Irregular Sampling and Function Spaces

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ABSTRACT

Sampling as a long tradition at NuHAG
(the Numerical Harmonic Analysis Group, Vienna),
from the beginning. It was always viewed as a field where different subjects have to come together:

- mathematical modeling;
- numerical algorithms
- theoretical foundations
- harmonic analysis
- function spaces
- good links to the real world;

Traditional function spaces

The theory of function spaces has become an independent branch of analysis (from where it arose) essentially in the second of of the last century, with the start of *approximation theory*, the use of e.g. *Sobolev spaces* in the theory of partial differential equations (elliptic equations describe an isomorphism between a Sobolev spaces of positive order and its dual, which is of negative order). Of course the development of classical Lebesgue theory, with the $\mathbf{L}^p$-spaces, $1 \leq p \leq \infty$, was significant for the early development of *functional analysis* in the first part of the last century, while *interpolation theory* was helping to understand that function spaces are coming in parameterized families.

Function Spaces for Fourier Analysis I

It is the current view-point in analysis that one needs to first develop the *Lebesgue integral* in order to have the spaces $(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1), (\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2), (\mathbf{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$, where for $p = 1, 2, \infty$ the size (norm) of a function is given by

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)| dx$$

$$\|f\|_2 = \sqrt{\int_{\mathbb{R}^d} |f(x)|^2 dx}$$

and

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|.$$

Function Spaces for Fourier Analysis II

Why are they considered useful?

- because the Fourier transform is an integral transform;
- because at first sight the definition of convolution requires again a finite integral (at least almost everywhere)
- the $\mathbf{L}^2$ allows to express "preservation of energy" under the Fourier transform by the equation

$$\|f\|_2 = \|\hat{f}\|_2.$$

And it is natural to *interpolate* (in the sense of complex interpolation of Banach spaces) in order to get the $\mathbf{L}^p$-spaces, for general $p \in [1, \infty]$.

The Hilbert Space $\mathbf{L}^2(\mathbb{R}^d)$

Although one often reads (in books and articles) that it is natural to consider the *Hilbert space* $\mathbf{L}^2(\mathbb{R}^d)$ because *natural signals have finite energy* the truth seems to be that the scalar product turns the linear space of signals into an *infinite dimensional Euclidean space*, with all the geometric features of $\mathbb{R}^3$, such as angles, orthonormal systems, etc..

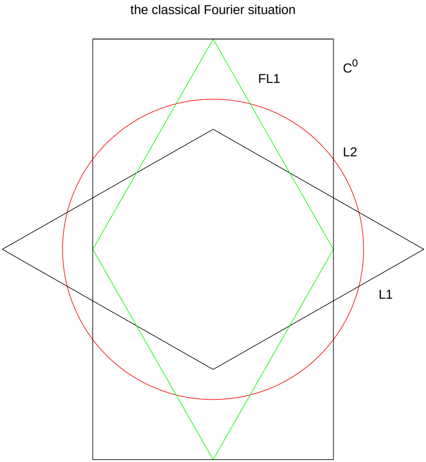
Still, the mathematical formulation using $\mathbf{L}^p$-spaces help neither to understand the Fourier transform of a "pure frequency" nor is the mathematically precise formulation in accepting the fact that for practical purpose there is *no problem* to work with band-limited signals of finite during (claimed to be "impossible" in mathematics!).

Traditional function spaces II

Aside from $\mathbf{L}^p$-spaces the literature contains various other function spaces which have their importance in analysis and sometimes to applications, just to mention

1. **Sobolev spaces**, where one takes function having certain number of derivatives in $\mathbf{L}^p$; (properly defined);
2. to make the scale continuous the **Bessel potential spaces** (defined using Fourier methods) are quite useful;
3. **Besov spaces** (and Triebel-Lizorkin spaces) allow to express fractional order smoothness by adding typically a Lipschitz condition on the highest existing derivative;

Traditional function spaces III

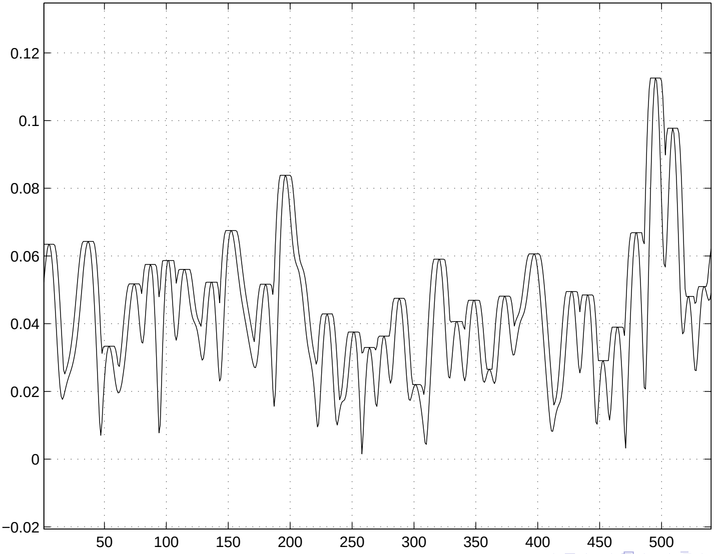


Traditional function spaces IV

Since ordinary $\mathbf{L}^p$-space on $\mathbb{R}^d$ have the problem that the are not order by inclusion the idea came up to describe function spaces by taking locally one possible norm (e.g. some $\mathbf{L}^p$-norm) and globally (e.g. over the integer sequence) some other ℓ^q space.
The following continuous embedding relations follow easily from the discrete characterization of Wiener amalgam spaces.

1. $W(B, \ell^p) \hookrightarrow W(B, \ell^r)$ if and only if $p \leq r$.
2. If $B^1_{loc} \hookrightarrow B^2_{loc}$ then $W(B^1, \ell^p) \hookrightarrow W(B^2, \ell^p)$ for $1 \leq p \leq \infty$.
3. $W(B, \ell^1) \hookrightarrow B \hookrightarrow W(B, \ell^\infty)$.
4. $L^p = W(L^p, \ell^p) \hookrightarrow W(L^1, \ell^p)$.
5. $W(C_0, \ell^p) \hookrightarrow L^p \cap C_o$.

Illustration of $W(C_0, \ell^p)$ spaces:



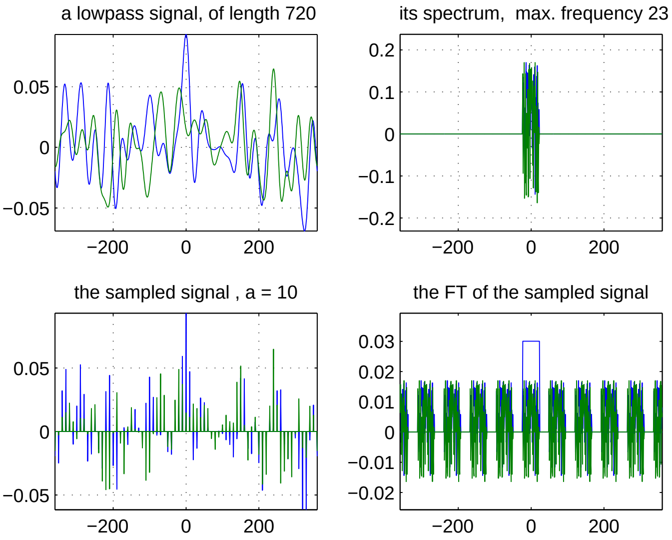
Function spaces for Sampling

There are of course *many* different function spaces that can be used, but the typical first theorem in sampling theory is Shannon's sampling theorem: Assume that $\Omega \subset \mathbb{R}^d$ is a bounded subset with the property Ω is disjoint from all its (non-trivial) $\Lambda^\perp$-translates, i.e. $\lambda^\perp + \Omega \cap \Omega = \emptyset$ for all $\lambda^\perp \neq 0, \lambda^\perp \in \Lambda^\perp$. Then every Ω-band-limited function $f \in \mathbf{L}^2(\mathbb{R}^d)$ can be written as:

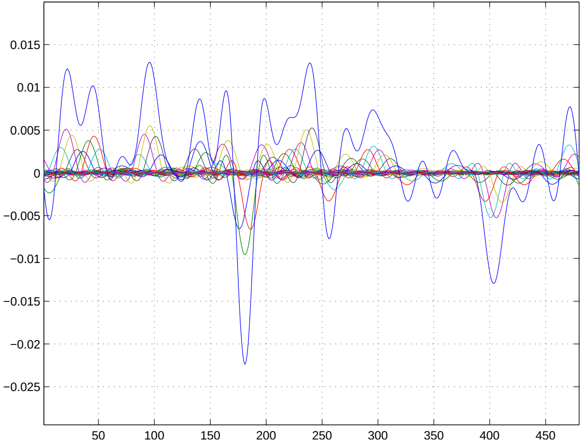
$$f(t) = C_\Lambda \sum_{\lambda \in \Lambda} f(\lambda) T_\lambda SINC_\Omega(t) = C_\Lambda \sum_{\lambda \in \Lambda} f(\lambda) SINC_\Omega(t - \lambda), \quad (1)$$

where $SINC_\Omega = IFFT(\mathbf{1}_\Omega)$. For $\Omega = [-1/2, 1/2]$ this is the classical SINC function, known to have bad localization.
!! Convergence is in the $\mathbf{L}^p$-sense $(1 < p < \infty)$ and uniformly.

Shannon's sampling: illustration I



Shannon's sampling: illustration II



But the speed/form of convergence depends (potentially) sensitively on on the norm chosen.

Wiener Amalgam spaces I

Mostly the function space $W(\mathbf{C}_0, \ell^p)$ are useful in the context of sampling because they guarantee that not only the (smooth) function is in $\mathbf{L}^p$, but also the samples (taken e.g. over some lattice) are in the corresponding ℓ^p-space.
Since this space, endowed with its natural norm, is continuously embedded into both the bounded and continuous functions (with the sup-norm $\|f\|_\infty$) and into $\mathbf{L}^p$ convergence of the Shannon series in the $W(\mathbf{C}_0, \ell^p)$ implies both types of convergence.
For the case $p = 2$ one can in fact show that

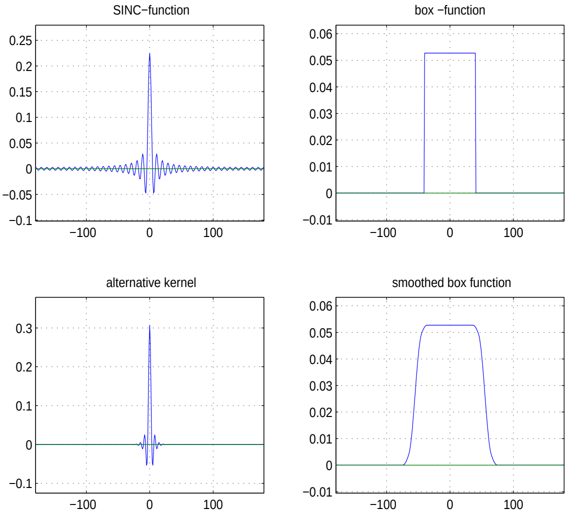
1. any band-limited function;
2. any function in a Sobolev space of order $s > d/2$

belongs to the space $W(\mathbf{C}_0, \ell^2)$. HENCE these space are reproducing kernel Hilbert spaces.

Reproducing kernel Hilbert spaces

However, having RKHs (Reproducing kernel Hilbert spaces) one has a whole reservoir of Hilbert space methods and at the same time concrete starting points for possible reconstruction:
By **interpreting the point-values** of the given set of samples as *scalar products* with the corresponding reproducing kernels one can of course (up to some stability questions) recover the sampled function in our model space by linear combinations of the involved kernels.
Note that one has shifted SINC-functions in the case of the band-limited functions in $\mathbf{L}^2(\mathbb{R}^d)$. But knowing their bad localization it is of course interesting to explore the freedom gained by slight oversampling, which in turn allows for improved locality of reconstruction.

Reproducing kernel Hilbert spaces II



Reproducing kernel Hilbert spaces III

For Sobolev spaces resp. $\mathbf{L}^2$-Bessel potential spaces, which can be characterized as weighted $\mathbf{L}^2$-spaces on the Fourier transform side a similar, and in a way better (because better localized) reproducing kernel can be used.
As it is shown in a joint paper the ([fewe02]) the best = minimal norm reconstruction of a function in the sense of such a Sobolev space (of order $s > d/2$) depends in a very good and continuous way on the parameter s and the lattice constant (for regular sampling).

Reproducing kernel Hilbert spaces IV

A new type of function space introduced in the PhD thesis of Roza Aceska (2009, [acfe11]) appears to be suitable to handle the variable case, i.e. the case where smoothness (in the sense of local maximal frequency) is varying as a function on the location. Since strictly band-limited are analytic they do not have any "local behavior", resp. must coincide globally if they would coincide locally on some open set. Nevertheless a Gabor approach was developed using time-frequency methods to describe such spaces in terms of growths conditions on the Gabor coefficients (of some/any good Gabor system).

Reproducing kernel Hilbert spaces V

Essentially the weight depends on a band-width function which does not change rapidly, and which "punishes" the "out of band Gabor coefficients". Clearly the order of growth has be satisfy the Sobolev embedding condition $s > d/2$ and should be rather large. For this setting the so-called coorbit theory (developed by Fei/Groch) can be used to show the independence of this characterization from the individual (good) Gabor system. Moreover, as as been shown recently, these variable bandwidth spaces have good reproducing kernels, and we expect/prepare sampling results, involving the notion of a *local Nyquist rate* [acfe13]).

Reproducing kernel Hilbert spaces VI

It is known from Shannon's sampling theorem that for the members of suitable subspaces, e.g. for the band-limited functions with

$$\text{spec}(f) \subseteq [-1/2 + \delta, 1/2 - \delta]$$

the extra amount of freedom (oversampling rate) allows to replace in the Shannon sampling theorem by a more localized function. Especially for the variable situation we have in mind for *field reconstruction* the **locality** of reconstruction is one of the important points, and SINC reconstructions are definitely inadequate.

Reconstruction Strategies

Depending on the signal space model one has to apply reconstruction methods, which are most of the time iterative in nature. There are different reasons for this:

- Size constraints (solving a linear system with too many variables simply requires to look for alternatives to simple matrix [pseudo-] inversion)
- Real-time constraints: one may not have all the data to start the computation, but needs at least good approximate answer from the available data

The current work of Peter Berger (see poster) is e.g. investigating the applicability of Kaczmarz methods methods ([stve09]), among others because this approach (POCS methods) shows a great amount of flexibility.

Alternative Signal Space Models

Although the assumption of band-limitedness is natural for many applications there are many other situations where the reconstruction strategies developed in the last two decades at NuHAG should be applicable.

The most important direction (originating in early work of Aldroubi and Feichtinger, see the survey of Aldroubi and Gröchenig [algr01]) is that of **spline-type spaces** (also called **shift-invariant function spaces**). Spaces of linear, quadratic or cubic splines are just a special case.

Given a suitable form of equicontinuity (expressed essentially in the norms $W(\mathbf{C}_0, \ell^2)$) of the members of such spaces (following from minimal smoothness assumptions on the generators) one can again use iterative methods for reconstruction.

Alternative Signal Space Models II

The theory used/developed so far is mostly concentrating on the reconstruction of functions in such spline-type space from regular samples (up to a shift the same lattice is used for sampling as for the generation the space as closed linear span of translates of the template, e.g. a cubic B-spline) [a classical setting], or irregular setting (now well understood). Function space theory allows to provide *claims on robustness* (e.g. with respect to jitter errors or model errors, i.e. the use of a slightly incorrect assumption about the generator/atom).

Alternative Signal Space Models III

Here two aspects come in:

- As explained in the introduction **signals will not always be so nice** to belong to $\mathbf{L}^2(\mathbb{R}^d)$ or some other $\mathbf{L}^p(\mathbb{R}^d)$, they may be slightly growing, or have good decay, or behave approximately stationary, but not in a strictly periodic sense; signal reconstruction strategies have to be able to cope with these situations
- The error description concerning the atoms has to be undertaken at a much more refined level compared to the signal description. Even for users interested in $\mathbf{L}^2$-theory only the model error has to be small in the sense of a Wiener algebra norm $\mathbf{W}(\mathbf{C}_0, \mathbf{L}^1)(\mathbb{R}^d)$ (norm equals essentiall the *upper* Riemannian sum to $\|f\|_1 = \int_{\mathbb{R}^d} |f(x)| dx$.

NuHAG contributions to that aspect

In this sense the PhD thesis of Sebastian Schmutzhard entitled "Galerkin methods for the numerical evaluation of the prolate spheroidal wave functions" is a contribution to the efficient computation of optimally localized building blocks for signal composition.

He has a poster on this subject.

The prolate spheroidal functions are interesting for many reasons, in our setting mostly for

- optimal locality of reconstruction
- is a possible building block for function spaces, e.g. spline type spaces, adapted to given situations

Elmar Pauwels PhD thesis I

The PhD thesis of Elmar Pauwels [pa11-4] (defended December 2011, and submitted partially as journal paper now) on **Pseudodifferential Operators, Wireless Communications and Sampling Theorems** is another example showing the usefulness of (the appropriate) function spaces for the analysis of an applied problem.

Based on experience within the team (mostly from the WWTF project MOHAWI, on modern harmonic analysis and wireless communication, also under K. Gröchenig and with Hlawatsch/Matz as cooperation [1] partners) we had learned to understand that the problem of channel identification and channel decoding can be put in a clean mathematical setting using suitable function spaces.

[1] Two patents based on that project, with S. Das and T. Hrycak are presently been moved to the US market.

Elmar Pauwels PhD thesis II

Mathematically speaking the tools required to model mobile channels with a given maximal time delay and Doppler shift as **underspread operators**, i.e. as operators having a *spreading representation* confined to a rectangular box Ω which is known by assumption (based on physical considerations). One should think of the spreading mapping as an operator analogue of the Fourier transform,

Clearly the compact support of the spreading function implies that its 2D-Fourier transform is a band-limited function and can be recovered using Shannon's theorem from sufficiently dense samples as long as the Nyquist criterion is satisfied (the shifts of Ω should not overlap). For the applications one takes the *symplectic* Fourier transform and then the object obtained is the so-called *Kohn-Nirenberg symbol* of the operator (the channel), which is well defined (at least distributionally).

Elmar Pauwels PhD thesis III

The key observation of the PhD thesis is to relate the problem of identifying a slowly varying (underspread) channel from the receive pilot tones.
Described more explicitly: The sending station is sending (in between the data packages) pilot tones, known to the receiver. The distortion observed by him/her are used to identify the channel.
Mathematically the channel (hopefully invertible and hence not a Hilbert Schmidt operator) has to be reconstructed from a filtered version of the Kohn-Nirenberg symbol. In fact, the density of pilot tones (arranged along a lattice in the TF-plane) has to be at Nyquist rate.

Elmar Pauwels PhD thesis IV

Again the appropriate function spaces and an established natural correspondence to families of linear operators can be used:

- the small class of regularizing operators having nice kernels, which are test functions of two variables ($\mathbf{S}_0(\mathbb{R}^{2d})$).
- the Hilbert space of Hilbert Schmidt operators with KNS or kernel in $\mathbf{L}^2(\mathbb{R}^{2d})$;
- the general class of operators $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$

The smoothing kernel describing the filter of the KNS symbol, depends on the window used for the pilot tones (ideally a Gauss-like atom, part of the system design) has to be free of zeros over the rectangle Ω.

Elmar Pauwels PhD thesis V

The topic of the thesis is in fact closely related to another research topic that has been pushed by HGFei in the last decade: Gabor multipliers.
The corresponding problem is then to **recover the upper symbol of the Gabor multiplier** (i.e. the amplitudes used by the audio-engineer in his device, slowly changing the contribution of the different frequencies) **from the lower symbol** (i.e. essentially the strength (and phase) of the received pilot tones.
Instead of Shannon one needs there a sampling theory for spline-type spaces (Gabor multipliers are elements from 2D spline-type spaces, in the KNS setting).

Other related projects at NuHAG

- The EU-network UnlocX (2009-2013) is concerned with uncertainty and localization. NuHAG is identifying optimal sampling strategies (also with slowly varying Gabor windows);
- The FWF Project by Maurice de Gosson deals with Hamiltionian time-frequency analysis, where Gabor atoms are slowly changing according to the Hamiltonian flow;
- Recent work with D. Onchis considers the problem of multi-window spline-type spaces and irregular sampling;
- some ongoing master theses and PhD theses or concerned with related topics;
- Jose L. Romero (surgery of frames);

Concluding Remarks

For scattered data or irregular sampling function spaces are indispensible, because

- they are needed to describe convergence
- measure the error
- allow to build a variety of function space models
- e.g. spline-type spaces

FINAL SLIDE

THANK YOU FOR YOUR ATTENTION