NuHAG Numerical Harmonic Analysis Group

Gabor Analysis in 2D: From Theory to Applications

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Standard Approach to Gabor Analysis

The standard introduction to Gabor Analysis would be to recall the claim made by Denis Gabor in his seminal paper of 1946 ([6]), where he conjectured that "every signal can be expanded in a unique way as a double sum (now called Gabor series expansions) of time-frequency shifted Gauss-functions, shifted along the unit lattice $\mathbb{Z} \times \mathbb{Z}$, i.e. of atoms (from the non-orthogonal) system of functions

$$g_{k,n}(t) := e^{2\pi ikt} \cdot g_0(t-n); \;\; k, n \in \mathbb{Z}.$$

A modern version would be: Every signal can be written as an infinite series of such Gabor atoms, resp. a DSP can be programmed to produce an arbitrary sound form elementary sound-atoms, with Gaussian envelopes, played at a fixed time-frequency-lattice.

Standard Approach to Gabor Analysis II

Of course such a claim reminds us of Fourier's statement that *every periodic function* $f(t)$ has a representation as the infinite sum of pure frequencies, with uniquely determined coefficients known as Fourier coefficients. We also know that it took decades unit mathematicians had developed proper tools (including set theory and Lebesgue's integration theory) to give this claim a meaning, finally with the insight that in some cases a "distributional interpretation" is most appropriate.
We do not have the time here to discuss the historical approach to Gabor analysis, but in short: he was slightly too optimistic concerning the possibilities of such a representation. On the other hand it is clear by now that redundant and stable representations are possible for lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, if $ab < 1$.

Standard Approach to Gabor Analysis III

The fact, that even after replacing the Gauss function by any possible other well-localized and smooth functions such a stable and unique representation is never possible for $ab = 1$ is meanwhile known as Balian-Low theorem.

The case $ab > 1$ (undersampling case, in which one has a stable Riesz basis for a closed subspace of $(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2)$) is not uninteresting, because it can be used (in principle) for wireless communication, where TF-shifted Gaussians are considered as *approximate eigenvectors to slowly varying channels*.

Finally one has so-called Gabor frames for the case $ab < 1$, which guarantee stable, but non-unique signal representations, not only for $\mathbf{L}^2$-functions, but much more general distributions, and has become a prototype for abstract frame theory.

Standard Approach to Gabor Analysis IV

The *usual way* of introducing Gabor frames would be to define them as frames within the (infinite dimensional) Hilbert space $\mathcal{H} = \left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, and then to show that such system have a lot of properties that even make the computation of the *minimal norm coefficients* possible, even in a numerically efficient way. Sometimes an easy way of computing "suitable coefficients" can be carried out in a *painless way* ([2], practically used within MP3), but the non-orthogonality appears to produce a lot of difficulties, making this signal representation not so useful as one may think.

!! Wrong!

However, we will choose for this presentation an approach which is starting with examples and tries to build the theory, starting from a Linear Algebra view-point!

General Aspects of Gabor Analysis

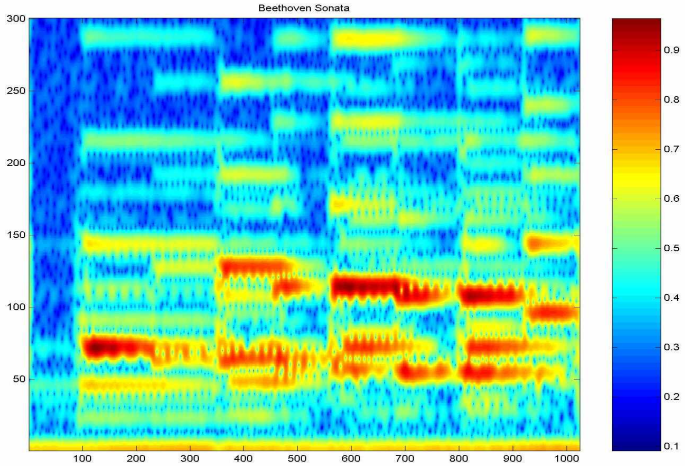
- **Linear Aspects**: dual frame = pseudo-inverse, Gabor Riesz basis: biorthogonal system
- **Algebraic Aspects**: lattice (Abelian Groups) act on a Hilbert space of signals via some *projective* representation, *commutation property* of Gabor frame operator, i.e. $[\pi(\lambda), S] == 0$, imply specific sparsity structure of S.
- **Functional Analytic features**: Convergence of double sums (sums over lattice) appear to be complicated (Bessel condition, unconditional convergence, modulation spaces, etc.)
- Specific properties of the acting **Weyl-Heisenberg group** (!phase factors) or the validity of Poisson's formula for the symplectic Fourier transform (in contrast to wavelets);

Overview

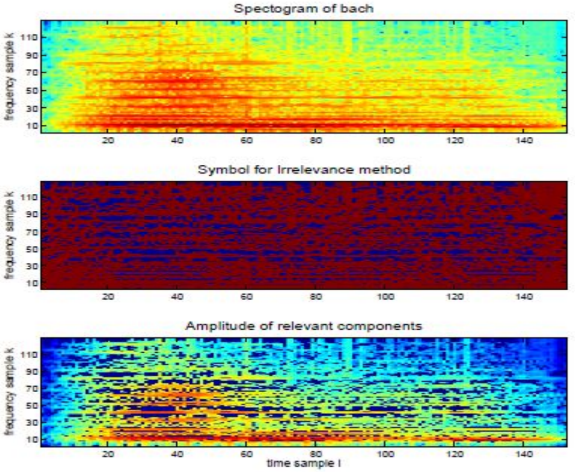
- Concepts of Time-Frequency Analysis (music, audio);
- The idea of Gabor expansions (discretization);
- The Linear Algebra behind Gabor Analysis;
- Regular (and non-regular) Gabor families;
- Duality theory (Frames and Riesz bases);
- The corresponding 2D version;
- Function spaces, Banach Gelfand Triples;
- Approximation of continuous setting by finite groups;

LET US START WITH AN STX-emo (ARI, OEAW)
Downloadable from the ARI website! (free demo-version)

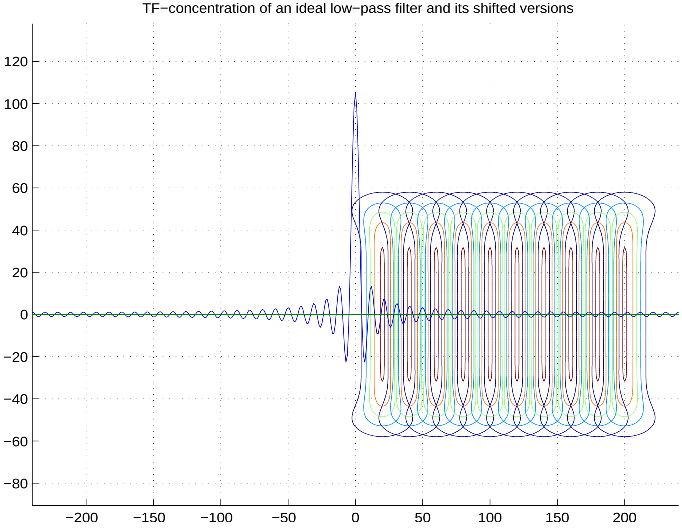
A Typical Musical STFT



Relevance of STFT for MP3 Coding



TF-concentration of shifted SINC-functions



The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

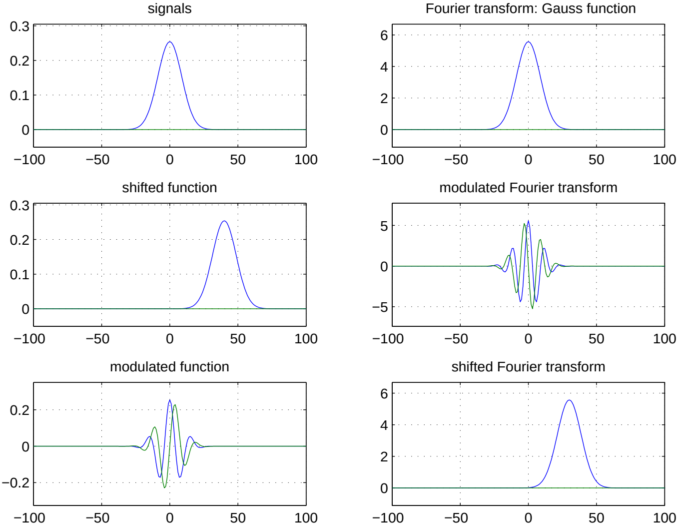
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

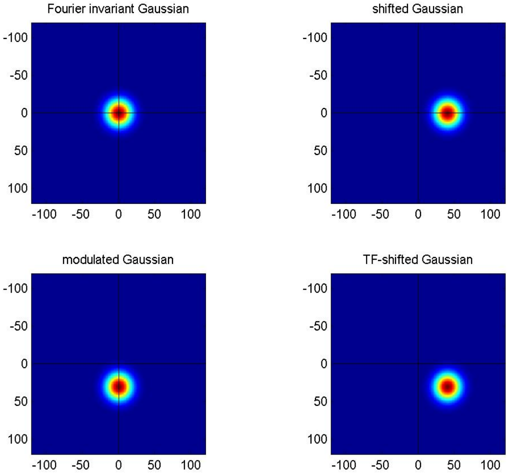
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

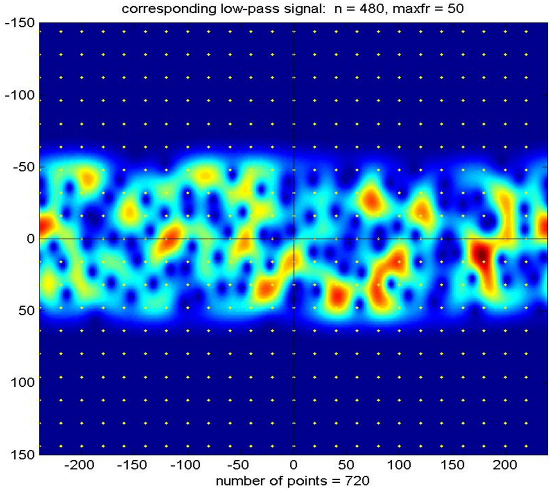
Time-Frequency Shift Operators on Signals



Time Frequency Shifts in Spectrogram



Spectrogram of a Lowpass Signal



Recalling concepts from linear algebra

We *believe that we know* how to translate concepts from linear algebra into the infinite dimensional setting (using finite sets): An indexed family $(g_i)_{i \in I}$ (in a finite of infinite dimensional vector space) is called linear independent if for every finite subset $F \subset I$ one has:

$$\sum_{i \in F} c_i g_i = 0 \Rightarrow c_i = 0 \quad \forall i \in F.$$

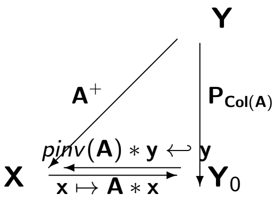
In this sense ANY family $M_{kb} T_{na} g_0$ IS linear independent, even if ab is small!

The concept of a "generating system" appears to be properly generalized by the concept of a *total set*: A family $(g_i)_{i \in I}$ is total in $\mathcal{H}$ if every $f \in \mathcal{H}$ can be arbitrarily well approximated by finite linear combinations (exactly for $ab \leq 1$):

$$\|f - \sum_{i \in F} c_i g_i\|_{\mathcal{H}} < \varepsilon.$$

Matrix Multiplication: linear independent columns:

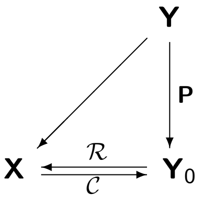
It is better to view a collection of vectors as a matrix, resp. identify it with the linear mapping induced by the corresponding matrix. Then our questions concern injectivity resp. surjectivity of such a mapping:



Comment: For linear independent columns the synthesis map $\mathbf{x} \mapsto \mathbf{A} * \mathbf{x}$ has a trivial nullspace, hence the row-space $Row(\mathbf{A})$ equals all of $\mathbb{R}^n$, and the matrix $\mathbf{A}$ has maximal rank.

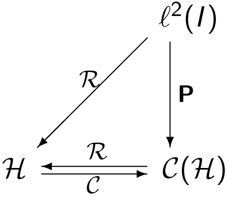
Fames and Riesz Bases: the Diagram

If on the other hand we have $m \geq n$ vectors in $\mathbb{R}^n$ in a matrix F (with columns (f_i)) then the mapping $\mathcal{C} : \mathbf{x} \to F' * \mathbf{x} = (\langle \mathbf{x}, f_i \rangle)$ is injective if and only if the family (f_i) generates all of $\mathbb{R}^m$.
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y} = \mathbb{R}^m$ onto the range $\mathbf{Y}_0$ (the row-space of $\mathbf{A}$) of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:

The situation then can be generalized to Hilbert spaces, with the target space now playing the role of $\mathbb{R}^m$. Now it is not true anymore that every total family (the closed linear span is all of $\mathcal{H}$) has a closed range within $(\ell^2(I), \|\cdot\|_2)$, but we may this fact into a definition:



The frame diagram for Hilbert spaces

The usual way of defining a so-called **frame** (you may call it a stable generating system) is then to require that the following pair of inequalities holds true:
There exist positive constants $A, B > 0$ such that one has:

$$A\|f\|_{\mathcal{H}}^2 \leq \sum_{i\in I} |\langle f, f_i\rangle|^2 \leq B\|f\|_{\mathcal{H}} \quad \forall f \in \mathcal{H}.$$

Similarly the family (g_j) is a **Riesz basic sequence** (Riesz basis for its closed linear span) if for some $C, D > 0$ on has

$$C\|\mathbf{c}\|_{ltI}^2 \leq \|\sum_{j\in I} c_j g_j k\|_{\mathcal{H}} \leq D\|\mathbf{c}\|_{\ell^2(I)}^2 \quad \forall \mathbf{c} \in \ell^2(I).$$

Some terminology concerning Gabor analysis

The fact, that the mapping $\lambda = (t, \omega) \mapsto \pi(\lambda) = M_\omega T_t$ defines a *projective representation* of the Abelian group $\mathbf{G} \times \widehat{\mathbf{G}}$ on the Hilbert space $\mathcal{H} = \mathbf{L}^2(G)$ makes it interesting to choose as index sets for disrete Gabor families not just a collection of well-spread points, but in particular lattices satisfying some density requirements.

We call a Gabor family $(g_\lambda)_{\lambda \in \Lambda}$ a **regular Gabor family** if the index set Λ is a discrete subgroup of $\mathbf{G} \times \widehat{\mathbf{G}}$. We call a regular Gabor family a **separable** one, of $\Lambda = \Lambda_1 \times \Lambda_2$, where $\Lambda_1 \lhd \mathbf{G}$ and $\Lambda_2 \lhd \widehat{\mathbf{G}}$, e.g. $a\mathbb{Z}^d \times b\mathbb{Z}^d \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
Regularity is sufficient to guarantee that the so-called Gabor frame operator (which is pos. definite in the frame case)

$$S = S_{g,\Lambda} : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g$$

commutes with any $\pi(\lambda), \lambda \in \Lambda$.

Some terminology concerning Gabor analysis

The combination of these observations tells us a lot about the rich (sparse, but also algebraic) structure of the problem.
First of all it is obvious that one has for general frames

$$f = S^{-1}S(f) = f = S^{-1}S(f),$$

or writing $\tilde{g} := S^{-1}(g)$ for the (canonical) **dual window**:

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle S^{-1}(\pi(\lambda)g) = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)\tilde{g},$$

respectively the **atomic composition** of f using $(\pi(\lambda)g)_{\lambda \in \Lambda}$:

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)\tilde{g} \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda} STFT(f, \tilde{g})(\lambda)\pi(\lambda)g.$$

Some terminology concerning Gabor analysis

Sometimes the assymetry in the role of g resp. $\tilde{g}$ is a disadvantage (e.g. of one wants to build self-adjoint operators from real-valued Gabor multipliers, in a kind of discrete variant of the Anti-Wick calculus) and for this reason one sometimes prefers to make use of the **canonical tight window** $h := S^{-1/2}(g)$, for which one has the representation

$$f = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) h \rangle \pi(\lambda) h;$$

which in fact *looks like an orthonormal expansion* although it is just a tight frame (hence in fact the orthonormal projection of some orthonormal basis in an ambient Hilbert space).

Typical **Gabor multipliers** are then of the form

$$f \mapsto \sum_{\lambda \in \Lambda} m(\lambda) \langle f, \pi(\lambda) h \rangle \pi(\lambda) h.$$

Wavelets versus Gabor Expansions

... while wavelet theory was gaining immediately recognition due to results on Calderon-Zygmund type operators the corresponding problem with respect to the STFT and Gabor system had not been investigated equally well in the early days of wavelet theory, but meanwhile we know (BECAUSE WE HAVE CORRESPONDING PATENTS) that Gabor building blocks are good joint approximate eigenvectors for slowly variant channels as they arise in mobile communication (due to multi-path propagation and limited Doppler effects, e.g. in car-to-car communication).
The corresponding matrix algebras are by now even better understood, because their index set are discrete, Abelian groups, even though the good Gabor families are no bases, but just frames and therefore even invertible operators are represented by infinite, non-invertible matrices (so properties of the pseudo-inverse come into play).

Recent Developments in Gabor Analysis

In Gabor analysis **localization theory** is well established, which means that the dual frame of a Gabor family generated from an atom which is well localized has similar properties (expressed in terms of decay of the Gabor coefficients for any nice Gabor frame). The same claim (in fact equivalent due to the Ron-Shen duality) can be made for Gaborian Riesz bases.

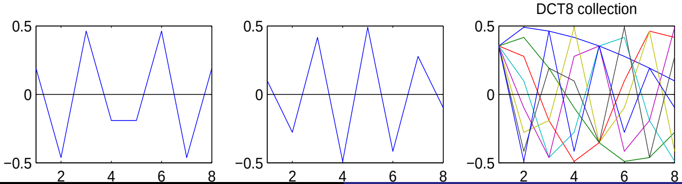
There are important consequences in the theory of pseudo-differential and Fourier integral operators, where e.g. Sjoestrand's class has now received a new interpretation in terms of modulation spaces. Although introduced already 1983 by the author they received only attention through the book [7] (see also [3]).

Modulation Spaces and Gabor Expansions

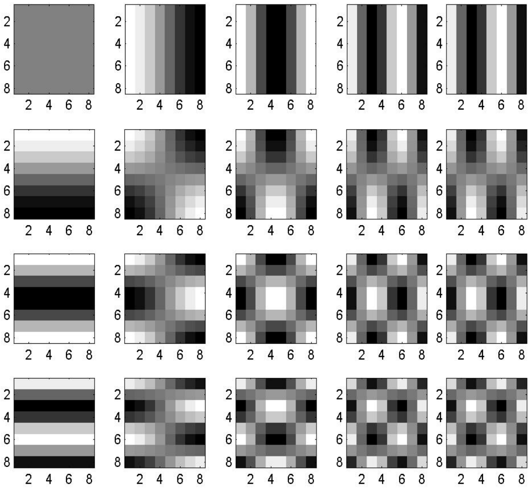
The usefulness of modulation spaces, in particular of the Segal algebra $\mathbf{S}_0(\mathbb{R}^d) = M_0^{1,1}(\mathbb{R}^d)$ (Feichtinger's algebra) and its dual are not limited just to time-frequency analysis, but are playing more and more an important role in classical Fourier analysis, especially when one is interested in the constructive realization of algorithms.

Using the concept of Banach Gelfand Triples, specifically $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$. It can be used to describe the Fourier transform as "the mapping identifying pure frequencies with Dirac measures", but also the spreading representation (the symplectic FT of the Kohn-Nirenberg symbol mapping) as the mapping identifying pure TF-shifts with Dirac measures in phase space (see [1]).

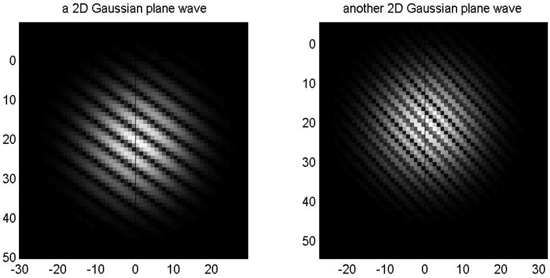
DCT building blocks



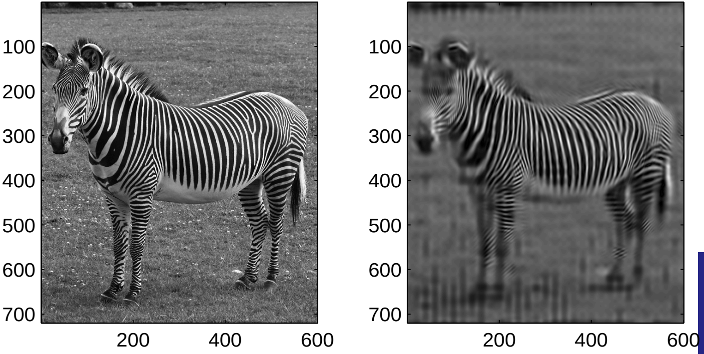
DCT2 building blocks



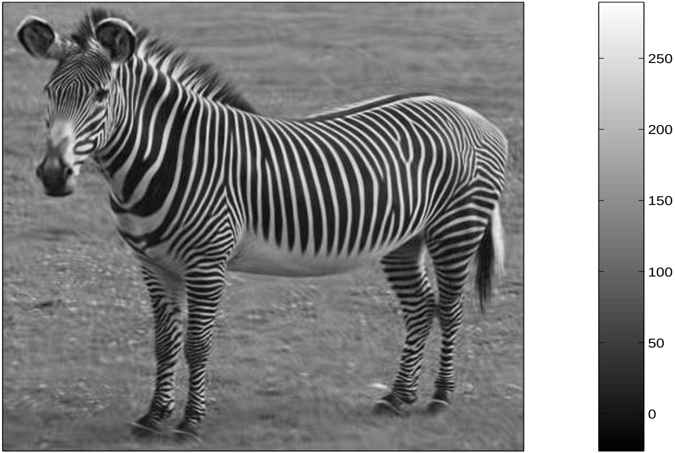
Building blocks Image Processing



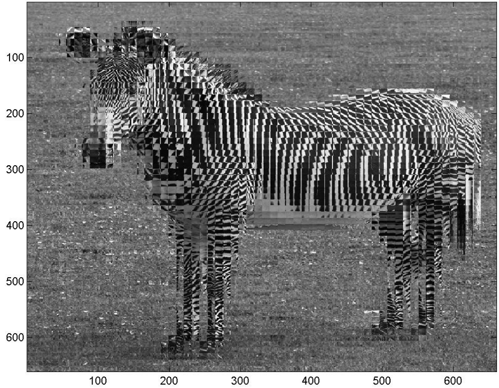
Applications to Image Processing



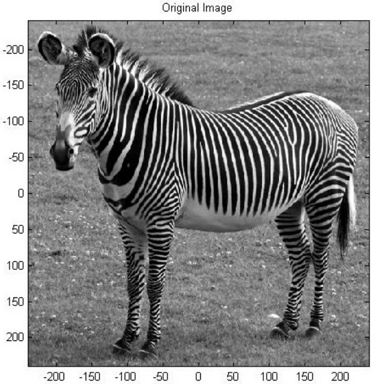
Applications to Image Processing 2

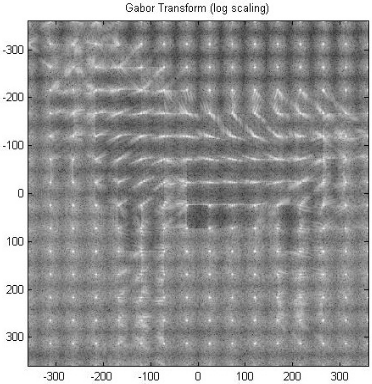


Applications to Image Processing



The Gabor Coefficients of the Zebra





CONCLUSION

I hope that the talk (and the slides, even if not all of the have been presented) convey the main spirit of Gabor Analysis:

- **Gabor expansions**, if properly done, provide slightly redundant, but well structured (obviously non-orthogonal) frame expansions for a big class of signals;
- **Gabor multipliers** are in fact a representative class of slowly varying time-variant filters (and their inversion is non-trivial);
- The decay properties of the **Gabor coefficients** indicate the concentration of energy in phase space and allow to characterize membership of distributions in certain modulation spaces;
- Such function spaces also allow to address questions of **approximation** of continuous operations by finite ones;

Not addressed here, hints to literature

There are of course many aspects that could not be addressed in a survey talk like this. The setting of the Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ is developed broadly in [1].
Finally, the articles [5] indicates how to reach an understanding of Gabor analysis coming from the **linear algebra** side (bottom up).
The algebraic side of Gabor Analysis is explained in the paper [4], dealing with general **finite Abelian groups**. Of course FFT2 is interpreting a pixel image of size $M \times N$ as a function on the finite group $\mathbb{Z}_M \times \mathbb{Z}_N$.
All the papers are downloadable from the NuHAG site
`www.nuhag.eu` resp. the BIBTEX collection at
`http://www.univie.ac.at/nuhag-php/bibtex/index.php`

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