Adaptive wavelet Galerkin methods: Extension to unbounded domains and fast evaluation of system matrices

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joint work with Prof. Dr. Karsten Urban and Prof. Dr. Rob Stevenson

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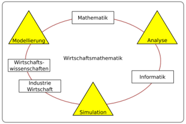

DFG Research Training Group 1100

DFG Priority Program 1324

Overview

Motivation: Wavelet methods in numerical finance

Adaptive wavelet (Galerkin) methods on unbounded domains

Fast evaluation of system matrices

Challenges in numerical solutions of PIDEs arising in numerical finance

Computation of European option prices via numer. solution of PIDE problem ([RSW10])

$$\begin{cases} \partial_\tau u(\tau, x) - \mathcal{B}^X[u](\tau, x) = 0, & x \in \mathbb{R}^n,\ \tau \in (0, T], \\ u(0, x) = h(x), & x \in \mathbb{R}^n, \end{cases}$$

where $X = (X_t^1, \ldots, X_t^n)_{t \geq 0}$ is Lévy process with **infinitesimal generator**

$$\mathcal{B}^X[w] := \frac{1}{2} \sum_{i,j=1}^{n} \mathcal{Q}_{ij} \frac{\partial^2 w}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \gamma_i \frac{\partial w}{\partial x_i} + \int_{\mathbb{R}^n} \left(w(\cdot + y) - w + \mathbf{1}_{\{|y| \leq 1\}} \sum_{i=1}^{n} y_i \frac{\partial w}{\partial x_i} \right) \nu(\mathrm{d}y).$$

- **Unbounded** domain:
  ⇝ A priori versus adaptive truncation.
- **Non-local** operator $\mathcal{B}^X$:
  ⇝ Wavelet compression.
- **Spatial dimension**:
  ⇝ Avoid curse of dimension by sparse grids or adaptive methods.

[RSW10] N. Reich, C. Schwab, C. Winter. *On Kolmogorov equations for anisotropic multivariate Lévy processes*. Finance and Stochastics, 2010

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For the computation of European option prices in multi-dimensional Lévy models

$$\begin{cases} \partial_\tau u_R(\tau,x) - \mathcal{B}^X[u_R](\tau,x) = 0, & x \in (-R,R)^n,\ \tau \in (0,T], \\ u_R(0,x) = h(x), & x \in (-R,R)^n, \\ u_R(\tau,x) = 0, & x \in \mathbb{R}^n \setminus (-R,R)^n,\ \tau \in (0,T], \end{cases}$$

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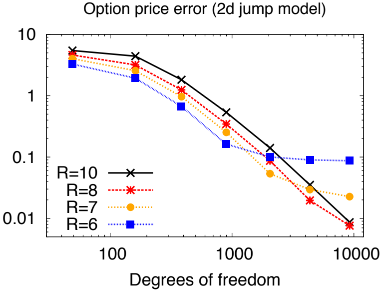

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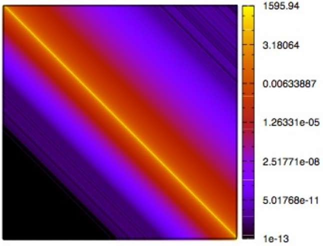

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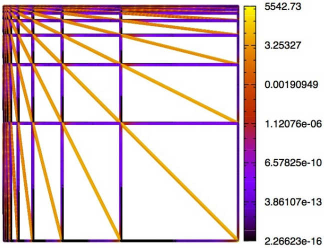

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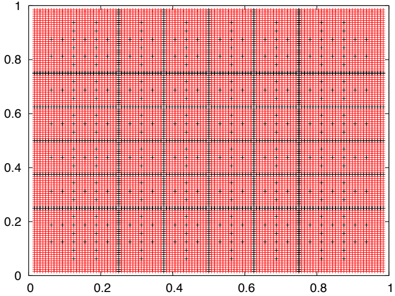

Adaptive solution of the option pricing problem

$$\begin{cases} \partial_\tau u(\tau,x) - \mathcal{B}^X[u](\tau,x) = 0, & x \in \mathbb{R}^n,\ \tau \in (0,T], \\ u(0,x) = h(x), & x \in \mathbb{R}^n. \end{cases}$$

For the discretization of $\mathcal{B}^X$, a common assumption is (compare, e.g., [Hep11]):

$$\langle v_1 \otimes \cdots \otimes v_n, \mathcal{B}^X[w_1 \otimes \cdots \otimes w_n]\rangle = \Big(\sum_{m=1}^{M} \alpha_m \cdot a_m^{(1)}(v_1, w_1) \otimes \cdots \otimes a_m^{(n)}(v_n, w_n)\Big),$$

where for $i \in \{1, \dots, n\}$ at most **one** univariate bilinear form $a_m^{(i)}$ is non-local.

Where wavelets find applications...

- Fast approximate evaluation of non-local bilinear forms ($\rightsquigarrow$ wavelet compression).
- Non-smooth payoff h requires adaptive refinement (see also [BHPS12]).
- **Treatment of unbounded domain $\mathbb{R}^n$:** Towards optimal balancing of truncation and discretization error.
- **Fast exact evaluation** of **local** bilinear forms.

[Hep11] P. Hepperger. *Option pricing in Hilbert space-valued jump diffusion models using partial integro-differential equations*. SIAM Journal on Financial Mathematics, 2011.

[BHPS12] H.-J. Bungartz, A. Heinecke, D. Pflüger and S. Schraufstetter. *Option pricing with a direct adaptive sparse grid approach*. Journal of Computational and Applied Mathematics, 2012.

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Fast evaluation of system matrices

Given $f \in \mathcal{X}'$, consider a linear, well-posed operator equation:

$$\mathcal{A}[u] = f \text{ in } \mathcal{X}',$$

- $\mathcal{X}$ is a Sobolev space over an unbounded domain $\boldsymbol{\Omega}$ (e.g., $\mathcal{X} = H^1(\mathbb{R}^n)$),
- $\mathcal{A} : \mathcal{X} \to \mathcal{X}'$ is boundedly invertible, self-adjoint.

Requirements for numerical scheme

- Adaptive domain truncation and local refinement.
- Optimal convergence rate under weak smoothness assumptions.
- Linear complexity for a large class of operators $\mathcal{A}$.

New approach

Existing methods (Infinite Elements, BEM, ...) do either not cover *all* of these requirements or are only applicable for special classes of $\mathcal{A}$.

- *Equivalent formulation* in an infinite-dimensional sequence space ℓ_2 of *wavelet coefficients*,

$$\mathcal{A}[u] = f \text{ in } \mathcal{X}' \quad \Longleftrightarrow \quad \mathbf{Au} = \mathbf{f} \text{ in } \ell_2.$$

- Approximation of $\mathbf{u}$ by means of *adaptive wavelet methods*.

Univariate Riesz wavelet bases

Let $\Omega \subseteq \mathbb{R}$ be a domain (possibly unbounded).

We consider a **Riesz** wavelet basis $\Psi = \{\psi_\lambda : \lambda \in \mathcal{J}\}$ that characterizes univariate Sobolev spaces $\mathcal{H}^s(\Omega)$ (possibly incorporating essential homogeneous bc's)

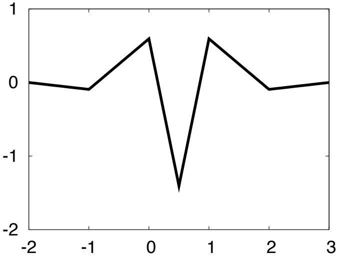
$$\underline{\mathrm{C}}^{\Psi}_{\mathcal{H}^s}\|\mathbf{v}\|^2_{\ell_2(\mathcal{J})} \leq \Big\| \sum_{\lambda\in\mathcal{J}} v_\lambda \psi_\lambda / \|\psi_\lambda\|_{H^s} \Big\|^2_{H^s(\Omega)} \leq \overline{\mathrm{C}}^{\Psi}_{\mathcal{H}^s}\|\mathbf{v}\|^2_{\ell_2(\mathcal{J})}, \quad \forall v = \mathbf{v}^\top \mathrm{D}^s\,\Psi \in \mathcal{H}^s(\Omega),$$

where $-\tilde{\gamma} < s < \gamma$. Here, Ψ is a column vector and $\mathrm{D}^s = \mathrm{diag}\left[(\|\psi_\lambda\|^{-1}_{H^s})_{\lambda\in\mathcal{J}}\right]$ a bi-infinite diagonal matrix.

Standard wavelet assumptions / notations

- $\psi_\lambda := 2^{j/2}\psi^{(i)}(2^j \cdot - k)$, $\lambda = (i, j, k)$.
- ψ_λ are piecewise polynomials of order d.
- Local support: $\mathrm{diam\,supp}\,\psi_\lambda \eqsim 2^{-j}$.
- Example:

$$\begin{aligned}\Psi_{L_2(\mathbb{R})} := \quad & \{2^{j_0/2}\phi(2^{j_0}\cdot - k) : k \in \mathbb{Z}\} \\ \cup \quad & \{2^{j/2}\psi(2^j \cdot - k) : j \geq j_0, k \in \mathbb{Z}\}\end{aligned}$$



Tensor product Riesz wavelet bases

Let now $\boldsymbol{\Omega} := \Omega_1 \times \cdots \times \Omega_n$ be a **product domain** (possibly unbounded).
With n univariate Riesz wavelet bases $\Psi^{(i)}$ for $L_2(\Omega_i)$ ($i \in \{1, \ldots, n\}$),

$$\boldsymbol{\Psi} := \Psi^{(1)} \otimes \cdots \otimes \Psi^{(n)} = \{\psi_{\boldsymbol{\lambda}} := \psi_{\lambda_1} \otimes \cdots \otimes \psi_{\lambda_n} : \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \boldsymbol{\mathcal{J}}\}$$

is a Riesz wavelet basis for $L_2(\boldsymbol{\Omega})$ where $\boldsymbol{\mathcal{J}} := \mathcal{J}^{(1)} \times \cdots \times \mathcal{J}^{(n)}$.
For Sobolev spaces $\mathcal{X}$ over $\boldsymbol{\Omega}$ that can be characterized by (intersections of) tensor products of univariate Sobolev spaces,

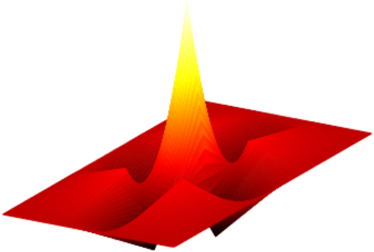
$$\boldsymbol{\Psi}^{\mathcal{X}} := \mathbf{D}^{\mathcal{X}} \boldsymbol{\Psi} := \{\psi_{\boldsymbol{\lambda}} / \|\psi_{\boldsymbol{\lambda}}\|_{\mathcal{X}} : \boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}\}, \quad \mathbf{D}^{\mathcal{X}} := \operatorname{diag}\left[(\|\psi_{\boldsymbol{\lambda}}\|_{\mathcal{X}}^{-1})_{\boldsymbol{\lambda} \in \boldsymbol{\mathcal{J}}}\right],$$

is a tensor product Riesz wavelet basis for $\mathcal{X}$.

Tensor product wavelets

- $\psi_{\boldsymbol{\lambda}}$ are piecewise polynomials.
- Local **anisotropic** support:

$$|\operatorname{supp} \psi_{\boldsymbol{\lambda}}| \eqsim 2^{-(|\lambda_1| + \cdots + |\lambda_n|)}.$$



Wavelet discretization of operator equations

Unique expansion of u in $\mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi}$, $u = \mathbf{u}^\top(\mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi}) := \sum_{\boldsymbol{\lambda}\in\mathcal{J}} \mathbf{u}_{\boldsymbol{\lambda}} \mathbf{D}^{\mathcal{X}}_{\boldsymbol{\lambda}} \psi_{\boldsymbol{\lambda}}$, yields ([CDD01])

$$\langle v, \mathcal{A}[u]\rangle = \langle v, f\rangle,\ \forall v \in \mathcal{X} \ \Leftrightarrow\ \underbrace{\langle \mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi}, \mathcal{A}[\mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi}]\rangle}_{\mathbf{A}}\, \mathbf{u} = \underbrace{\langle \mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi}, f\rangle}_{\mathbf{f}} \Leftrightarrow\ \mathbf{A}\mathbf{u} = \mathbf{f} \text{ in } \ell_2(\mathcal{J}).$$

Infinite load vector $\mathbf{f} = \mathbf{D}^{\mathcal{X}}\left[(\langle \psi_{\boldsymbol{\lambda}}, f\rangle)_{\boldsymbol{\lambda}\in\mathcal{J}}\right] \in \ell_2(\mathcal{J})$.

Boundedly invertible **bi-infinite** system matrix $\mathbf{A} = \mathbf{D}^{\mathcal{X}}\left[(\langle \psi_{\boldsymbol{\lambda}}, \mathcal{A}[\psi_{\boldsymbol{\mu}}]\rangle)_{\boldsymbol{\lambda},\boldsymbol{\mu}\in\mathcal{J}}\right]\mathbf{D}^{\mathcal{X}}$.

$\Longrightarrow$: This discretization only requires a Riesz basis (independent of the domain).

Error estimate: Riesz basis property guarantees for **both bounded and unbounded** domains Ω that

$$\boxed{C\|\mathbf{u} - \mathbf{u}_{\boldsymbol{\Lambda}}\|_{\ell_2} \le \varepsilon \quad \Longrightarrow \quad \|u - \mathbf{u}_{\boldsymbol{\Lambda}}^\top(\mathbf{D}^{\mathcal{X}}\boldsymbol{\Psi})\|_{\mathcal{X}} \le \varepsilon}$$

$\Longrightarrow$: Local refinement and adaptive domain truncation via selecting significant wavelet indices from $\mathbf{u}$.

[CDD01] A. Cohen, W. Dahmen, R. DeVore. *Adaptive wavelet methods for elliptic operator equations: Convergence rates*. Mathematics of Computation, 2001.

Adaptive approximation of $\mathbf{A}\mathbf{u} = \mathbf{f}$

Solve **finite Galerkin systems** on nested index sets $\boldsymbol{\Lambda}_k \subset \boldsymbol{\Lambda}_{k+1} \subset \cdots \subset \mathcal{J}$ ([GHS07]).

AWGM[ε]
for $k = 0$; $\|\mathbf{r}_k\|_{\ell_2} \leq \varepsilon$; $k = k + 1$ **do**
Compute approx. solution $\mathbf{w}_{\boldsymbol{\Lambda}_k}$ of

$$\mathbf{A}_{\boldsymbol{\Lambda}_k} \mathbf{u}_{\boldsymbol{\Lambda}_k} = \mathbf{f}_{\boldsymbol{\Lambda}_k}.$$

Compute approximation $\mathbf{r}_k$ of the *infinite* residual

$$\mathbf{f} - \mathbf{A}\mathbf{w}_{\boldsymbol{\Lambda}_k}.$$

Compute smallest $\boldsymbol{\Lambda}_{k+1} \supset \boldsymbol{\Lambda}_k$ s.t.

$$\|\mathbf{P}_{\boldsymbol{\Lambda}_{k+1}} \mathbf{r}_k\|_{\ell_2} \geq \mu \, \|\mathbf{r}_k\|_{\ell_2} \qquad (1)$$

for $\mu \in (0, 1)$ where $\mathbf{P}_{\boldsymbol{\Lambda}} \mathbf{v} := \mathbf{v}|_{\boldsymbol{\Lambda}}$.
end for

[GHS07] T. Gantumur, H. Harbrecht, R. Stevenson. *An optimal adaptive wavelet method without coarsening of the iterands*. Mathematics of Computation, 2007.

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end for

Idea from [KU12]:

In (1), significant wavelet indices $\lambda \in \mathcal{J}$ are added for

- local refinement of singularities, **and**
- domain extension.

Approximation of the infinite residual

$$\rightsquigarrow \|\mathbf{APPLY}[\mathbf{w}_{\Lambda_k}, \eta] - \mathbf{A}\mathbf{w}_{\Lambda_k}\|_{\ell_2} \leq \eta,$$
$$\rightsquigarrow \|\mathbf{RHS}[\eta] - \mathbf{f}\|_{\ell_2} \leq \eta.$$

For suff. small η, define $\mathbf{r}_k$ as

$$\mathbf{r}_k := \mathbf{RHS}[\eta] - \mathbf{APPLY}[\mathbf{w}_{\Lambda_k}, \eta].$$

[KU12] S. K., K. Urban. *Adaptive wavelet methods on unbounded domains*. Journal of Scientific Computing, 2012.

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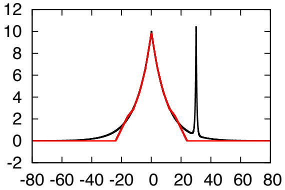
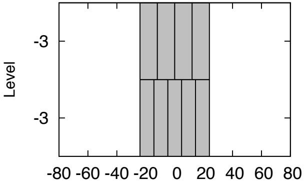
Exact / numerical solution



Index set Λ



[SU09] W. Sickel, T. Ullrich. *Tensor products of Sobolev-Besov spaces and applications to approximation from the hyperbolic cross*. Journal of Approximation Theory, 2009

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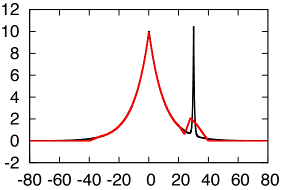

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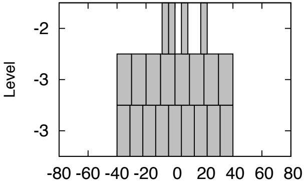

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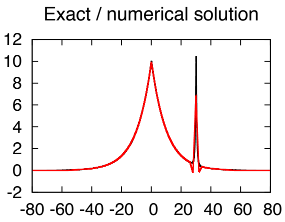

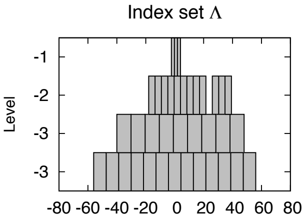

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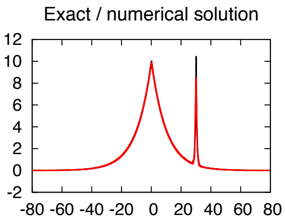

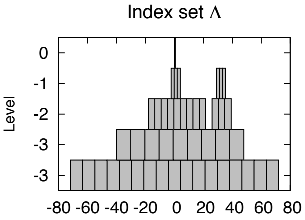

Adaptive approximation of $\mathbf{Au} = \mathbf{f}$

Solve **finite Galerkin systems** on nested index sets $\boldsymbol{\Lambda}_k \subset \boldsymbol{\Lambda}_{k+1} \subset \cdots \subset \mathcal{J}$ ([GHS07]).

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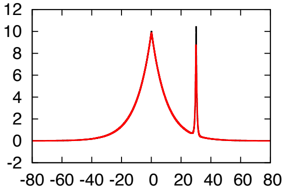
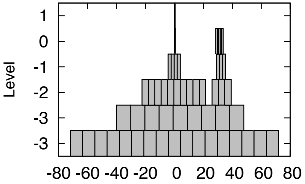
for $\mu \in (0,1)$ where $\mathbf{P}_{\boldsymbol{\Lambda}}\mathbf{v} := \mathbf{v}|_{\boldsymbol{\Lambda}}$.
end for

Exact / numerical solution



Index set Λ



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Adaptive approximation of $\mathbf{A}\mathbf{u} = \mathbf{f}$

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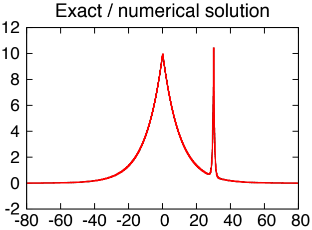

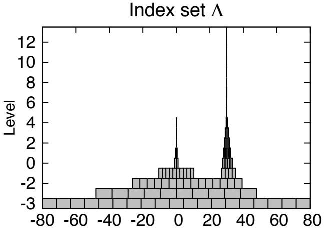

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end for

Best N-term approximation:

$$\|\mathbf{u} - \mathbf{u}_N\|_{\ell_2} \leq CN^{-s}.$$

$\rightsquigarrow$ (Tensor-)Besov regularity of u ([SU09]).
$\rightsquigarrow$ Asymptotic dimension-independent convergence rates are possible.

[GHS07, Theorem 2.7]

Given *suitable* routines **APPLY** and **RHS**:

- $\|\mathbf{u} - \mathbf{u}_{\mathbf{\Lambda}_k}\|_{\ell_2} \eqsim \|\mathbf{u} - \mathbf{u}_{N_k}\|_{\ell_2} \lesssim N_k^{-s}$ where $N_k := \operatorname{supp} \mathbf{w}_{\mathbf{\Lambda}_k}$.
- Linear complexity.

$\rightsquigarrow$ **Optimal scheme for bounded and <u>unbounded</u> domains**

Adaptive wavelet algorithms: From bounded to unbounded domains

Bounded domain: $\Omega = (a, b)$

$$\mathcal{J}^{\Omega} := \{(j,k) : j \geq j_0, k \in \mathcal{I}_j\}, j_0 \geq 0 \,.$$

- ✓ Fixed minimal level j_0.
- ✓ Realization of **RHS** (e.g. [GHS07]):
- ✓ Realization of **APPLY** (e.g., [Ste09], [Urb09]).
- ✓ Quantitative analysis (e.g. [DHS07]).

Unbounded domain: $\mathbb{R}$ (cf. [KU12])

$$\mathcal{J}^{\mathbb{R}} := \{(j,k) : j \geq j_0, k \in \mathbb{Z}\}, j_0 \in \mathbb{Z}.$$

- ✓ *Good* choice of j_0:
  $\rightsquigarrow$ Diameter initial domain: $\approx 2^{-j_0}$.
- ✓ Construct *finite* $\nabla_\eta \subset \mathcal{J}^{\mathbb{R}}$ with

$$\|\mathbf{f} - \mathbf{f}|_{\nabla_\eta}\|_{\ell_2} \leq \eta.$$

  $\rightsquigarrow$ Bound for translation indices.
- ✓ Special treatment of negative levels:

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{+-} & \mathbf{A}_{++} \\ \mathbf{A}_{--} & \mathbf{A}_{-+} \end{pmatrix}.$$

- ✓ Adaptive domain truncation, local refinement, convergence.

[DHS07] W. Dahmen, H. Harbrecht and R. Schneider. *Adaptive methods for boundary integral equations: complexity and convergence estimates*. Mathematics of Computation, 2007.
[Ste09] R. Stevenson. *Adaptive wavelet methods for solving operator equations: An overview*. Springer, 2009.
[Urb09] K. Urban. *Wavelet methods for elliptic partial differential equations*. Oxford University Press, 2009.

Adaptive approximation of $\mathbf{Au} = \mathbf{f}$: Numerical experiment

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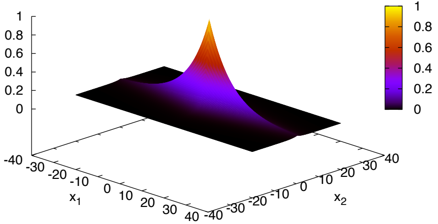
end for



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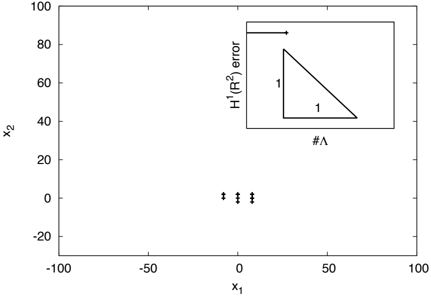

$+$: barycenter of supp $\psi_{\boldsymbol{\lambda}}$ for $\boldsymbol{\lambda} \in \Lambda_k$

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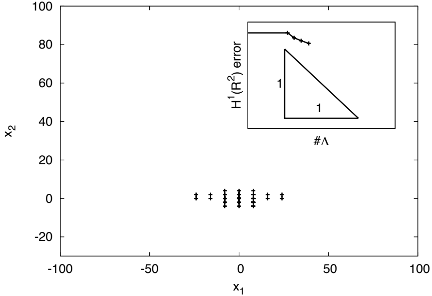

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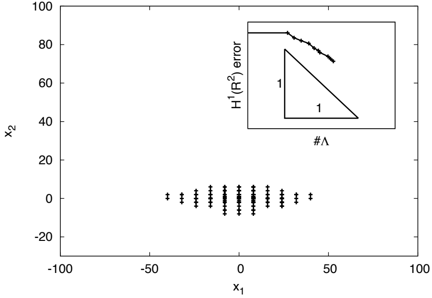

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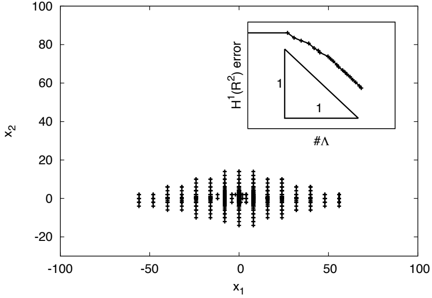

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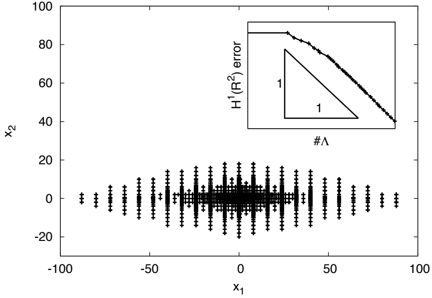

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Adaptive wavelet methods on unbounded domains: Remarks

Product domains are **not** mandatory.

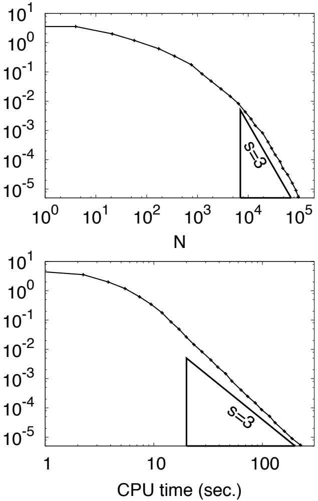
Works also within heuristic wavelet schemes (e.g. [BK06]).

Same proceeding for **non-linear** problems on unbounded domains (when isotropic wavelet bases are used).

Adaption of special multiwavelet bases for constant coefficient PDE operators (e.g. [DS10]) is possible (cf. [K.12]). On the right, applied to the problem

$$-\Delta u + u = f \text{ in } H^{-1}(\mathbb{R}^2).$$

Work in progress: **Non-local** operators.



[BK06] S. Berrone, T. Kozubek. *An adaptive WEM algorithm for solving elliptic boundary value problems in fairly general domains*. SIAM Journal on Scientific Computing, 2006.
[DS10] T. Dijkema, R. Stevenson. *A sparse Laplacian in tensor product wavelet coordinates*. Numerische Mathematik, 2009.
[K.12] *A special multiwavelet basis for unbounded product domains*. Proceedings of ENUMATH 2011, 2012.

Overview

Motivation: Wavelet methods in numerical finance

Adaptive wavelet (Galerkin) methods on unbounded domains

Fast evaluation of system matrices

Tensor structure of the system matrix A

Consider $\mathbf{\Psi} = \Psi^{(1)} \otimes \cdots \otimes \Psi^{(n)}$ and let $\mathscr{B}(v, w) := \langle v, \mathcal{A}[w] \rangle$ be such that

$$\mathscr{B}(\otimes_{i=1}^n v_i, \otimes_{i=1}^n w_i) := \sum_{m=1}^{M} \prod_{i=1}^{n} a_m^{(i)}(v_i, w_i),$$

where $a_m^{(i)}$ are **local, univariate** bilinear forms related to coordinate direction $\mathbf{e}_i$. Now,

$$\mathbf{A} = \mathbf{D}\mathscr{B}(\mathbf{\Psi}, \mathbf{\Psi})\mathbf{D} = \mathbf{D}\left[\sum_{m=1}^{M} \bigotimes_{i=1}^{n} \vec{S}_m^{(i)}\right]\mathbf{D}, \quad \vec{S}_m^{(i)} := a_m^{(i)}(\Psi^{(i)}, \Psi^{(i)}).$$

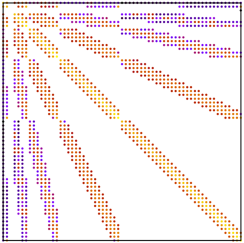
Poisson's equation $\left(n = 2, \mathbf{\Omega} = (0, 1)^2\right)$

$$\mathscr{B}(\mathbf{\Psi}, \mathbf{\Psi}) = \vec{A} \otimes \vec{M} + \vec{M} \otimes \vec{A},$$

where $\Psi^{(1)} = \Psi^{(2)}$ and

$$\vec{A} := \left[\int_0^1 \partial\psi_\lambda \partial\psi_\mu\right]_{\lambda,\mu\in\mathcal{J}}, \quad \vec{M} := \left[\int_0^1 \psi_\lambda \psi_\mu\right]_{\lambda,\mu\in\mathcal{J}}.$$

$\Longrightarrow \mathscr{B}(\mathbf{\Psi}, \mathbf{\Psi})$ **is not sparse!**



Three fundamental principles

Splitting into unidirectional operations

$$\vec{S} \otimes \vec{S} = (\vec{S} \otimes \vec{\mathrm{Id}}) \circ (\vec{\mathrm{Id}} \otimes \vec{S}) = (\vec{\mathrm{Id}} \otimes \vec{S}) \circ (\vec{S} \otimes \vec{\mathrm{Id}}).$$

Sequential application of up- and down operations

$$\vec{S} = [a(\psi_\lambda, \psi_\mu)]_{\lambda,\mu \in \mathcal{J}} = \vec{L} + \vec{U}, \quad \vec{L} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| > |\mu|}, \quad \vec{U} = [a(\psi_\lambda, \psi_\mu)]_{|\lambda| \leq |\mu|}.$$

Multi-level structure of univariate wavelet bases

$$\left.\begin{array}{l} \Psi = \bigcup_{\ell \in \mathbb{N}_0} \Psi_j, \quad \Psi_\ell := \{\psi_\lambda : \lambda \in \mathcal{J}, \, |\lambda| = \ell\} \\ \mathrm{clos}_{L_2}\left(\mathrm{span} \bigcup_{0 \leq \ell \leq j} \Psi_\ell\right) = \mathrm{clos}_{L_2}\left(\mathrm{span}\, \Phi_j\right) \end{array}\right\} \text{Bi-directional transformations in linear complexity: } \textbf{FWT}, \textbf{IFWT}.$$

These principles have been introduced in **sparse grid** settings (using in particular **hierarchical** bases), see, e.g., [BG04, Bun92 BZ96, Zei11, Zen91].

[BG04] H.-J. Bungartz and M. Griebel. *Sparse grids*. Acta Numerica, 2004.
[Bun92] H.-J. Bungartz. *Dünne Gitter und deren Anwendung bei der adaptiven Lösung der dreidimensionalen Poisson-Gleichung*. PhD Thesis (TU München), 1992.
[BZ96] R. Balder and C. Zenger. *The solution of multidimensional real Helmholtz equations on sparse grids*. SIAM Journal on Scientific Computing, 1996.
[Zei11] A. Zeiser. *Fast matrix-vector multiplication in the sparse-grid Galerkin method*. Journal of Scientific Computing, 2011.
[Zen92] C. Zenger. *Sparse grids*. Parallel algorithms for fluid mechanics, 1991.

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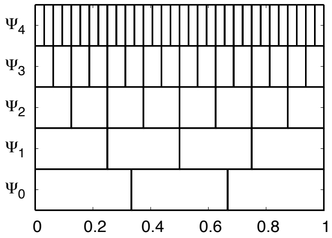
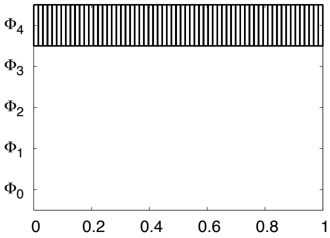
Bi-directional transformations in linear complexity: **FWT**, **IFWT**.





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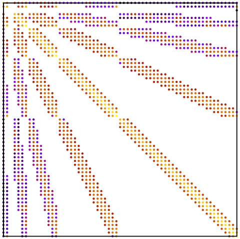
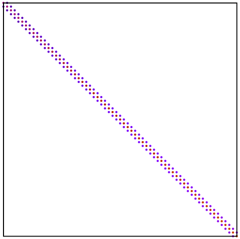
Multi-level structure of univariate wavelet bases

$$\left.\begin{array}{l} \Psi = \bigcup_{\ell \in \mathbb{N}_0} \Psi_j, \quad \Psi_\ell := \{\psi_\lambda : \lambda \in \mathcal{J}, \, |\lambda| = \ell\} \\ \mathrm{clos}_{L_2}\left(\mathrm{span} \bigcup_{0 \leq \ell \leq j} \Psi_\ell\right) = \mathrm{clos}_{L_2}\left(\mathrm{span}\, \Phi_j\right) \end{array}\right\}$$ Bi-directional transformations in linear complexity: **FWT**, **IFWT**.

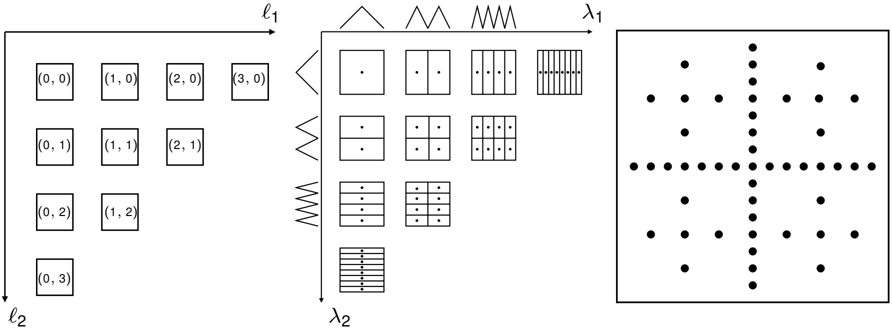




Example: Matrix-vector multiplication on sparse grids

Given a refinement level $j \in \mathbb{N}_0$, consider the (two-dimensional) sparse grid space

$$\boldsymbol{\Lambda}_j := \bigcup_{\ell_1+\ell_2 \leq j} \boldsymbol{\Lambda}_{(\ell_1,\ell_2)}, \quad \boldsymbol{\Lambda}_{(\ell_1,\ell_2)} := \{(\lambda_1,\lambda_2) : |\lambda_1| = \ell_1, |\lambda_2| = \ell_2\}, \quad \#\boldsymbol{\Lambda}_j \approx (j+1)\, 2^j.$$



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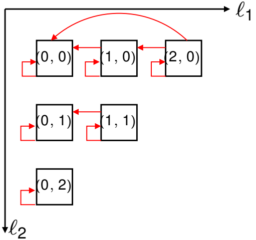
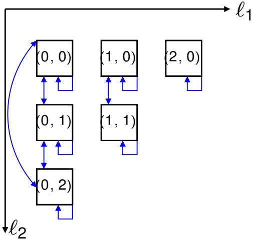
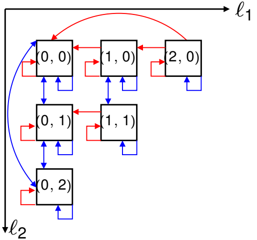
Within the matrix-vector multiplication on a sparse grid, by the splitting $\vec{S} = \vec{L} + \vec{U}$,

$$\mathbf{P}_{\mathbf{\Lambda}_j}(\vec{S} \otimes \vec{S})\mathbf{E}_{\mathbf{\Lambda}_j} = \mathbf{P}_{\mathbf{\Lambda}_j}(\vec{L} \otimes \vec{\mathrm{Id}})\mathbf{E}_{\mathbf{\Lambda}_j}\,\mathbf{P}_{\mathbf{\Lambda}_j}(\vec{\mathrm{Id}} \otimes \vec{S})\mathbf{E}_{\mathbf{\Lambda}_j} + \boxed{\mathbf{P}_{\mathbf{\Lambda}_j}(\vec{\mathrm{Id}} \otimes \vec{S})\mathbf{E}_{\mathbf{\Lambda}_j}\,\mathbf{P}_{\mathbf{\Lambda}_j}(\vec{U} \otimes \vec{\mathrm{Id}})\mathbf{E}_{\mathbf{\Lambda}_j}}$$

we do **not** leave the sparse grid index set $\mathbf{\Lambda}_j$!







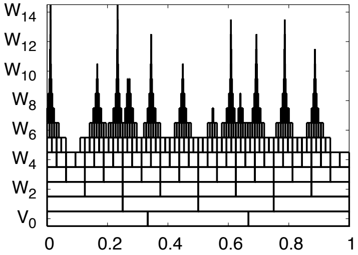
Towards adaptivity: multi-tree structured index sets

A (univariate) index set $\Lambda \subset \mathcal{J}$ is a (univariate) **tree** when for all $\lambda \in \Lambda$

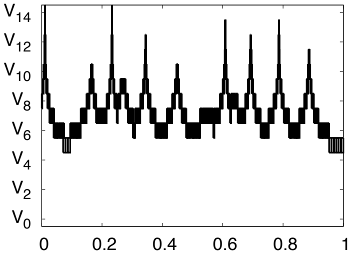
$$\operatorname{supp} \psi_\lambda \subset \bigcup_{\mu \in \Lambda, |\mu| = |\lambda| - 1} \operatorname{supp} \psi_\mu .$$

Theorem ([KS12a]): When $\tilde{\Lambda}, \Lambda \subset \mathcal{J}$ are trees, then $\mathbf{P}_{\tilde{\Lambda}} \vec{X} \mathbf{E}_\Lambda = \big[a(\psi_\lambda, \psi_\mu)\big]_{\lambda \in \tilde{\Lambda}, \mu \in \Lambda}$ for $\vec{X} \in \{\vec{S}, \vec{L}, \vec{U}\}$ can be applied within $\mathcal{O}(\#\tilde{\Lambda} + \#\Lambda)$ operations.

multi-scale representation



local single-scale representation



[KS12a] S. K., R. Stevenson *Fast evaluation of system matrices w.r.t. multi-tree collections of refineable tensor product basis functions*. Preprint (submitted), 2012.

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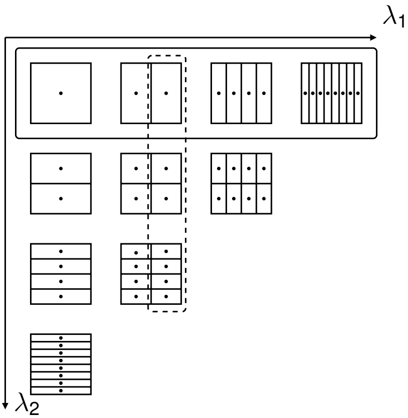
$\boldsymbol{\Lambda} \subset \boldsymbol{\mathcal{J}}$ is a **multi-tree** (cf. [KS12a]) when for any $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}, \ldots, \lambda_n)$

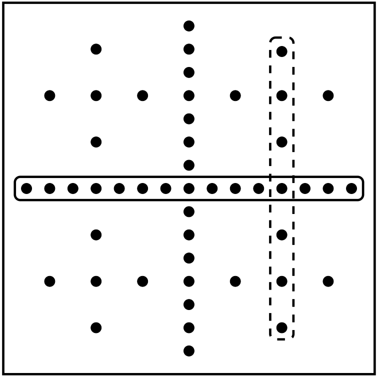
$$\boldsymbol{\Lambda}_{\mathbf{e}_i, \boldsymbol{\lambda}} := \{\mu \in \mathcal{J} : (\lambda_1, \ldots, \lambda_{i-1}, \mu, \lambda_{i+1}, \ldots, \lambda_n) \in \boldsymbol{\Lambda}\}, \quad i \in \{1, \ldots, n\},$$

is a tree (or the empty set). “A multi-tree $\boldsymbol{\Lambda}$, when frozen in $n-1$ coordinate directions, is a tree in the remaining coordinate.”

[KS12a] S. K., R. Stevenson *Fast evaluation of system matrices w.r.t. multi-tree collections of refineable tensor product basis functions*. Preprint (submitted), 2012.

Towards adaptivity: multi-tree structured index sets





$\mathbf{\Lambda} \subset \mathcal{J} = \mathcal{J} \otimes \mathcal{J}$ is a **multi-tree** when for any $\lambda \in \mathcal{J}$

$$\mathbf{\Lambda}_{\mathbf{e}_1,\lambda} := \{\mu \in \mathcal{J} : (\mu, \lambda) \in \mathbf{\Lambda}\}, \quad \mathbf{\Lambda}_{\mathbf{e}_2,\lambda} := \{\mu \in \mathcal{J} : (\lambda, \mu) \in \mathbf{\Lambda}\},$$

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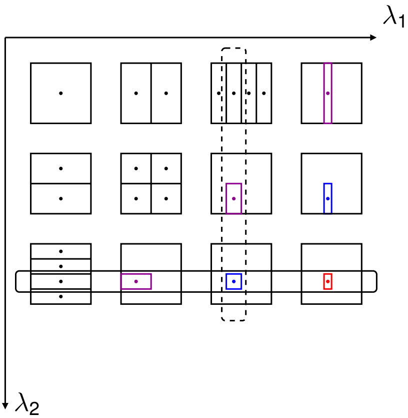
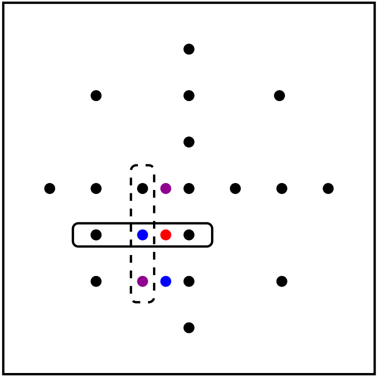
Towards adaptivity: multi-tree structured index sets





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is a tree or the empty set.

Theorem ([KS12a]): Consider two (poss. different) tensor product Riesz wavelet bases

$$\check{\boldsymbol{\Psi}} := \{\check{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \check{\mathcal{J}}\} := \check{\Psi}^{(1)} \otimes \cdots \otimes \check{\Psi}^{(n)}, \quad \hat{\boldsymbol{\Psi}} := \{\hat{\psi}_{\boldsymbol{\lambda}} : \boldsymbol{\lambda} \in \hat{\mathcal{J}}\} := \hat{\Psi}^{(1)} \otimes \cdots \otimes \hat{\Psi}^{(n)}$$

for $L_2(\boldsymbol{\Omega})$. If $\check{\boldsymbol{\Lambda}} \subset \check{\mathcal{J}}$, $\hat{\boldsymbol{\Lambda}} \subset \hat{\mathcal{J}}$ are multi-trees, then the matrix-vector multiplication w.r.t.

$$\mathbf{P}_{\check{\boldsymbol{\Lambda}}}\, \mathscr{B}(\check{\boldsymbol{\Psi}}, \hat{\boldsymbol{\Psi}})\, \mathbf{E}_{\hat{\boldsymbol{\Lambda}}} := \mathbf{P}_{\check{\boldsymbol{\Lambda}}} \Big[\sum_{m=1}^{M} \prod_{i=1}^{n} a_m^{(i)}(\check{\Psi}^{(i)}, \hat{\Psi}^{(i)}) \Big] \mathbf{E}_{\hat{\boldsymbol{\Lambda}}}$$

can be computed within $\mathcal{O}(\#\check{\boldsymbol{\Lambda}} + \#\hat{\boldsymbol{\Lambda}})$ operations.

This result **generalizes** results from adaptive sparse grids (see, e.g., [Pfl10]):

- **Different** trial- and test bases ($\rightsquigarrow$ Petrov-Galerkin methods).
- Very general **tree** concept.
- Different input and output sets.
- New approximate residual approximation (cf. [KS12b]).

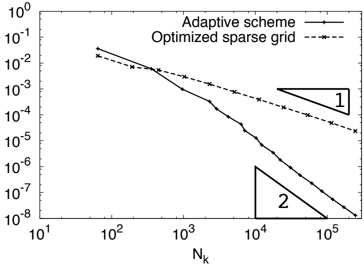
The resulting algorithm uses the decomposition $\vec{S} = \vec{L} + \vec{U}$ analogously to sparse grid schemes and is **recursive** in the dimension n.

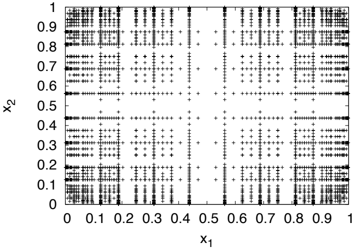
[Pfl10] D. Pflüger *Spatially adaptive sparse grids for high-dimensional problems*. PhD Thesis, 2010.
[KS12b] S. K., R. Stevenson *An efficient approximate residual evaluation in the adaptive tensor product wavelet method*. Preprint (submitted), 2012.

Numerical experiment 1: Poisson's equation with variable coefficients

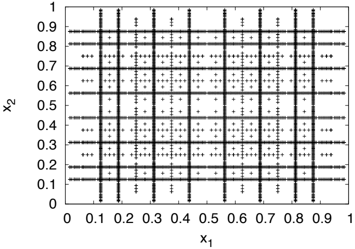
$$\begin{cases} \nabla \cdot (\mathbf{p}\nabla u) = f \text{ on } \square := (0,1)^2 \\ u|_{\partial\square} = 0 \end{cases}$$

- Constant right-hand side $f \equiv 20$.
- **Biorthogonal** wavelets ($d = 3$).
- **AWGM** with multi-tree constraint and new approximate residual.





Multi-tree obtained by the **AWGM**.



Optimized sparse grid index set.

Numerical experiments realized with **L**ibrary for **A**adaptive **W**wavelet **A**pplications (http://lawa.sourceforge.net)

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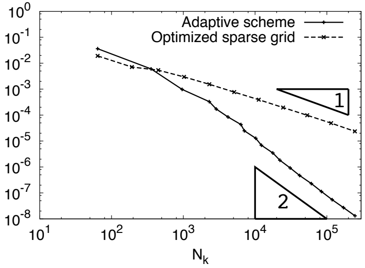
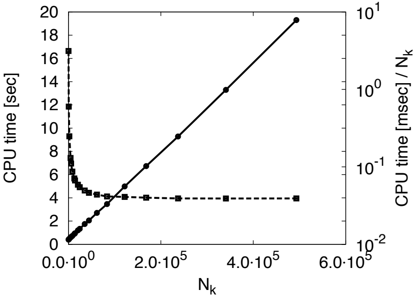
Computation times for matrix-vector multiplication (Galerkin system) w.r.t.

$$\mathbf{P}_{\mathbf{\Lambda}_k}(\vec{A} \otimes \vec{M} + \vec{M} \otimes \vec{A})\mathbf{E}_{\mathbf{\Lambda}_k},$$

where $N_k := \#\mathbf{\Lambda}_k + \#\mathbf{\Lambda}_k$.





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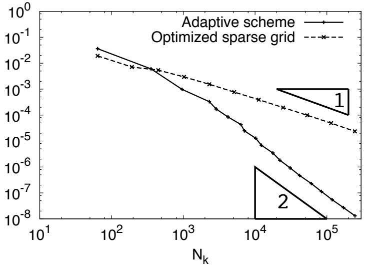

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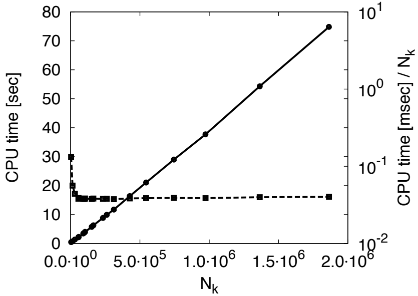

Numerical experiment 2: Poisson's equation with constant coefficients

$$\begin{cases} \nabla \cdot (\nabla u) = f \text{ on } \square := (0,1)^3 \\ u|_{\partial \square} = 0 \end{cases}$$

- Constant right-hand side $f \equiv 100$.
- L_2-**orthonormal** multiwavelets.
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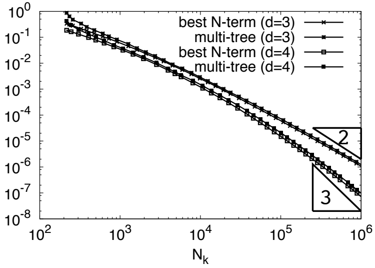
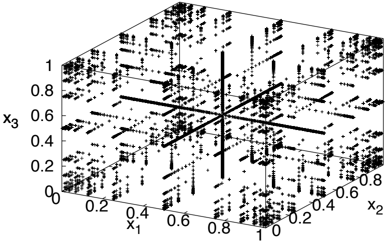
Multitree in 3d obtained by **AWGM** with refinements in the corners and along the edges.





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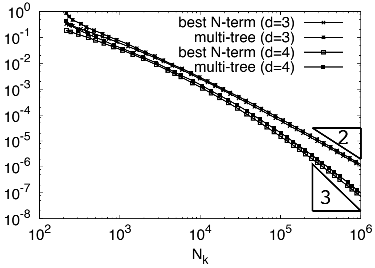

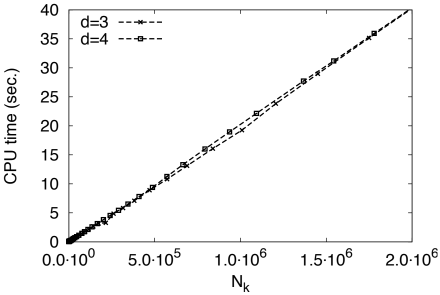

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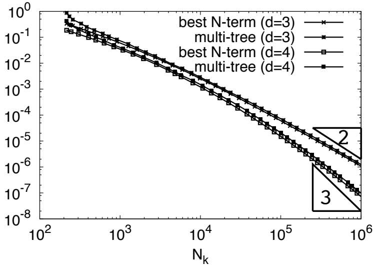

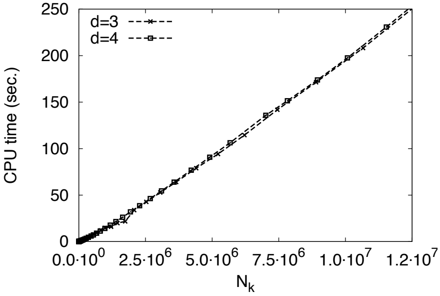

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Thank you for your attention!

Questions / Remarks . . .