NuHAG Numerical Harmonic Analysis Group

Group theoretical methods and wavelet theory (coorbit theory and applications)

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

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OUTLINE

The main theme of this talk is the usefulness of (locally compact) groups and the function spaces defined over such groups for a good understanding of signal representation, be it via the Fourier transform (relevant for Abelian groups), or the wavelet transform, or equally for non-orthogonal Gabor expansions of generalized functions.

They allow to choose the building blocks (freely!) from appropriate spaces of test functions, and guarantee convergence of the resulting expansions in suitable function spaces.

Finally one can claim that discrete groups (and hence the use of various forms of FFT) are behind many of the efficient algorithms developed in this field.

Personal background

What are the ingredients for *COORBIT THEORY* and why does this work (which is up to now the most cited work of both authors) have such an impact, over a period of more than 25 years?

- I had from the beginning an interest in function spaces over locally compact groups (under my advisor Hans Reiter);
- I had realized through interpolation theory that they come in families (strong influence of H. Triebel and J. Peetre);
- Harmonic Analysis was for me always s a subdiscipline of functional analysis, where duality is a crucial concept; therefore the use of generalized functions is obligatory;
- Faszination with smoothness described by Fourier methods (E. Stein, L. Hörmander).

Historical Viewpoint I

If one looks at the development of function space theory, Fourier analysis or even the theory of PDE one my notice an intensive **exchange between the needs of applications and the development of the appropriate mathematical tools**.
First it was the work of J.P. Fourier which kept generations of mathematicians busy with his claim that *any periodic function could be represented as a superposition of "pure frequencies"*. But what is a function, what does it mean to "represent" a function (since there are infinitely many of them, so the notion of convergence has to be specified!).
The work of B. Riemann, H. Lebesgue, or L. Schwartz gave us what is now viewed as Classical Fourier Analysis (using $\mathbf{L}^p$-spaces and integrals, resp. tempered distributions) to describe the properties of the Fourier transform.

Historical Viewpoint II

First the questions of function representation was taken serious by Dirichlet, who found conditions about the *convergence of the Fourier series at good points*.

The more refined study of points of convergence inspired the invention of *set theory*, and the determination of Fourier coefficients is of course based on a suitable concept of an integral, first the well-known *Riemann integral* and then the *Lebesgue integral* is the final development in this direction.

The convergence theorems valid for Lebesgue integrable functions imply that $\left(\mathbf{L}^1(\mathbb{U}), \|\cdot\|_1\right)$ is a complete (i.e. a Banach space), and that $\left(\mathbf{L}^2(\mathbb{U}), \|\cdot\|_2\right)$ is a Hilbert space. Parseval's identity (the infinite version of Pythagoras' theorem) is valid: the Fourier transform is energy preserving. On the other hand this makes convergence in the quadratic mean almost obvious.

Historical Viewpoint III

Such considerations lead to the analysis of $\mathbf{L}^p$-spaces, duality and the foundation of functional analysis in the first half of the last century.
The theory of Fourier transforms on $\mathbb{R}$ and $\mathbb{R}^d$ are good examples of a combination of functional analytic tools ($\mathbf{L}^p$-spaces) and concrete measure theoretic arguments in order to establish e.g. Plancherel's theorem: $\|\hat{f}\|_2 = \|f\|_2, \forall f \in \mathbf{L}^2(\mathbb{R}^d)$.
The difficulty now being that the discrete sum is replaced by a continuous integral, which needs some extra properties to be valid (and are not satisfied by the simple SINC function, because it is not integrable). Another difficulty is the fact that pure frequencies do not belong to $\mathbf{L}^2(\mathbb{R})$.

Historical Viewpoint IV

It is interesting to observe that the convolution theorem, telling us that the Fourier transform converts convolution into pointwise multiplication, plays only a little role in Zygmund's monumental treatise [7], while it is important e.g. in probability theory: The characteristic function of the distribution of a sum of two independent random variables (Fourier Stieltjes transform of their distributions) is the product of the individual characteristic functions.

Of course, any LCA (locally compact Abelian) group G carries a translation invariant Haar measure, which in turn is the basis for defining $\left(\mathbf{L}^1(G), \|\cdot\|_1\right)$ and a Fourier transform, using characters. Again there we have Plancherel's theorem and a convolution theorem, but also the Hausdorff-Young inequality, telling us that $\mathcal{F}_G(\mathbf{L}^p(G)) \subset \mathbf{L}^q(\widehat{G})$ for $1 \leq p \leq 2$, with $1/q = 1 - 1/p$.

Historical Viewpoint V

This view-point appears as the culmination of this development. As Andre Weil was presenting it: The realm of LCA-groups is the natural setting of what has been later called (by E. Hewitt) Abstract Harmonic Analysis.
An interesting aspect is the connection to the theory of Banach algebras, specifically to the work of I.M. Gelfand, with the insight that the dual group can be identified with the spectrum of the Banach algebra $\left(\mathbf{L}^1(G), \|\cdot\|_1\right)$, endowed with convolution. In fact, for every non-zero *multiplicative linear functional* σ on

$$(\mathbf{L}^1(G), *, \|\cdot\|_1)$$

there exists a unique point in $\chi \in \widehat{G}$ such that

$$\sigma(f) = \hat{f}(\chi) \quad \forall f \in \mathbf{L}^1(G).$$

Historical Viewpoint VI

With the (academically) significant paper by L. Carleson (1966) concerning the almost everywhere convergence of the inverse Fourier transform the most important results concerning the Fourier transform in the most general setting had been settled, and the theory of maximal functions, Muckenhoupt weights, and function spaces started to gain interest.
For the applications the extension of the Fourier transform to tempered distributions through Laurent Schwartz was probably much more important. Its subsequent intensive use by Lars Hörmander for the study partial differential equations (including the development of microlocal analysis) has changed the field.
Finally one can of course combine the setting to work with distributions over LCA groups or even consider generalized stochastic processes over LCA groups.

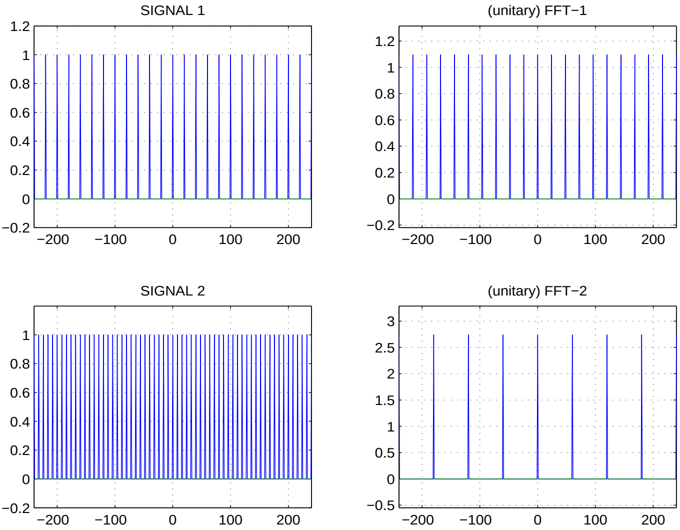
Historical Viewpoint VII

Even for engineers the abstract harmonic analysis view-point has big advantages. It puts the underlying group in the focus, and consequently function on a group can be shifted. It also tells us, that in each case there are the characters, i.e. objects which are invariant with respect to translation, and that the theory of Fourier transform just provides a kind of representation of general functions on the group, specifically those from $\mathbf{L}^2(G)$, as a (possible continuous) superposition of those pure frequencies. Depending on the group, whether it is discrete or "continuous", one- or multidimensional, or just finite (with cyclic shifts then), there is always only one natural way to define convolution and a Fourier transform satisfying the convolution theorem. Thus the DFT (FFT) is the Fourier transform for the cyclic group of unit roots of order N (resp. products of such groups).

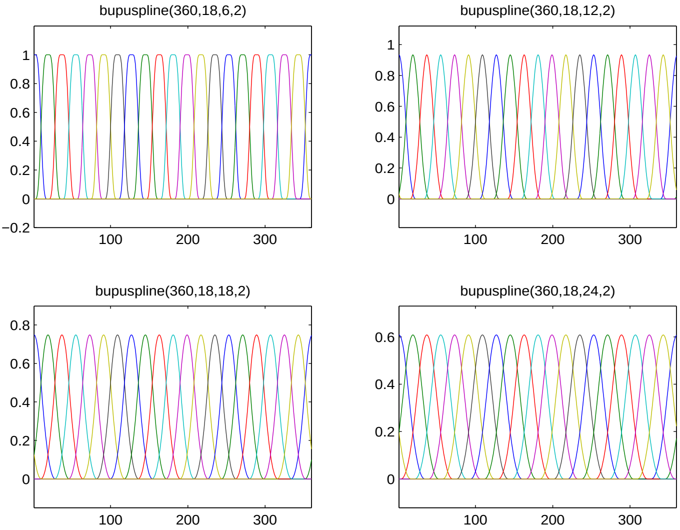
Groups in Fourier Analysis

- Poisson's formula
- Shannon's sampling theorem
- the slice theorem ($>$ tomography)
- cubic splines $>$ spline-type spaces
  (with applications in irregular sampling, see [1]);
- minimal norm interpolation in Sobolev spaces
- (via the Kohn-Nirenberg symbol):
  best approximation of operators by Gabor multipliers;

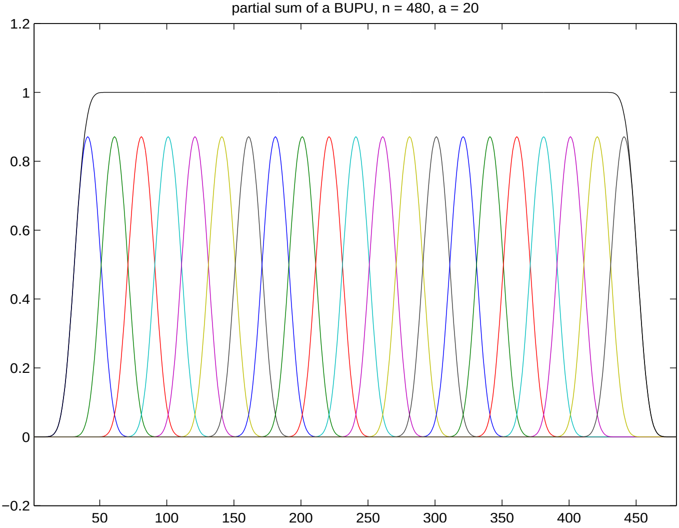
Groups in Fourier Analysis: Illustration



Groups in Fourier Analysis: Illustration



Groups in Fourier Analysis: Illustration



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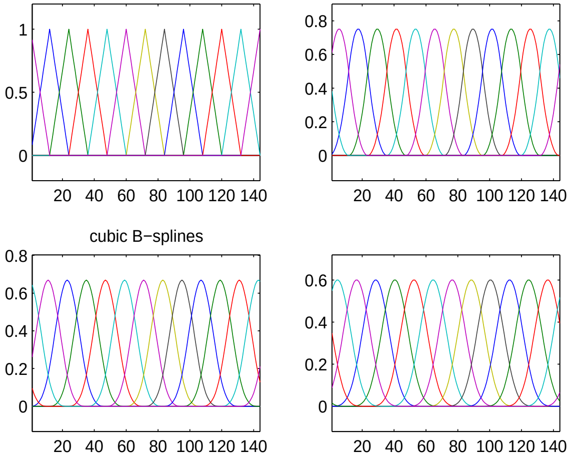

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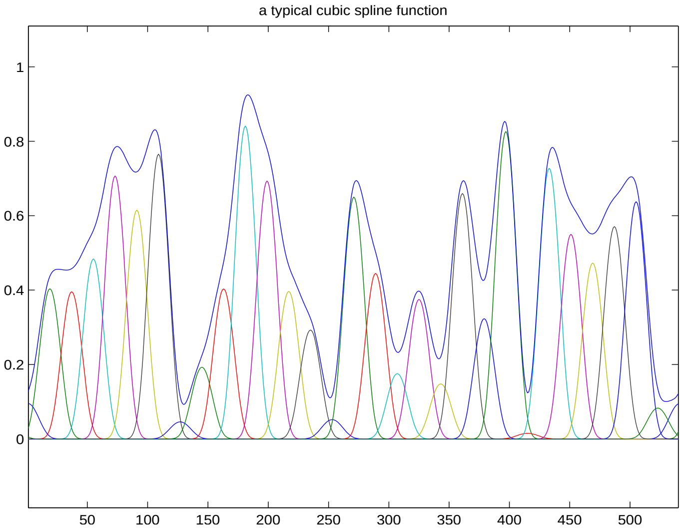
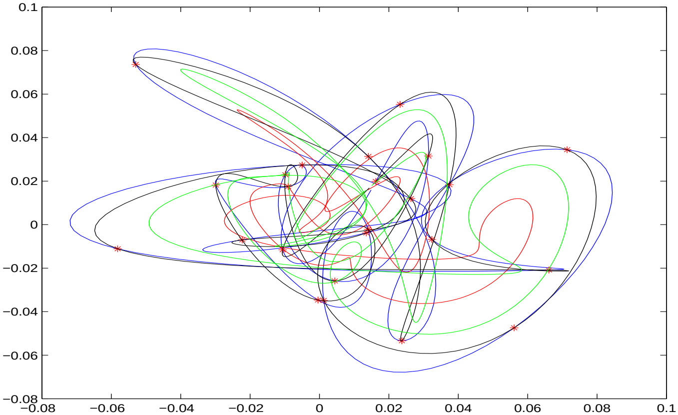


Groups in Fourier Analysis: Illustration



A

Band-limited complex-valued function, several spline- approximations and spline interpolation

Groups in Fourier Analysis: Hints

It maybe worthwhile noticing, that certain practical conventions, which influence e.g. the choice of formats for digital displays, can be also explained using group theory:

Why do we have a standard HiFi sampling rate of $N = 44100$?

First of all because we have to obeye the Nyquist criterion (with some oversamplig), so ca. $40.000 * (1 + \delta)$ would be fine.

Looking for numbers in this range, which are also rich in divisors (resp. subgroups of Z_N) you will find that there are 81 of them, because

$$44100 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$$

has many *different* small prime factors.

Similar arguments apply to most display formats for your electronic devices, just test them! (e.g. 1366×768).

Groups in Fourier Analysis: Problems

Before turning to *non-commutative groups* (where one cannot expect to have a joint diagonalization of all convolution operators, because the do not commute anymore, even if the group is finite!) let us point out some problems with the Fourier transform, from a more technical view-point:

- The Fourier inversion does not apply to all elements in the Fourier algebra $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ (only to $\mathbf{L}^1(\mathbb{R}^d) \cap \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$);
- Even that space is too big for Poisson's formula;
- Later on we will see that some pairs of functions in $\mathbf{L}^2(\mathbb{R}^d)$ the sampled STFT may not belong to $\ell^2(\mathbb{Z}^{2d})$.

So somehow despite their deceptive "naturality" for the use in Fourier Analysis the $\mathbf{L}^p$-spaces are not so useful for the discussion of Fourier analysis.

From Abelian to Non-Ablian Groups

Having reviewed the basic facts about commutative Harmonic Analysis let us turn (still keeping a group-theoretical setup of our mind) to non-commutative.
Note that direct products of commutative groups are again commutative groups and hence have been covered already. In fact, the FFT2 is just the tensor product of the FFT action (on rows and columns of a matrix) and is thus the basis of digital image processing.

Non-Abelian Groups, the two Main examples

One natural way to obtain non-Abelian groups is the combine commutative groups in a different way, namely either as a semi-direct product. This is the way how one can obtain the *affine group*, the so-called ax+b-group. It is composed from the additive group of translations (i.e. $G_1 = (\mathbb{R}, +)$) and the multiplicative group of positive reals with multiplication (i.e. $G_2 = (\mathbb{R}^*_+, \cdot)$), which acts of course as group of automorphisms $x \mapsto ax$ on G_1.

Another way to construct the more-or-less least non-commutative (namely nilpotent, step-1) group is to combine the commutative group of time-shifts with the commutative group of frequency shifts (modulation operators $f \mapsto \chi \cdot f, \ \chi \in \widehat{G}$). From an engineering point we talk about time-frequency (TF-) shifts), which create a *projective representation* of phase space, i.e. of $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ on $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, via unitary mappings.

Non-Abelian Groups

Of course a price has to be paid for the nontrivial commutation relations for TF-shifts (they commute only up to phase factors, which are equal to 1 only under specific conditions). Gabor Analysis, i.e. the idea of representing signals as superposition of TF-shifted copies of a Gabor atom) is a perfect example showing that projective group representations work almost like ordinary group representations, where addition of group elements corresponds exactly the composition of the corresponding unitary operators.

One way out of this defect in the composition law is to add as a third component a torus, i.e. to form $\mathbb{R}^d \times \widehat{\mathbb{R}}^d \times \mathbb{U}$, the *reduced Heisenberg group*. Its representation of this enlarged group is then called the *Schrödinger representation*.

Non-Abelian Groups

In summary: The continuous wavelet transform resp. the spectrogram (short-time or sliding window Fourier transform) are representatives of transforms, obtained from the action of suitable non-Abelian groups on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$.

The theory of **coorbit space** then tried to unify this situation and take an abstract approach, describing both situations in the following way. The setting envisaged is then then following one:

Non-Abelian Groups

In the definition of voice-transform a *unitary and irreducible representation* π of the group $(\mathbf{G}, \cdot)$ on a Hilbert space $\mathcal{H}$ is used, which is supposed to be strongly continuous, i.e. $\pi(x)g$ depends continuously on $x \in \mathbf{G}$ for every $g \in \mathcal{H}$. The *voice transform* of $f \in \mathcal{H}$ generated by the representation π and by the *analyzing window* or atom $g \in \mathcal{H}$ is the (possibly complex-valued) continuous and uniformly bounded (by Cauchy Schwarz) function on $\mathbf{G}$ defined by

$$(V_g f)(x) := \langle f, \pi(x)g \rangle \quad (x \in \mathbf{G},\ f, g \in \mathcal{H}). \tag{1}$$

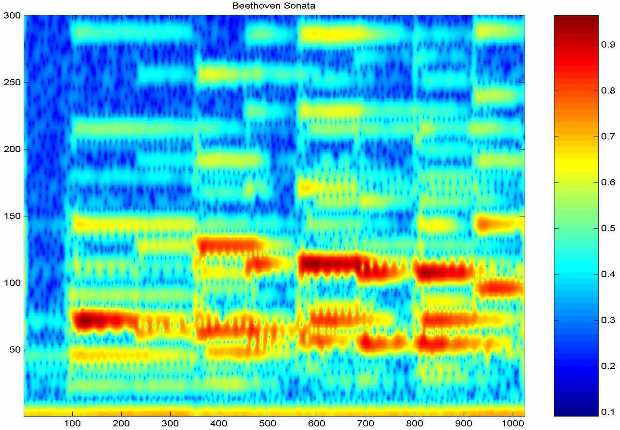
The prototypes are the CWT and the STFT!

Non-Abelian Groups: Coorbit spaces

The insight which arose by comparing the characterization of modulation spaces (at the time of the invention of coorbit spaces only known, with the present view-point to the author) through the behaviour of the STFT of their elements, quite similar to the characterization of the classical function spaces (Triebel-Lizorkin and Besov spaces) through the CWT (continuous wavelet transform), was very helpful in the design of coorbit spaces.
In order to have the most natural and most general setting coorbit spaces are defined as spaces of (e.g. tempered) distributions, with a certain behavior of their voice transform, describing typically their smoothness or decay at infinity. It then easy define coorbit spaces (with respect to π) by

$$\mathcal{C}\mathbf{o}_\pi(\mathbf{Y}) := \{f \mid V_g(f) \in \mathbf{Y}\}.$$

Gabor Analysis: Beethoven Piano Sonata



Usefulness of Gabor Expansions: Audio

Gabor multipliers are just time-variant filterbanks:

Usefulness of Gabor Expandions: Audio

The interpretation of the spectrogram (resp. STFT, or sliding window Fourier transform) makes it natural to use it for various purposes in (musical) audio-processing. Note that for this purpose one does not have to store only the absolute values of the STFT ($|V_g(f)|^2$ really can be viewed as energy distribution, but not in the pointwise sense) but also the phase.
Either one wants to do denoising, knowing in which are in phase space the "bird was making extra noise in the open air concert".
But for concrete applications one may want to use not just the standard Gaussian atoms or B-splines as atoms, but maybe generalized i.e. chirped Gaussians, where the ambiguity function $V_g g$ has elliptic contour lines, and correspondingly the corresponding Gabor expansions use sheared lattices.
On the other hand MP3 compression used by everybody can be seen as STFT-thresholding to compress audio data.

Abstract Coorbit Space Theory

$$\mathcal{C}\mathbf{o}_\pi(\mathbf{Y}) := \{f \mid V_g(f) \in \mathbf{Y}\}.$$

To make those Banach spaces we need a few basic facts:

- Does the definition depend on the choice of the analyzing window? (of course the same condition on g should apply for the full range of $p \in [1, \infty]$);
- What about other (solid) function spaces on **G**, instead of $\mathbf{L}^p(G)$, for example weighted mixed-norm spaces?
- Motivated by the theory of coherent states one may ask: Can one build the elements of those coorbit spaces from the Orbit of an atom under the group action?

Non-Abelian Groups, Different Types

The technicalities behind this abstract approach my look a bit complicated, but this should not be a surprise given the fact that a variety of special cases already requires non-trivial arguments.
For example, one can attribute the so-called *admissibility condition* to be imposed on the analyzing wavelet to the fact that the ax+b-group is *not unimodular*, hence convolution from the right by some $\mathbf{L}^1(G)$-function does not necessarily define a bounded operator on $\mathbf{L}^2(G)$ resp. on other $\mathbf{L}^p$-spaces over $\mathbf{G}$.
In contrast, the reduced Heisenberg group relevant for *modulation spaces* is a nilpotent and unimodular group, even an [IN]-group, i.e. having compact invariant neighborhoods Q of the identity, with $yQ = Qy, \forall y$, e.g. $Q = [-\delta, \delta]^2 \times \mathbb{U}$.

Non-Abelian Groups, Function Spaces

One of the possible technicalities observed by the novice interested in the coorbit spaces may be the degree of generality concerning the space $(\mathbf{Y}, \|\cdot\|_{\mathbf{Y}})$ (on $\mathbf{G}$) used in the construction.
It turns out that it is sufficient to assume that **Y** is solid (i.e. only "seize matters": Given a function in **Y** any other function which is dominated - in terms of absolute values - also belongs to the spaces and has smaller norm) and that is translation invariant.
Due to the covariance properties of the voice-transform this implies that with a given g also any linear combination of windows of the form $g_1 = \sum c_i \pi(\lambda_i) g$ can be used, and those atoms $h \in \mathcal{H}$ such that $V_h(f)$ is dominated by $|V_{g_1} f|$, and this is enough to establish independence of the definition from g for a large collection of atoms g.

Non-Abelian Groups, Mixed Norm Spaces

The most popular function spaces to be used are mixed norm spaces with weights. On our groups those weights are typically related to polynomial growth in the "smoothness direction". In fact, such a weight will grow like r^s for $r > 0$ in the direction of scale space, and the (homogeneous) Besov spaces are characterized as coorbit spaces for spaces with respect to the natural representation of the ax+b group on $\mathbf{L}^2(\mathbb{R}^d)$. All wavelets with sufficiently many vanishing moments and good polynomial decay at infinity will qualify.

In the design of *modulation spaces* care has been taken to design them in a quite similar fashion. This is why the "classical modulation spaces" $\mathbf{M}^s_{p,q}(\mathbb{R}^d)$ have the same parameters as the Besov family $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$ (and quite comparable properties).

Atomic Characterization via Series Expansions

One of the key features of Coorbit Theory it the fact that it allows to answer two more or less equivalent questions:

1. Can one recover a voice transform from it sufficiently dense samples at points $(x_i)_{i \in I}$, i.e. from the values $(V_g f(x_i))_{i \in I}$?
2. Can on expand all the elements in $\mathbf{Co}_\pi(\mathbf{Y})$ as a non-orthogonal, but unconditionally convergent sum of the form

$$f = \sum_i c_i \pi(x_i) g,$$

with coefficients coming from a disrete version $\mathbf{Y}_d$ of the function spaces $(\mathbf{Y}, \|\cdot\|_\mathbf{Y})$.

In a linear algebra situation one would say: YES, if the family $(\pi(x_i)g)$ generates the spaces! (here: is a Banach frame!)

Atomic Characterization via Series Expansions II

It is plausible that one may have to make still further assumptions on the atoms (slightly stronger than the one for independent characterization of the coorbit spaces), at least in some cases. Slightly more surprising this *extra assumption are automatically satisfied in the case of modulation spaces!*. Here we have the pleasant situation (we restrict our attention now to the unweighted case, i.e. to the modulation spaces $\mathbf{M}^{p,q}(\mathbb{R}^d)$), that the atom belongs to the minimal space in this family, namely $M_0^{1,1}(\mathbb{R}^d)$ or $\mathbf{M}^1(\mathbb{R}^d) = \mathbf{S}_0(\mathbb{R}^d)$, the minimal Segal algebra with respect to the property of being isometrically invariant under TF-shifts, i.e. the Schrödinger representation. It is the [IN]-property of the acting group together with convolution relations for Wiener amalgam spaces which allows to make this conclusion.

Atomic Characterization via Series Expansions III

The final result can then be summarized as follows:

Theorem

Given a family of coorbit spaces with a uniform control on the norm of the operators $\pi(x)$, acting on them [a] one can guarantee that for all sufficiently "decent" atoms g (with $V_g g \in \mathbf{L}^1_w(G)$) the answer to the above questions (in fact to both of them) is positive, for any sufficiently dense family $(x_i)_{i \in I}$, meaning that a covering of "balls centered at those points is possible.

[a] Uniformity meant for each fixed $x \in \mathbf{G}$ with respect to the family, by a submultiplicative weight $w(x)$ on $\mathbf{G}$.

Atomic Characterization via Series Expansions IV

For a given atom or window g and a family of spaces, "morally" the result can be compared with a kind of Nyquist criterion:
If there the density is high enough the voice-transforms show uniform "smoothness" and stable recovery is possible, in the sense that whenever samples from a function in $f \in \mathbf{Co}_{\pi}(\mathbf{Y})$ the recovery (typically iterative) will be convergent in this space, at a geometric rate. The higher the "sampling rate" resp. the quality of g the better the guaranteed rate of convergence.
If we take the final result we can say: there is a linear mapping from the space of samples back to the coorbit space, which in fact is robust with respect to jitter error or slightly incomplete information with respect to the window g or additive noise (if it is small in $\mathbf{Y}_d$).

Atomic Characterization via Series Expansions V

If we go to back to the Hilbert space case, typically $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$, then the situation is more or less comparable with the problem of a MNLSQ-problem. For a given family $(x_i)_{i \in I}$ the set of samples is a closed subspace of $\ell^2(I)$. Hence for any sequence in $\ell^2(I)$ (be it in the range of the mapping $f \mapsto (V_g f(x_i))_{i \in I}$ or not) there exists a projection on that space, which is in a one-to-one correspondence to the original space, i.e. $\mathcal{H} = \mathbf{Co}(\mathbf{L}^2(G))$.
While the most natural left inverse to the sampling operator is the identification of the "least norm" solution to the modified problem (consistent system of linear equations), resp. to Moore-Penrose pseudo-inverse, such a thing does not exist in the general setting and therefore constructive, iterative methods to solve the problem "in the same spirit" are a good working solution. It is also important that the method of recovery does not require that the user knows which space $\mathbf{Y}$ is involved.

The Role of Groups for Gabor Expansions

Let us recall that the theory of Gabor expansions is concerned with the recovery of functions from samples of the STFT (Short-Time Fourier Transform). For simplicity let us assume that we have signals on the real line, and consequently $V_g f$ is a function over phase space, resp. the complex plane, of you want.
Any non-zero Schwartz-function g will be fine for practically all relevant modulation spaces (including the Shubin classes $Q_s(\mathbb{R}^d)$).
If the sampling is coming from a lattice, i.e. a discrete subgroup Λ, which is dense enough, we know (see above) that stable recovery is possible.
In the classical case on takes $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$, and then it is enough that $max(a, b) \leq \delta_0$, for a suitable value of $\delta_0 > 0$.
For the Gauss function (suggested by D. Gabor) one even knows that $ab < 1$ is a sufficient and necessary condition.

The Role of Groups for Gabor Expansions II

The advantage of the *regular case*, i.e. the case that the sampling is is not just a general, sufficiently dense set, but even a discrete subgroup of phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ (taken as an additive group, isomorphic to $\mathbb{R}^{2d}$) is the fact that one can write the recovery in a Shannon-like form (convergence in e.g. $\mathcal{H}$:

$$V_g f = \sum_{\lambda \in \Lambda} V_g f(\lambda) \pi(\lambda) \tilde{g},$$

for a well choosen "dual Gabor window $\tilde{g}$, which can be obtained by solving the positive definite equation $S_{g,\Lambda}\tilde{g} = g$ for $\tilde{g}$, where

$$S_{g,\Lambda}(f) = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g,$$

the so-called Gabor frame operator (which is pos. definite if and only if the family is a Gabor frame).

The Role of Groups for Gabor Expansions III

Depending on the density of the lattice there are many alternative options to chose instead of the *canonical dual window* (which is the solution of the problem with minimal norm, or also the solution closest to g in the $\mathbf{L}^2(\mathbb{R}^d)$-sense) alternative dual windows. This is comparable to the situation of regular sampling with some redundancy, where the not-so-localized SINC function can be replaced by a better concentrated function in the recovery. Typical questions arising in this context are then

1. For which given pairs (g, Λ) can one guarantee reconstruction (i.e. when is the family $(\pi(\lambda)g)_{\lambda \in \Lambda}$ a Gabor frame)
2. What can be said about the condition number of the frame operator $S_{g,\Lambda}$?
3. How well localized is the expansion, which has a lot to do with the concentration of $V_{\tilde{g}}\tilde{g}$ around the origin.

The Role of Groups for Gabor Expansions IV

Answers to the question of localization, at least at a qualitative level arise from so-called localization theory, introduced by K. Gröchenig: One can show, nowadays mostly using Banach algebra methods, that whenever the Gabor frame operator is invertible on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ and the atom g is well concentrated (in both time and frequency), then the same can be said for the canonical dual $\tilde{g}$.

Technically such results are very much in the spirit of Wiener's inversion theorem, showing that the pointwise inverse of a function having an absolutely convergent Fourier series has the same property.

Unfortunately there is typically now control on the constants involved in the most general case, but one can invoke the Neumann series argument in many practically relevant case.

The Role of Groups for Gabor Expansions V

When it comes to analyze the quality of (regular) Gabor frames one can define various measure of quality of a Gabor family, such as

- the condition number of the Gabor frame operator;
- the concentration of the dual atom;
- the $\mathbf{S}_0$-norm of the dual window;
- the quotient between minimal covering and maximal paving radius (for Gauss functions, resp. circles).

Fortunately systematic experiments have resulted in the insight, that the relative ranking is not very sensitive with respect to the choice of the criterion. Furthermore, the most hexagonal like lattices are the best one for the (discretized) Gauss function.

The Role of Groups for Gabor Expansions VI

The studies undertaken in this directions had again to do with investigations concerning the signal size N that one should use (again: N should be rich of divisors, because in this case also the number of subgroups of $\mathbb{Z}_N \times \mathbb{Z}_N$ is large!)
For a given signal size and any possible divisor of N^2 (i.e. any possible cardinality of groups) we are now able to produce an exhausting list of subgroup of equal redundancy (Gabor frames with localized windows have to have $> N$ elements) and use our fast algorithms in order to determine the figures of merit mentioned above. As it turns out one can find quite good Gabor families at redundancy in the range of $[1.2, 1.75]$, if the lattice and the window match well.

The Role of Groups for Gabor Expansions VII

A slightly different criterion was considered in a recent master thesis (Kirian Doepfner, Vienna, 2012), who investigated the ability, to approximate (in the Hilbert Schmidt sense) a given slowly-varying channel by a Gabor multipier. Such a time-variant filter can be realized typically as an STFT-multiplier. Here it is better to look out for tight Gabor families, i.e. atoms h with

$$f = \sum_{\lambda \in \Lambda} \langle f, h_\lambda \rangle h_\lambda,$$

and one can expect that very smooth STFT-multipliers can be well approximated by Gabor multipliers.
As it turned out the problem can be reformulated as an approximation problem of the (smooth) STFT-multiplier (upper or Anti-Wick symbol) by shifted bump-functions along the lattice used, obtained as Kohn-Nirenberg symbols of the rank 1 operator $f \mapsto \langle f, h \rangle h$. Redundancies around 4 are OK.

What made Wavelets so quickly important?

Why did wavelets have an immediate impact in the field of Calderon Zygmund operators?
An a posteriori explanation from today's point of view can the based on the following list of facts:

- Wavelet systems are good bases for the function spaces under consideration (which are in fat the right ones), namely $\mathbf{L}^p$-spaces, and Besov or Triebel-Lizorkin spaces, but also the real Hardy space and **BMO**;
- CZ-operators have a matrix representation which is strongly diagonally concentrated, i.e. good wavelets are joint approximate eigenvalues for CZ-operators!

The Role of Groups for Gabor Expansions VIII

The corresponding problem with respect to the STFT and Gabor system had not been investigated equally well in the early days of wavelet theory, but meanwhile we know (BECAUSE WE HAVE CORRESPONDING PATENTS) that Gabor building blocks are good joint approximate eigenvectors for slowly variant channels as they arise in mobile communication (due to multi-path propagation and limited Doppler effects, e.g. in car-to-car communication). The corresponding matrix algebras are by now even better understood, because their index set are discrete, Abelian groups, even though the good Gabor families are no bases, but just frames and therefore even invertible operators are represented by infinite, non-invertible matrices (so properties of the pseudo-inverse come into play).

The Role of Groups for Gabor Expansions IX

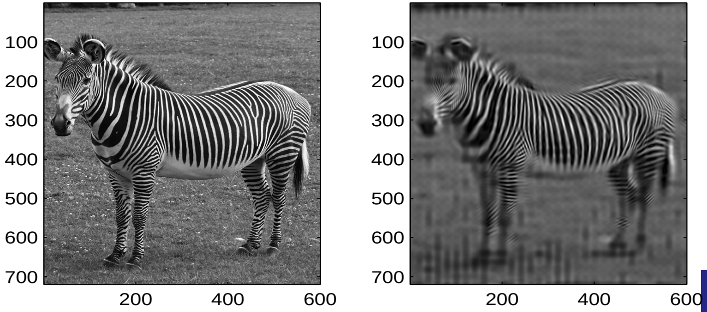
In Gabor analysis localization theory is well established, which means that the dual frame of a Gabor family generated from an atom which is well localized has similar properties (expressed in terms of decay of the Gabor coefficients for any nice Gabor frame). The same claim (in fact equivalent due to the Ron-Shen duality) can be made for Gaborian Riesz bases.
There are important consequences in the theory of pseudo-differential and Fourier integral operators, where e.g. Sjoestrand's class has now received a new interpretation in terms of modulation spaces.

The Role of Groups for Gabor Expansions X

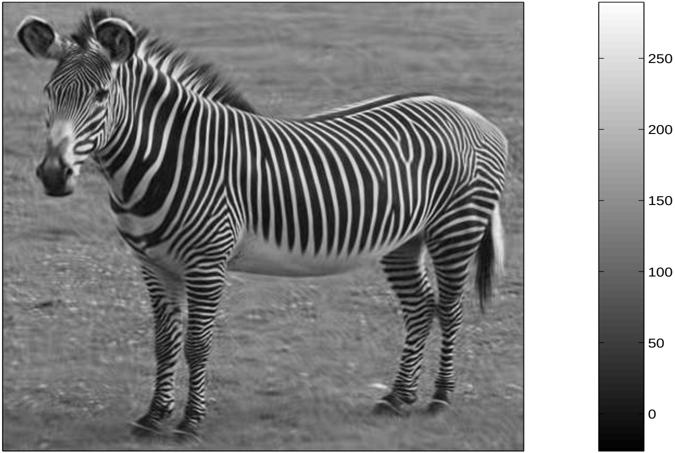
The usefulness of modulation spaces, in particular of the Segal algebra $\mathbf{S}_0(\mathbb{R}^d) = M_0^{1,1}(\mathbb{R}^d)$ (Feichtinger's algebra) and its dual are not limited just to time-frequency analysis, but are playing more and more an important role in classical Fourier analysis, especially when one is interested in the constructive realization of algorithms.

Using the concept of Banach Gelfand Triples, specifically $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$. It can be used to describe the Fourier transform as "the mapping identifying pure frequencies with Dirac measures", but also the spreading representation (the symplectic FT of the Kohn-Nirenberg symbol mapping) as the mapping identifying pure TF-shifts with Dirac measures in phase space (see [2]).

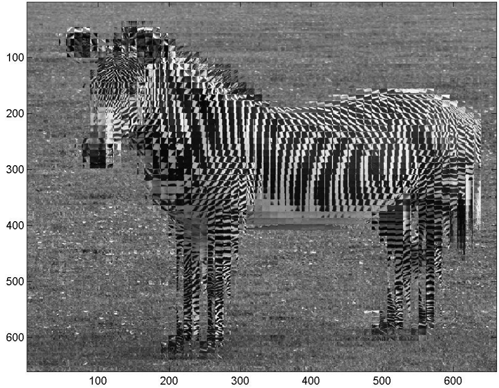
Applications to Image Processing



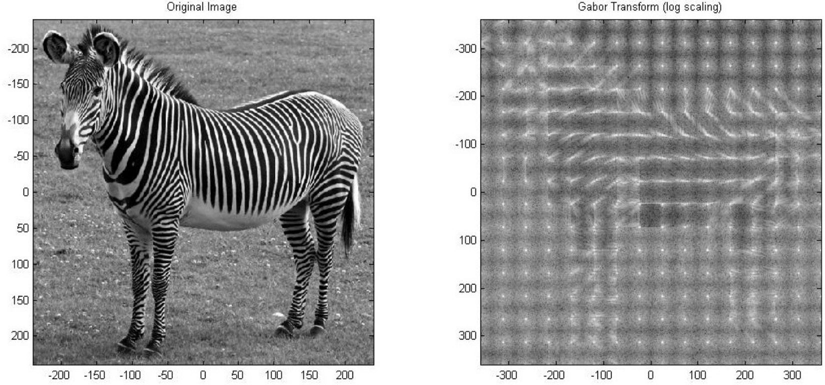
Applications to Image Processing 2



Applications to Image Processing



The Gabor Coefficients of the Zebra



CONCLUSION

In summary we have seen that

- group theoretical methods allow for a unified treatment;
- group theoretical methods yield efficient algorithms
- Banach spaces of functions and distributions often have their natural expanding system (of coherent-like atoms);
- there are more groups to be studied besides the ax+b-group and the Heisenberg group, e.g. shearlets
- function spaces are not only of academic interest but are relevant for the validity and efficiency of computations carried out with finite vectors.

References

A long list of references is given in the Conference Proceedings. Let us just mention the main paper on *coorbit spaces* together with K. Gröchenig ([4]), or the paper [5] where he formally introduced the concept of Banach frames.

The most convenient source to learn about the basic facts concerning modulation spaces is [6], while the historical paper (finished in 1983) was reproduced in [3].

For Banach Gelfand Triples view [2].

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