$ (injectivity of $T >>$ RIESZ BASIS for subspaces) or the columns of $\mathbf{A}$ span all of $\mathbb{R}^m$ (i.e. $Null(A') = \{0\}$): FRAME SETTING!

Linear Independence: > Riesz Basic Sequence

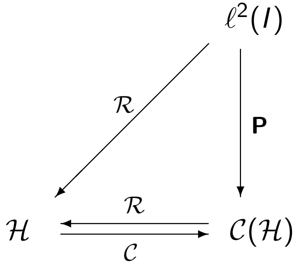


Generating Family > Frame



The frame diagram for Hilbert spaces:

If we consider **A** as a collection of column vectors, then the role of **A**$'$ is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (15).

Riesz basic sequences in Hilbert spaces:

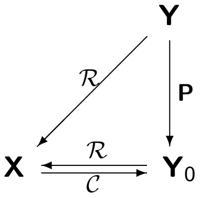
The diagram for a Riesz basis (for a subspace), nowadays called a Riesz basic sequence looks quite the same.
In fact, from an abstract sequence there is no! difference, just like there is no difference (from an abstract viewpoint) between a matrix $\mathbf{A}$ and the transpose matrix $\mathbf{A}'$.
However, it makes a lot of sense to think that in one case the collection of vectors (making up a Riesz BS) spans the (closed) subspace of $\mathcal{H}$ by just taking all the infinite linear combinations (series) with ℓ^2-coefficients.
In this way the synthesis mapping $\mathbf{c} \mapsto \sum_i c_i g_i$ from $\ell^2(I)$ into the closed linear span is *surjective*, while in the frame case the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ from $\mathcal{H}$ into $\ell^2(I)$ is injective (with bounded inverse).

Frames versus atomic decompositions

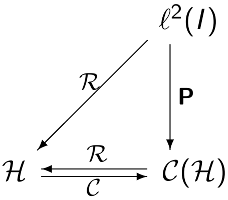
Although the *definition of frames in Hilbert spaces* emphasizes the aspect, that the frame elements define (via the Riesz representation theorem) an injective analysis mapping, the usefulness of frame theory rather comes from the fact that frames allow for atomic decompositions of arbitrary elements $f \in \mathcal{H}$.
One could even replace the lower frame bound inequality in the definition of frames by assuming that one has a Bessel sequence (i.e. that the upper frame bound is valid) with the property that the synthesis mapping from $\ell^2(I)$ into $\mathcal{H}$, given by $\mathbf{c} \mapsto \sum_i c_i g_i$ is *surjective* onto *all of* $\mathcal{H}$.
Analogously one can find Riesz bases interesting (just like linear independent sets) because they allow to uniquely determine the coefficients of f in their closed linear span on that closed subspace of $\mathcal{H}$.

Frames and Riesz Bases: the Diagram

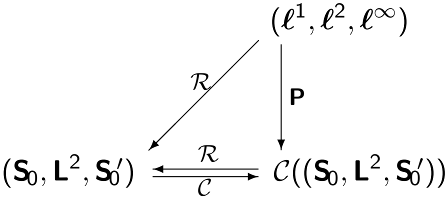
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ. Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping $\mathcal{C}$ from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse $\mathcal{R}$ (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.

In practice it means, that the dual Gabor atom $\tilde{g}$ is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.

Gabor expansions and Diagrams

Theorem

Assume that a Gabor family generated from a window/atom in $\mathbf{S}_0(\mathbb{R}^d)$ *and a lattice* $\Lambda = \mathbf{A}(\mathbb{Z}^{2d})$ *, i.e.*

$$\{\pi(\mathbf{A}(k))g \mid k \in \ Ztd\}$$

is a frame for $\mathbf{L}^2(\mathbb{R}^d)$*.*
Then the mapping

$$f \mapsto V_g f(\lambda), \lambda \in \Lambda\}$$

is an injective mapping from the BGTR $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ *into the standard BGTR* $(\ell^1, \ell^2, \ell^\infty)$*.*
Equivalently, the synthesis mapping from $(\ell^1, \ell^2, \ell^\infty)$ *to* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ *is surjective in this case.*
Altogether, distributions belong to $\mathbf{S}_0, \mathbf{L}^2$ *or* $\mathbf{S}_0'$ *respectively if and only they have an appropriate representation with coefficients in the corresponding sequence space within* $(\ell^1, \ell^2, \ell^\infty)$*.*

Generalized Stochastic Processes I

Definition

1. Let $\mathcal{H}$ be an arbitrary Hilbert space. A bounded linear mapping $\rho : S_0(G) \to \mathcal{H}$ is called a **generalized stochastic process** (GSP).
2. A GSP ρ is called **(wide sense time-) stationary**, if
$$(\rho(f)|\rho(g)) = (\rho(T_x f)|\rho(T_x g)) \quad \forall x \in G \text{ and } \forall f, g \in S_0(G).$$
3. A GSP ρ is called **(wide sense) frequency stationary**, if
$$(\rho(f)|\rho(g)) = (\rho(M_t f)|\rho(M_t g)) \quad \forall t \in \hat{G} \text{ and } \forall f, g \in S_0(G).$$
4. A time- and frequency-stationary GSP is called **white noise**.

Generalized Stochastic Processes II

Definition

1. A GSP ρ is called **bounded**, if $\exists\, c > 0$ such that
$$\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_\infty \quad \forall\, f \in S_0(G).$$
2. A GSP ρ is called **variation-bounded (V-bounded)**, if $\exists\, c > 0$ such that
$$\|\rho(f)\|_{\mathcal{H}} \leq c\|\hat{f}\|_\infty \quad \forall\, f \in S_0(G).$$
3. A GSP ρ is called **orthogonally scattered** if
$$\mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset \Longrightarrow \rho(f) \perp \rho(g) \quad \text{for } f, g \in S_0(G).$$

Generalized Stochastic Processes III

Due to the **tensor product property** of S_0 :

$$S_0(G)\hat{\otimes}S_0(G) = S_0(G \times G)$$

(cf. [3] Theorem 7 D) it is justified to hope that the following definition determines an element of $S_0(G \times G)$.

Definition 6: Let ρ be a GSP. The **autocovariance** (or auto-correlation) distribution σ_ρ is defined as:
$\langle \sigma_\rho, f \otimes g \rangle := (\rho(f)|\rho(\bar{g})) \ \forall f, g \in S_0(G)$.

Generalized Stochastic Processes IV

Theorem 1: For a GSP ρ the following properties are equivalent:
a) ρ stationary $\iff \sigma_\rho$ diagonally invariant, i.e.

$$L_{(x,x)}\sigma_\rho = \sigma_\rho \;\forall\, x \in G;$$

b) ρ bounded $\iff \sigma_\rho$
extends in a unique way to a bimeasure on $G \times G$;
c) ρ orthogonally scattered
$\iff \sigma_\rho$ is supported by the diagonal, i.e.
$\mathrm{supp}(\sigma_\rho) \subseteq \Delta_G := \{(x,x) \mid x \in G\}$;
$\iff \exists$ positive and translation bounded measure τ_ρ with:

$$\langle \sigma_\rho, f \otimes g\rangle = \langle \tau_\rho, fg\rangle \;\forall\, f,g \in S_0(G).$$

Further applications areas

There is a large variety of topics, where the same setting appears to be most appropriate, mostly because it is technically easier than the use of **Schwartz-Bruhat** functions, and on the other hand more general ($\mathcal{S}(G) \subset \mathbf{S}_0(G)$), in particular work on non-commutative geometry by Connes and Rieffel (via the work of Franz Luef);

THANK YOU!

Thank you for your attention!

Most of the referred papers of NuHAG can be downloaded from
http://www.univie.ac.at/nuhag-php/bibtex/

Furthermore there are various talks given in the last few years on related topics (e.g. Gelfand triples), that can be found by searching by title or by name in
http://www.univie.ac.at/nuhag-php/nuhag_talks/

Selection of bibliographic items, see www.nuhag.eu

J. J. Benedetto and W. Czaja.
Integration and Modern Analysis.
Birkhäuser, 2009.

T. Chonavel.
Statistical Signal Processing. Modelling and Estimation. Transl. from the French by Janet Ormrod. Incl. 1 CD-ROM.
Springer, London, 2002.

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

G. B. Folland.
Harmonic Analysis in Phase Space.
Princeton University Press, Princeton, N.J., 1989.

G. B. Folland.
A Course in Abstract Harmonic Analysis.
CRC Press, Boca Raton, FL, 1995.

A hierarchy of conditions 1

While the following conditions are equivalent in the case of a finite dimensional vector space (we discuss the frame-like situation) one has to put more assumptions in the case of separable Hilbert spaces and even more in the case of Banach spaces.
Note that one has in the case of an infinite-dimensional Hilbert space: A set of vectors $(f_i)_{i \in I}$ is total in $\mathcal{H}$ if and only if the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ is injective. In contrast to the frame condition nothing is said about a series expansion, and in fact for better approximation of $f \in \mathcal{H}$ a completely different finite linear combination of $g_i's$ can be used, without any control on the ℓ^2-norm of the corresponding coefficients.
THEREFORE one has to make the assumption that the range of the coefficient mapping has to be a *closed subspace* of $\ell^2(I)$ in the discussion of *frames in Hilbert spaces*.

A hierarchy of conditions 2

In the case of Banach spaces one even has to go one step further. Taking the norm equivalence between some Banach space norm and a corresponding sequence space norm in a suitable Banach space of sequences over the index set I (replacing $\ell^2(I)$ for the Hilbert space) is not enough!
In fact, making such a definition would come back to the assumption that the coefficient mapping $\mathcal{C} : f \mapsto (\langle f, g_i \rangle)$ allows to identify with some closed subspace of that Banach space of sequences. Although in principle this might be a useful concept it would not cover typical operations, such as taking Gabor coefficients and applying localization or thresholding, as the modified sequence is then typically not in the range of the sampled STFT, but resynthesis should work!

A hierarchy of conditions 3

What one really needs in order to have the diagram is the identification of the Banach space under consideration (modulation space, or Besov-Triebel-Lozirkin space in the case of wavelet frames) with a close and complemented subspace of a larger space of sequences (taking the abstract position of $\ell^2(I)$.
To assume the existence of a left inverse to the coefficient mapping allows to establish this fact in a natural way. Assume that $\mathcal{R}$ is the left inverse to $\mathcal{C}$. Then $\mathcal{C} \circ \mathcal{R}$ is providing the projection operator (the orthogonal projection in the case of $\ell^2(I)$, if the canonical dual frame is used for synthesis) onto the range of $\mathcal{C}$. The converse is an easy exercise: starting from a projection followed by the inverse on the range one obtains a right inverse operator $\mathcal{R}$.

A hierarchy of conditions 4

The above situation (assuming the validity of a diagram and the existence of the reconstruction mapping) is part of the definition of Banach frames as given by K. Gröchenig in [10].
Having the classical situation in mind, and the *spirit of frames in the Hilbert spaces case* one should however add two more conditions:
In order to avoid trivial examples of Banach frames one should assume that the associated Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ of sequences should be assumed to be solid, i.e. satisfy that $|a_i| \leq |b_i|$ for all $i \in I$ and $b \in B$ implies $a \in \mathbf{B}$ and $\|a\|_{\mathbf{B}} \leq \|b\|_{\mathbf{B}}$.
Then one could identify the reconstruction mapping $\mathcal{R}$ with the collection of images of unit vectors $h_i := \mathcal{R}(\vec{e}_i)$, where $\vec{e}_i$ is the unit vector at $i \in I$. Moreover, unconditional convergence of a series of the form $\sum_i c_i h_i$ would be automatic.

A hierarchy of conditions 6

Instead of going into this detail (including potentially the suggestion to talk about unconditional Banach frames) I would like to emphasize another aspect of the theory of Banach frames. According to *my personal opinion* it is not very interesting to discuss individual Banach frames, or the existence of *some Banach frames* with respect to *some abstract Banach space of sequences*, even if the above additional criteria apply.

The *interesting cases* concern situations, where the coefficient and synthesis mapping concern a whole family of related Banach spaces, the setting of Banach Gelfand triples being the minimal (and most natural) instance of such a situation.

A comparison: As the family, consisting of father, mother and the child is the foundation of our social system, Banach Gelfand Triples are the prototype of *families*, sometimes *scales of Banach spaces*, the "child" being of course our beloved Hilbert space.

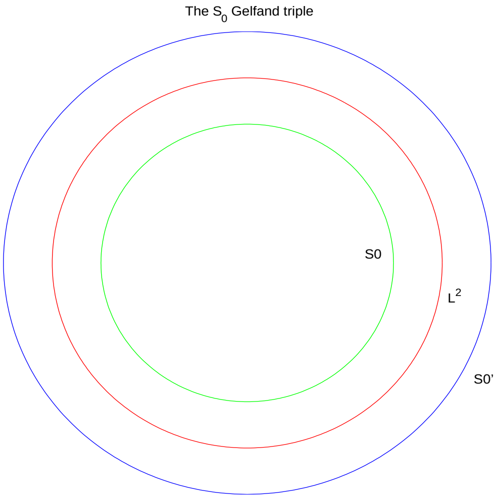
Banach Gelfand Triples and Rigged Hilbert space

The next term to be introduced are Banach Gelfand Triples.
There exists already and established terminology concerning triples of spaces, such as the Schwartz triple consisting of the spaces $(\mathcal{S}, \mathbf{L}^2, \mathcal{S}')(\mathbb{R}^d)$, or triples of weighted Hilbert spaces, such as $(\mathbf{L}^2_w, \mathbf{L}^2, \mathbf{L}^2_{1/w})$, where $w(t) = (1+|t|^2)^{s/2}$ for some $s > 0$, which is - via the Fourier transform isomorphic to another ("Hilbertian") Gelfand Triple of the form $(\mathcal{H}_s, \mathbf{L}^2, \mathcal{H}_s{}')$, with a Sobolev space and its dual space being used e.g. in order to describe the behaviour of elliptic partial differential operators.
The point to be made is that suitable Banach spaces, in fact imitating the prototypical Banach Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ allows to obtain a surprisingly large number of results resembling the finite dimensional situation.

A Classical Example related to Fourier Series

There is a well known and classical example related to the more general setting I want to describe, which - as so many things - go back to N. Wiener. He introduced (within $\mathbf{L}^2(\mathbb{U})$) the space $\left(\mathbf{A}(\mathbb{U}), \|\cdot\|_{\mathbf{A}}\right)$ of absolutely convergent Fourier series. Of course this space sits inside of $\left(\mathbf{L}^2(\mathbb{U}), \|\cdot\|_2\right)$ as a dense subspace, with the norm $\|f\|_{\mathbf{A}} := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)|$.
Later on the discussion about Fourier series and generalized functions led (as I believe naturally) to the concept of pseudo-measures, which are either the elements of the dual of $\left(\mathbf{A}(\mathbb{U}), \|\cdot\|_{\mathbf{A}}\right)$, or the (generalized) inverse Fourier transforms of bounded sequences, i.e. $\mathcal{F}^{-1}(\ell^\infty(\mathbb{Z}))$.
In other words, this extended view on the Fourier analysis operator $\mathcal{C} : f \mapsto (\widehat{f}(n)_{n\in\mathbb{Z}})$ on the BGT $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})$ into $(\ell^1, \ell^2, \ell^\infty)$ is the prototype of what we will call a BGT-isomorphism.

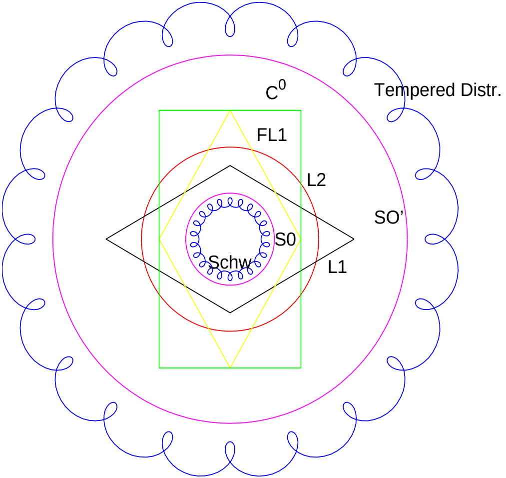
The visualization of a Banach Gelfand Triple



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as $5/7$ is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. But there are easier alternatives.

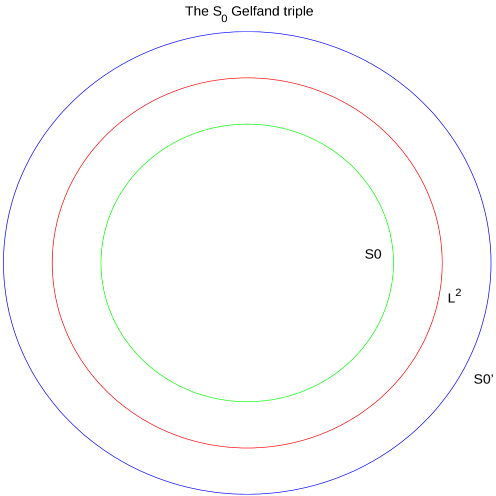
A schematic description of the situation



The Banach space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ or $(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1})$), which will serve our purpose. Its dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to Banach Gelfand Triples, mostly related to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

repeated: SOGELFTR



Gabor frame matrix representations

Much in the same way as basis in $\mathbb{C}^n$ are used in order to describe linear mappings as matrices we can also use Gabor frame expansions in order to describe (and analyze resp. better understand) certain linear operators T (slowly variant channels, operators in Sjoestrand's class, connected with another family of modulation spaces) by their frame matrix expansion. Working (for convenience) with a Gabor frame with atom $g \in \mathbf{S}_0(\mathbb{R}^d)$ (e.g. Gaussian atom, with $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$), and form for $\lambda, \mu \in \Lambda$ the infinite matrix

$$a_{\lambda,\mu} := [T(\pi(\lambda)g)](\pi(\mu)g).$$

This makes sense even if T maps only $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{S}_0'(\mathbb{R}^d)$!

Gabor frame matrix representations II

For any good Gabor family (tight or not) the mapping $T \mapsto \mathbf{A} = (a_{\lambda,\mu})$ is it self defining a frame representation, hence a retract diagram, from the operator BGT $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ into the $(\ell^1, \ell^2, \ell^\infty)$ over $\mathbb{Z}^{2d}$!
In other words, we can recognize whether an operator is regularizing, i.e. maps $\mathbf{S}_0'(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$ (with w^*-continuity) if and only if the matrix has coefficients in $\ell^1(\mathbb{Z}^{2d})$.
Note however, that invertibility of T is NOT equivalent to invertibility of $\mathbf{A}$! (one has to take the pseudo-inverse).

Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathbf{S}_0(\mathbb{R}^d)$ (typically one will take g and $\tilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity (> Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):[4]

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ), *with* $g \in \mathbf{S}_0(\mathbb{R}^d)$ *defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the* $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$*-sense on both parameters.*

Invertibility, Surjectivity and Injectivity

In another, very recent paper, Charly Groechenig has discovered that there is another analogy to the finite dimensional case: There one has: A square matrix is invertible if and only if it is surjective or injective (the other property then follows automatically).
We have a similar situation here (systematically describe in Charly's paper):
`K.Groechenig: Gabor frames without inequalities, Int. Math. Res. Not. IMRN, No.23, (2007).`

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\mathbf{B} = \mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$.

Theorem

There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. *This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. *Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators* $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

The w^*- topology: a natural alternative

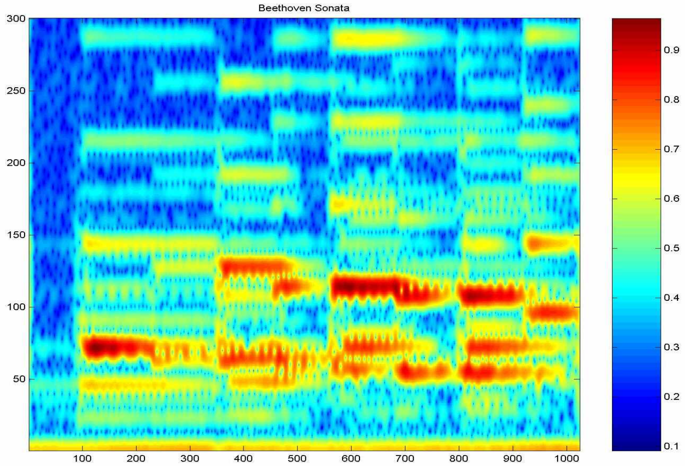
It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
Therefore it is interesting to check what the w^*-convergence looks like:

Lemma

For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ a sequence σ_n is w^*-convergent to σ_0 if and only the spectrograms $V_g(\sigma_n)$ converge uniformly over compact sets to the spectrogram $V_g(\sigma_0)$.

The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

A Typical Musical STFT



The w^*- topology: dense subfamilies

From the practical point of view this means, that one has to look at the spectrograms of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution $V_g(\sigma_0)$ over compact sets.
The approximation of elements from $\mathbf{S}_0'(\mathbb{R}^d)$ takes place by a bounded sequence.
Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^*-dense subsets of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.
Let us look at some concrete examples: Test-functions, finite discrete measures $\mu = \sum_i c_i \delta_{t_i}$, trigonometric polynomials $q(t) = \sum_i a_i e^{2\pi i \omega_i t}$, or discrete AND periodic measures (this class is invariant under the generalized Fourier transform and can be realized computationally using the FFT).

The w^*- topology: approximation strategies

- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \to \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an $\mathbf{L}^1$-Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite disrete sequence using quasi-interpolation (N. Kaiblinger): $Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x)$.