

From Numerical to Conceptual Harmonic Analysis
with: Numerical Aspects of Gabor Analysis

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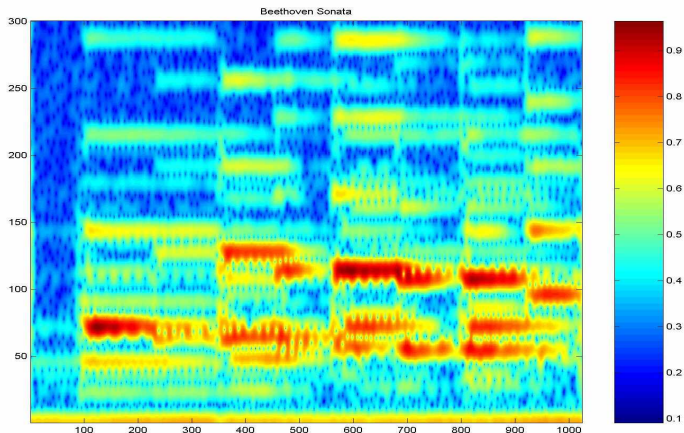
PAPERS: <http://www.nuhag.eu/bibtex>

TALKS: <http://www.nuhag.eu/talks>

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Gabor Analysis: Beethoven Piano Sonata



Hints to the literature

Reading about Banach Gelfand Triples:

<http://www.univie.ac.at/nuhag-php/home/db.php>

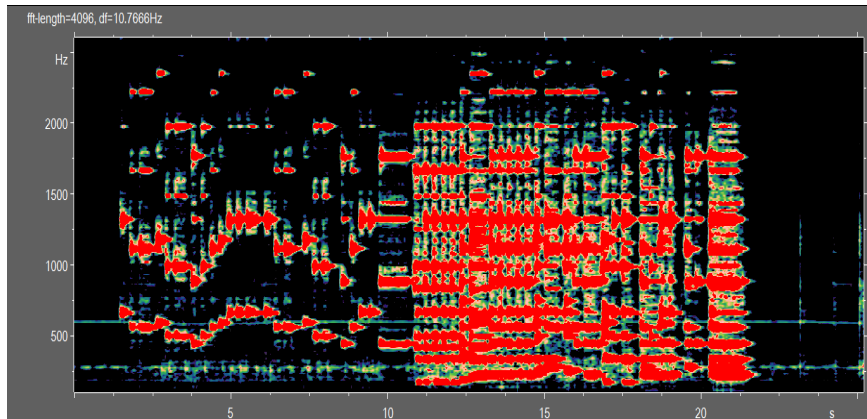
From there you can go into TALKS and search e.g. for Banach Gelfand (in the title), in particular the Bordeaux talk.

The BIBTEX section contains all our papers, including the ones with the code `feluwe07` (from Linear Algebra to Gabor Analysis) resp. `cofelu08` (Banach Gelfand triples), or `feko1u09` (Gabor Analysis over finite Abelian Groups).

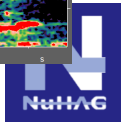
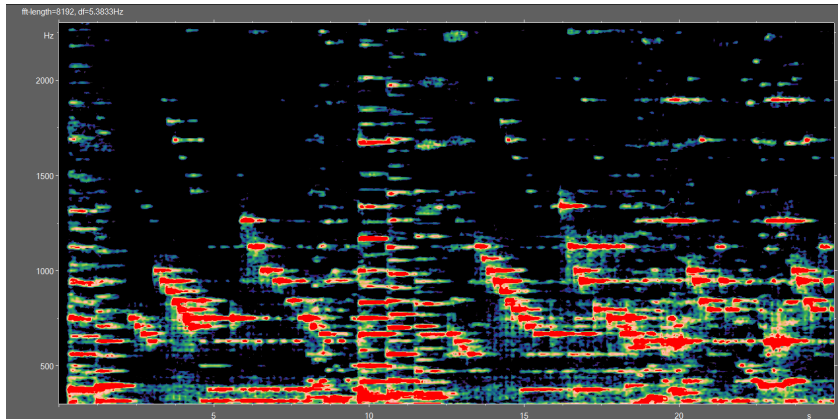
MATLAB code for all this can also be obtained from **hgfei**. The LTFAT toolbox is highly recommended (running on Windows or Linux, and Octave or MATLAB).



Guess the Children's Song Played!



From a Mozart Piano Trio



Aspects of Gabor Analysis

Gabor analysis is concerned with a very intuitive way of representing signals, also allowing to realize time-variant filtering (i.e. to do the computational analogue of the action of an audio engineer).

The classical literature emphasizes the functional-analytic subtleties of such **non-orthogonal expansions**. Describing Gabor Analysis from a **Numerical Linear Algebra and Harmonic Analysis** point of view however helps to separate the points to be observed and allows to also explain the right view-point to the “continuous case”.



Aspects of Gabor Analysis II

There is a variety of application areas of Gabor Analysis (similar and sometimes in competition with *wavelets* or *shearlets, curvelets*). Let us mention project related topics:

- audio signal processing (e.g. for electro-cars);
- image processing (see our contribution to the hand-book of image processing, [2], based on S. Paukner's master thesis);
- mobile communication (Gabor Riesz bases, ADSL, OFDM,...).

A short survey of the subject is given in Encyclopedia Applied Mathematics (see [3]).



AGENDA (for home-reading)

We would like to address the following questions:

- What is **Gabor analysis** (motivation, problems, applications)?
- What are the **numerical** challenges arising from this theory?
- In which sense provides a combination of arguments from **numerical linear algebra** combined with concepts from (abstract) **harmonic analysis** the foundation for the **DEVELOPMENT OF EFFICIENT ALGORITHMS**



Linear Signal Representations

Let us compare various classical settings where signal representations occur in a natural way:

- Fourier Series;
- Fourier transforms (on \mathbb{R}^d);
- FFT resp. DFT (Discrete Fourier Transform);

In the first case we have learned that the correct view-point is to put oneself into the *Hilbert space* $\mathbf{L}^2(\mathbb{T})$ and consider the sequence of pure frequencies as a CONB for this space.

In the last (FFT) case the sum is even a finite one and we just have a unitary change of basis (up to the normalization factor \sqrt{N}).



Linear Signal Representations II

Already on $\mathbf{L}^2(\mathbb{R}^d)$ the situation is much more delicate:
Fourier analysis is obtained by computing

$$\hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s \cdot t} dt,$$

which in fact is not even pointwise well defined for general $f \in \mathbf{L}^2(\mathbb{R}^d)$. On the other hand the Fourier inversion formula is known to be delicate (requiring summability) even for $f \in \mathbf{L}^1(\mathbb{R}^d)$; so only “somehow” one has with $\chi_s(t) = e^{2\pi i s \cdot t}$ with $\hat{f}(s) = \langle f, \chi_s \rangle$:

$$f = \int_{\mathbb{R}^d} \hat{f}(s) \chi_s ds = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \chi_s ds,$$

or in a pointwise sense (if we are lucky)

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(s) \chi_s(t) ds = \int_{\mathbb{R}^d} \hat{f}(s) e^{2\pi i s \cdot t} ds.$$



Linear Signal Representations III

Although the integration techniques are by now well developed we can learn that one has to be careful with the kind of signal representations that one is going to use, and as soon as infinite families of building blocks are involved the corresponding infinite sums have to be interpreted carefully (sometimes pointwise, sometimes in the \mathbf{L}^2 -sense, etc.).

For the Fourier inversion one can think of the family of characters $(\chi_s)_{s \in \mathbb{R}^d}$ as a kind of “continuous basis”, but the *strange thing* is that they do not belong to $\mathbf{L}^2(\mathbb{R}^d)$ and even less to $\mathbf{L}^1(\mathbb{R}^d)$, and so finite Riemannian sums corresponding to the written integrals do not make much sense in the Fourier inversion.

But in which sense are they “linear independent” and how is convergence properly defined!?



Linear Signal Representations IV

Just for the sake of comparison let us remind the audience of good wavelet bases: These are double indexed families $(\psi_{k,l})$ with the property of forming a CONB for $\mathbf{L}^2(\mathbb{R}^d)$. Therefore it is clear that every $f \in \mathbf{L}^2(\mathbb{R}^d)$ has an (unconditional) convergence of the form

$$f = \sum_{k,l} \langle f, \psi_{k,l} \rangle \psi_{k,l}.$$

Of course wavelet system would not have gained so quickly high recognition if they would not have some extra properties: *good wavelets*, i.e. well concentrated, smooth wavelets satisfying a few moment conditions allow to completely characterize the membership of f in the classical function spaces (e.g. Besov spaces), and convergence of the sum is taking place automatically on all those spaces (the form an unconditional basis!).



Recalling facts about Matrices

Let \mathbf{A} be a (potentially) rectangular $m \times n$ matrix, representing a linear mapping from \mathbb{C}^n to \mathbb{C}^m . It obviously has a null-space $\mathbf{Null}(\mathbf{A})$ and a range space, the column-space of \mathbf{A} , denoted by $\mathbf{Col}_\mathbf{A}$. Of course the same is true for the adjoint matrix $\mathbf{A}' := \text{conj}(\mathbf{A}^t)$ (using MATLAB notation). Since the columns of \mathbf{A}' are essentially the rows of \mathbf{A} we call their linear span the row-space $\mathbf{Row}_\mathbf{A}$ and refer to the space $\mathbf{Null}(\mathbf{A}')$ as the co-Null-space of \mathbf{A} . It is an elementary but important fact to observe that the $\mathbb{C}^n = \mathbf{Row}_\mathbf{A} \oplus \mathbf{Null}(\mathbf{A})$ and $\mathbb{C}^m = \mathbf{Col}_\mathbf{A} \oplus \mathbf{Null}(\mathbf{A}')$, and we should remember those *FOUR SPACES* (see Gilbert Strang). Without justifying all the details let us make the following observation: whatever matrix we are forming, starting from those two matrices, e.g. the Gramian matrix $G = \mathbf{A}' * \mathbf{A}$, or $S = \mathbf{A} * \mathbf{A}'$, or the pseudo-inverse \mathbf{A}^\perp , or $\mathbf{A}^\perp * \mathbf{A}$ and $\mathbf{A} * \mathbf{A}^\perp$ etc., all their Null-spaces or Range spaces are always one of those four spaces.



Harmonic Analysis and Algorithms

(?) Abstract versus(?) Computational Harmonic Analysis ¹ ??

- 1 Abstract Harmonic Analysis (AHA) provides a unified terminology for Fourier Analysis over general LCA² groups \mathbf{G} , be they continuous or discrete, compact or non-compact, finite or infinite, one- or high-dimensional;
- 2 Not only Fourier Analysis has its natural analogue over finite Abelian groups, but even all the ingredients of *time-frequency analysis* have their natural meaning for finite groups ([4]);
- 3 In contrast to the *valid analogy of concepts* AHA provide only little support from an approximation theoretic view-point, e.g. quantitative error estimations.

¹I like to call their combination *conceptual harmonic analysis*.

²Locally Compact Abelian!



LINEAR ALGEBRA: Gilbert Strang's FOUR SPACES

Let us now take a **LINEAR ALGEBRA POINT OF VIEW!**

We recall the *standard linear algebra situation*. We view a given $m \times n$ matrix \mathbf{A} either as a collection of *column* or as a collection of *row vectors*, generating $\text{Col}(\mathbf{A})$ and $\text{Row}(\mathbf{A})$. We have:

$$\mathbf{row-rank}(\mathbf{A}) = \mathbf{column-rank}(\mathbf{A})$$

Each *homogeneous linear system of equations* can be expressed in the form of scalar products³ we find that

$$\mathbf{Null}(\mathbf{A}) = \mathbf{Rowspace}(\mathbf{A})^\perp$$

and of course (by reasons of symmetry) for $\mathbf{A}' := \text{conj}(\mathbf{A}^t)$:

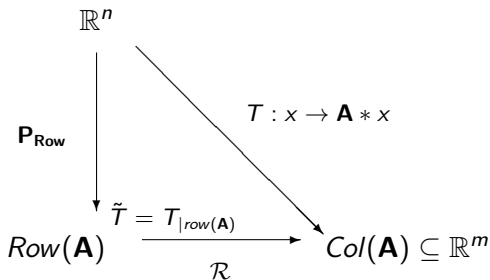
$$\mathbf{Null}(\mathbf{A}') = \mathbf{Colspace}(\mathbf{A})^\perp$$

³Think of $3x + 4y + 5z = 0$ is just another way to say that the vector $\mathbf{x} = [x, y, z]$ satisfies $\langle \mathbf{x}, [3, 4, 5] \rangle = 0$.

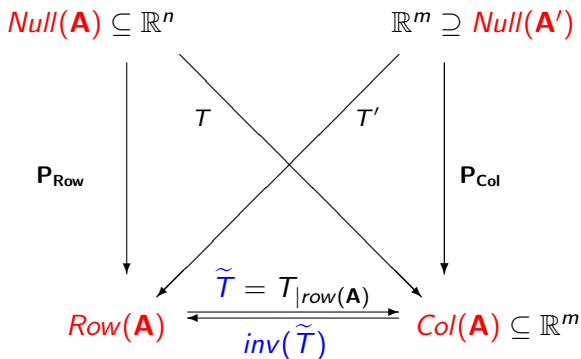


Geometric interpretation of matrix multiplication

Since *clearly* the restriction of the linear mapping $x \mapsto \mathbf{A} * x$ is injective we get an isomorphism \tilde{T} between $Row(\mathbf{A})$ and $Col(\mathbf{A})$.



Geometric interpretation of matrix multiplication



$$T = \tilde{T} \circ P_{Row}, \quad pinv(T) = inv(\tilde{T}) \circ P_{Col}.$$



Four spaces and the SVD

The **SVD** (the so-called Singular Value Decomposition) of a matrix, described in the MATLAB helpful as a way to write A as

$$A = U * S * V'$$

, where the columns of U form an ON-Basis in \mathbb{R}^m and the columns of V form an ON-basis for \mathbb{R}^n , and S is a (rectangular) diagonal matrix containing the non-negative *singular values* (σ_k) of A . We have $\sigma_1 \geq \sigma_2 \dots \sigma_r > 0$, for $r = \text{rank}(A)$, while $\sigma_s = 0$ for $s > r$. In standard description we have for A and $\text{pinv}(A) = A^+$:

$$A * x = \sum_{k=1}^r \sigma_k \langle x, v_k \rangle u_k, \quad A^+ * y = \sum_{k=1}^r \frac{1}{\sigma_k} \langle y, u_k \rangle v_k.$$



Generally known facts in this situation

The **Four Spaces** are well known from **LINEAR ALGEBRA**, e.g. in the **dimension formulas**:

ROW-Rank of \mathbf{A} equals COLUMN-Rank of \mathbf{A} .

The defect (i.e. the dimension of the Null-space of \mathbf{A}) plus the dimension of the range space of \mathbf{A} (i.e. the column space of \mathbf{A}) equals the dimension of the domain space \mathbb{R}^n . Or in terms of linear, homogeneous equations: The dimension of set of all solution to the homogeneous linear equations equals the number of variables minus the dimension of the column space of \mathbf{A} .

The SVD also shows, that the *isomorphism* \tilde{T} between the *Row-space and the Column-space* can be described by a diagonal matrix, if suitable orthonormal bases are used.



Consequences of the SVD

We can describe the quality of the isomorphism \tilde{T} by looking at its condition number, which is σ_1/σ_r , the so-called **Kato-condition number** of T .

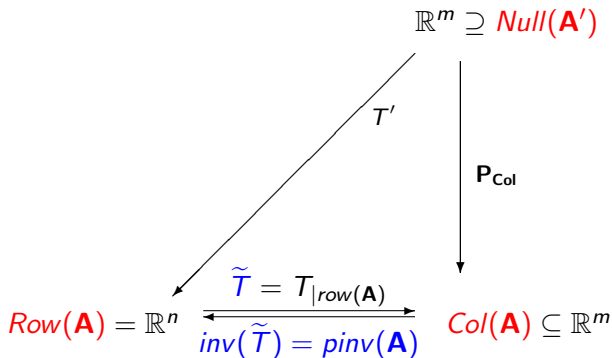
It is not surprising that for **normal matrices** with $A' * A = A * A'$ one can even have diagonalization, i.e. one can choose $U = V$, using the following simple argument:

$$\text{Null}(A) =_{\text{always}} \text{Null}(A' * A) = \text{Null}(A * A') = \text{Null}(A').$$

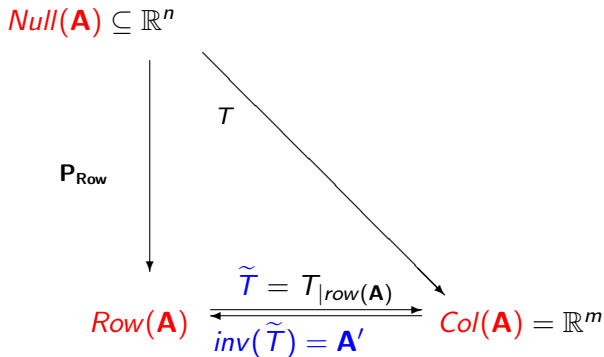
The most interesting cases appear if a matrix has **maximal rank**, i.e. if $\text{rank}(\mathbf{A}) = \min(m, n)$, or equivalently if one of the two Null-spaces is trivial. Then we have either linear independent columns of \mathbf{A} (injectivity of $T \gg$ **RIESZ BASIS** for subspaces) or the columns of \mathbf{A} span all of \mathbb{R}^m (i.e. surjectivity, resp. $\text{Null}(A') = \{0\}$): \gg **FRAME SETTING!**



Geometric interpretation: linear independent set $>$ R.B.

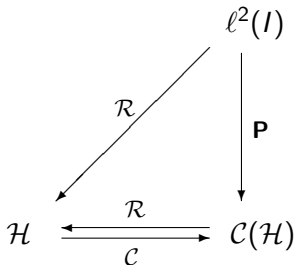


Geometric interpretation: generating set > FRAME



The frame diagram for Hilbert spaces:

If we consider \mathbf{A} as a collection of column vectors, then the role of \mathbf{A}' is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (??).



Riesz basic sequences in Hilbert spaces:

The diagram for a Riesz basis (for a subspace), nowadays called a **Riesz basic sequence** (RBS) looks quite the same ([1]).

In fact, from an abstract sequence there is no! difference, just like there is no difference (from an abstract viewpoint) between a matrix \mathbf{A} and the transpose matrix \mathbf{A}' .

In this way in the RBS case one has the **synthesis mapping** $\mathbf{c} \mapsto \sum_i c_i g_i$ from $\ell^2(I)$ into the Hilbert space \mathcal{H} is *injective*, while in the frame case the **analysis mapping** $f \mapsto (\langle f, g_i \rangle)$ from \mathcal{H} into $\ell^2(I)$ is injective (with bounded inverse).

Of course one can consider a RBS as a Riesz basis for the closed linear span of its elements, establishing an isomorphism between $\ell^2(I)$ and \mathcal{H} .



What have we seen so far:

- We can get a basic understanding of redundant signal representations by refreshing our **linear algebra** background, including SVD and PINV;
- That in principle one can expect that things work in the same way in a Hilbert space setting, and that this can be expressed either by **inequalities** or by **commutative diagrams**;
- That **Gaborian systems** have a particular structure, i.e. the vectors used in the Hilbert space $L^2(G)$ are obtained by applying TF-shifts (from a lattice $\Lambda \triangleleft G \times \widehat{G}$), which gives the problem additional invariance properties.



Facts about matrices

We have seen (as a consequence of SVD etc.) that associated with every matrix \mathbf{A} there are a couple of other matrices, such as the transpose conjugate \mathbf{A}' or the pseudo-inverse \mathbf{A}^+ , and that they all share the same range and column spaces.^a

In particular we have that $\mathbf{A}^+ * \mathbf{A}$ is the projection onto the row space of \mathbf{A} and $\mathbf{A} * \mathbf{A}^+$ is the projection onto the column space of \mathbf{A} (the range of $\mathbf{x} \rightarrow \mathbf{A} * \mathbf{x}$). And \mathbf{A} is injective (meaning that it has linear independent) columns if and only if the Gram-Matrix $\mathbf{A}' * \mathbf{A}$ is invertible, and \mathbf{A} is surjective (meaning that the columns of \mathbf{A} are a set of generators for \mathbb{C}^m , or that \mathbf{A}' is injective) if and only if $\mathbf{A} * \mathbf{A}'$ (the frame operator matrix) is invertible.



Facts about matrices II

There is a simple and quite useful formula (closely related to the way how the normal equation is used in order to derive the MNLSQ-principle, i.e. the PINV solution to $\mathbf{A} * \mathbf{b} = \mathbf{x}$):

$$\mathbf{A}^+ = (\mathbf{A}' * \mathbf{A})^+ * \mathbf{A} = \mathbf{A}' * (\mathbf{A} * \mathbf{A}')^+; \quad (1)$$

of course equivalently, the *dual system* (e.g. dual frame resp. biorthogonal system of vectors, which is $(\mathbf{A}^+)' = (\mathbf{A}')^+$) is

$$(\mathbf{A}^+)' = \mathbf{A}' * (\mathbf{A}' * \mathbf{A})^+ = (\mathbf{A} * \mathbf{A}')^+ * \mathbf{A}. \quad (2)$$

Recall that in the case of *matrices of maximal rank* we can the pseudo-inverse of those square matrices by their true inverse matrix, e.g. the inverse frame operator.



Facts about matrices III

Furthermore we have noticed that in each case there is a “most symmetric” version of the family under consideration available. In case the case of linear independent systems the *Loewdin orthonormalization* of the given system, and in the case of generating systems or frames the *canonical tight frame*. Both are maximally close to the original system in the Hilbert-Schmidt Frobenius norm. Written for the maximal rank case Loewdin reads:

$$\mathbf{L} = \mathbf{A} * (\mathbf{A}' * \mathbf{A})^{-1/2}. \quad (3)$$

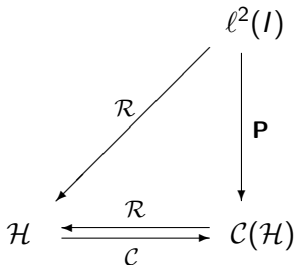
while the canonical tight frame is just

$$\mathbf{H} = (\mathbf{A} * \mathbf{A}')^{-1/2} * \mathbf{A} = \mathbf{S}^{-1/2}(\mathbf{A}).$$



The frame diagram for Hilbert spaces:

If we consider \mathbf{A} as a collection of column vectors, then the role of \mathbf{A}' is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (??).



Frames defined by a Pair of Inequalities

HHH

Definition

A family $(f_i)_{i \in I}$ in a Hilbert space \mathbf{H} is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathbf{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

It is well known that condition (1) is satisfied if and only if the associated frame operator is positive definite:



Frames and Frame Operators II

Definition

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for } f \in \mathbf{H},$$

is invertible. The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\tilde{f}_i)_{i \in I}$, defined by $\tilde{f}_i := S^{-1}(f_i)$ has the property that one has for $f \in \mathbf{H}$:

$$f = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i \quad (5)$$

These formulas emphasize either the reconstruction (sampling) point of view or the *atomic composition* aspect of frames.

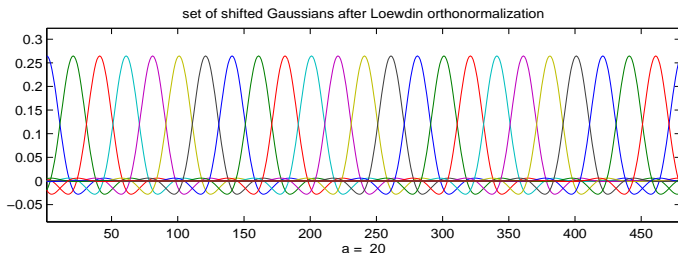
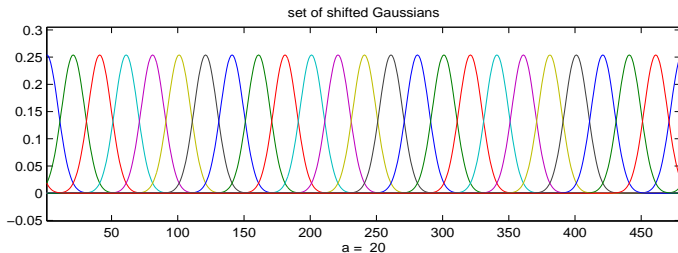


Harmonic Analysis: Operators and Invariance

When you study wavelet theory you probably have found two equivalent definitions of MRAs (multiresolution analysis). In any case you assume that one has a *scale space* (I call it spline-type space) which has a *Riesz basis* of translates. Others require an ONB for their closed linear span. In his wavelet book Y. Meyer talks of the application of $G^{-1/2}$ to the original system because it allows orthogonalization in a way compatible with the translation structure.



Illustration of Loewdin orthonormalization



Invariance Properties of Frame operator

SIMILARLY we can prove that a *Gabor frame operator*

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$$

commutes with the whole family of TF-shifts used from Λ :

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

This has many striking consequences, others:

Lemma

Given a regular Gabor system (induced by a lattice Λ) which is a frame or a Riesz basis, then the corresponding dual system is also a Gabor system.

The spreading representation

The TF-view-point also provides a Fourier-like decomposition of matrices as a sum of TF-operators.

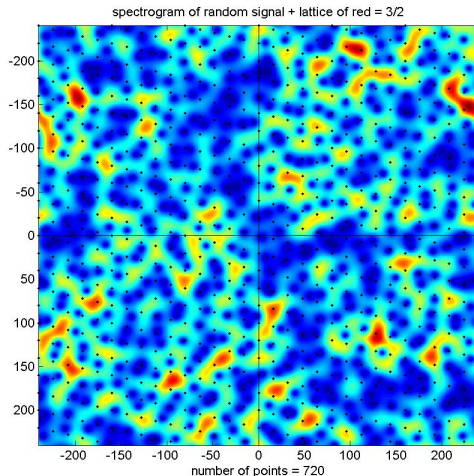
Recall that we have n cyclic shift operators on \mathbb{Z}_n and an equal number of frequency shifts, so altogether n^2 TF-shifts of the form $M_j T_k, 0 \leq j, k \leq n - 1$.

Asking the question, whether those n^2 special matrices are perhaps a basis for the vector space of all $n \times n$ -matrices one comes to the surprising answer that they are indeed an ONB with respect to the Frobenius scalar product (Euclidean structure coming from \mathbb{C}^{n^2}).

Definition

The mapping from the $n \times n$ -matrices into itself, which maps \mathbf{A} to the corresponding coefficients in this matrix is called the **spreading mapping**.
$$\mathbf{A} = \frac{1}{n} \sum_{\lambda \in \mathbb{Z}_n \times \mathbb{Z}_n} \langle \mathbf{A}, \pi(\lambda) \rangle_{\mathcal{HS}} \pi(\lambda).$$

Sampling point of view for Gabor



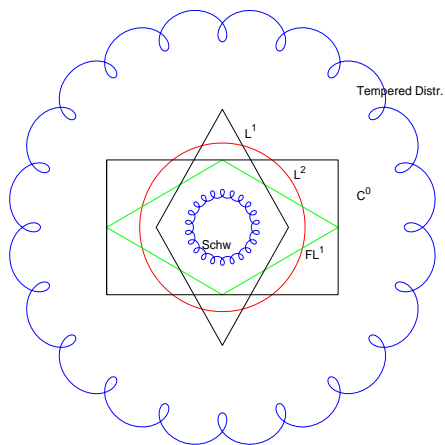
Sampling point of view for Gabor II

FIRST recall some basic facts about TF-analysis.

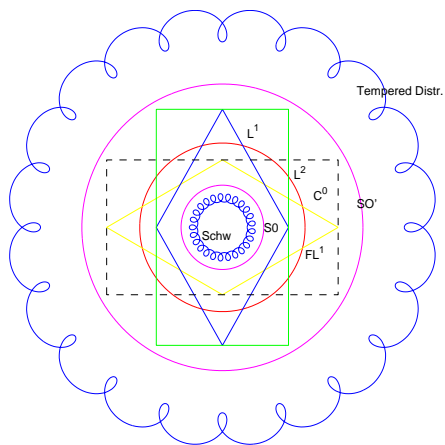
When we read this last picture now in a continuous setting we have to verify that the frame operator is bounded, but because it is a composition of the analysis operator $f \mapsto V_g(f)|_\Lambda$ and its adjoint, which is given explicitly by $\mathbf{c} \mapsto \sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda)g$ it is enough to first control the boundedness the sampling operator from $\mathbf{L}^2(\mathbb{R}^d)$ to $\ell^2(\Lambda)$.

On the positive side we can see that for any $g \in \mathbf{L}^2(\mathbb{R}^d)$ the STFT mapping $f \rightarrow V_g f$ is giving us a function in $\mathbf{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, hence a uniformly continuous function, which is also in $\mathbf{L}^2(\mathbb{R}^d)$, with $\|V_g f\|_2 = \|f\|_2 \|g\|_2$. Of course for this reason one assumes $\|g\|_2 = 1$ (e.g. the Gauss function) because then $f \rightarrow V_g f$ is isometric! But this does NOT HELP.





repeated: SOPLCLASS



Functional Analytic Aspects

From a modeling point of view real world signals are **analogue** while their representation in the computer are **digital**. Sound signals are sampled at 44.1kHz , digital cameras turn images in the optical lens into (stacks of 3) matrices (R-G-B).

Ignoring the (non-linear) problem of appropriate quantization a good recording device (and then a system to perform digital signal processing on the recorded signal) we realize that we are facing an **approximation theoretical problem**, which in turn brings us to **functional analysis** (measuring the errors by some norms) and **function spaces**.



Functional Analytic Aspects II

Analyzing more carefully what the typical situation is we are facing various steps:

- 1 Describe according to **which measure (norm)** the result should be “optimal” (e.g. forming a simulation routine should provide good approximation of the “real output” up to a given error, in some norm, and e.g. stochastically);
- 2 *Approximation theory* provides general possibilities, **constructive approximation theory** is outlining a concrete method, but at the end **realization** on a given computer has to be carried out!
- 3 Ideally one should try to demonstrate that the chosen strategy is close to **optimal**.



Functional Analytic Aspects III

Without going into details let us mention that the classical repertoire of function spaces is by no means satisfactory. Within the large zoo of possible function space norms the most popular ones in “hard analysis”, namely the spaces $(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p)$ are not really important for applications, except of course $p = 1, 2, \infty$.

While there is a variety of norms which describe the smoothness of functions only the **classical Sobolev spaces** $\mathcal{H}^s(\mathbb{R}^d)$ are really important for PDE applications. For $s \in \mathbb{N}$ they can be described as \mathbf{L}^2 -functions with s (distributional) derivatives in $\mathbf{L}^2(\mathbb{R}^d)$.



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t).$$

Behavior under Fourier transform

$$(T_x f)^\wedge = M_{-x} \hat{f} \quad (M_\omega f)^\wedge = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \quad \lambda = (t, \omega);$$



Algebraic Properties of Gabor Frames

The *algebraic properties* allowing to use appropriate (non-commutative) Banach algebras of sparse matrices can thus be explained in the context of **linear algebra and finite groups**. In a second step the transition to the infinite-dimensional situation can be done using **functional analytic arguments** (some new questions arise) and the proper function space setting (namely modulation spaces, which contain ordinary Sobolev spaces and Shubin classes, arising in the study of the harmonic oscillator).

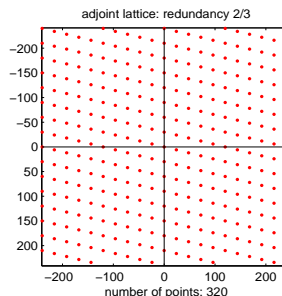
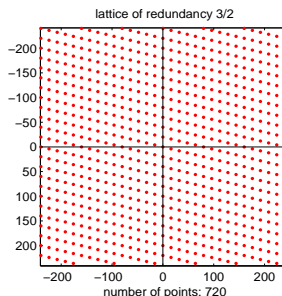


The Gabor Frame Operator for (g, Λ)

Main properties of the Gabor frame operator

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda)g \rangle \pi(\lambda)g = \sum_{\lambda \in \Lambda} \langle f, g_{\lambda} \rangle g_{\lambda}, f \in \mathbf{L}^2(\mathbb{R}^d).$$

A typical example: every point of the left lattice below (Λ) corresponds to one “atom centered at $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ ”:



Commutation Rules in a Non-Comm. Setting

The commutation relation

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

implies that the matrix/operator can be written as a superposition of TF-shift operators from the [adjoint lattice](#). This is called the Janssen representation of the Gabor frame operator.

$$S_{g,\gamma,\Lambda} = \text{red}(\Lambda) \cdot \sum_{\lambda^\circ \in \Lambda^\circ} V_\gamma g(\lambda^\circ) \pi(\lambda^\circ).$$

Note the explicit form of the coefficients. Good decay and smoothness imply that for $\gamma = g$ the invertibility of $S_{g,\Lambda}$ follows from concentration of $V_g(g)$ around zero.



The Ron-Shen Principle

From the Janssen criterion one finds that (g, Λ) generates a Gabor frame (i.e. S is invertible on $\mathbf{L}^2(\mathbb{R}^d)$) if and only if there exists $\gamma \in \mathbf{L}^2(\mathbb{R}^d)$ such that $V_g \gamma(\lambda^\circ) = \delta_{0, \lambda^\circ}$. In fact, if g is normalized with $\|g\|_2 = 1$ the zero-element $\pi(0, 0) = Id$ takes a dominant role within the Janssen expansions and guarantees invertibility (not only over $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$).

In particular, invertibility is granted if Λ° is coarse enough or equivalently if Λ is dense enough.

Theorem

$G(g, \Lambda)$ is a frame if and only if the Gabor system $G(g, \Lambda^\circ)$ is a Riesz basis for its linear span. Moreover, the condition number of the frame operator for $G(g, \Lambda)$ coincides with the condition number for the Gramian matrix for the system $G(g, \Lambda^\circ)$.

Solving the Biorthogonality Problem

The Ron-Shen principle shows that one can replace the inversion of the frame operator S by the inversion of the Gram matrix for the system $(g_{\lambda^\circ})_{\lambda^\circ \in \Lambda^\circ}$, which is smaller.

For the finite setting, e.g. $n = 480$, $red = 3/2$ we have 720 Gabor atoms for the space \mathbb{C}^n , and the Gram-matrix has only size 320×320 .

The invariance properties mentioned allow to solve the problem to solve the equation

$$S(h) = g$$

for $h \in \mathbf{L}^2(\mathbb{R}^d)$. In fact one obtains the **canonical dual atom** by inverting the positive definite and sparse matrix.



Function spaces based on TF-analysis

Function spaces resulting from TF-analysis⁴ turn out to be more useful for a variety of applications, among them the Segal algebra $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ (functions from $\mathbf{L}^1 \cap L^\infty(\mathbb{R}^d)$ with a STFT $V_g(f) \in \mathbf{L}^1(\mathbb{R}^{2d})$), which is the smallest Banach space of functions with an isometrically translation invariant norm and also Fourier invariant.

Together with the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ and its dual space $\mathbf{S}'_0(\mathbb{R}^d)$, which can be characterized as the space of all (tempered) distributions with uniformly bounded spectrogram one obtains the **Banach Gelfand triple** $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)$, which is naturally isomorphic (via **Wilson bases**, to the BGTriple $(\ell^1, \ell^2, \ell^\infty)$, endowed with three types of norm convergence plus also w^* = coordinatewise convergence in ℓ^∞ .

⁴i.e. analyzing a distribution by looking at its Short-time Fourier transform or spectrogram, which is a continuous function over phase-space anyway! ▶



Signal Representation and Mobile Communication

The Ron-Shen principle also says that the **stability** of the two related families, namely the Gabor frame $(g_\lambda)_{\lambda \in \Lambda}$, expressed by the **condition number of the Gabor frame operator** S is exactly the same as the quality of the (linear independent) Riesz basic sequence $(g_{\lambda^\circ})_{\lambda^\circ \in \Lambda^\circ}$ (for its closed linear span), i.e. the condition number of the corresponding Gram matrix.

While frames are good for the representation of “arbitrary signals” (functions or even tempered distributions) the good stability of Gaborian Riesz bases, which provide **approximate eigenvectors to slowly variant channels** (linear operators).

Our **patents** concern efficient algorithms to identify such operators (from the received **pilot tones**) and to do a fast approximate inversion (*channel identification and decoding*).



Further Numerical Issues

In addition to the general structural properties of Gaborian families (frame resp. Riesz basic sequences) we have studied and implemented methods considering:

- 1 preconditioners, double preconditioners (obtained by inverting e.g. the diagonal or circulant “component” of S , resp. commutative subalgebras!)
- 2 functional analytic (spectral - Banach algebra methods) allow to show good properties of the *atom* g (decay at infinity and smoothness) imply corresponding properties for the dual atom $\tilde{g} = h$ (as above), which indicates that a **local biorthogonality problem** will/can give good approximate dual window;
- 3 Locality allows to go for a theory where regularity is only valid locally (but not globally).



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the “window”) in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{\mathbf{S}_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathcal{S}(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.



SOGTr-results in Banach Triple terminology

In the setting of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)$ a quite similar result is due to Gröchenig and coauthors:

Theorem

Assume that for some $g \in \mathbf{S}_0$ the Gabor frame operator $S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ is invertible at the Hilbert space level, then S defines automatically an automorphism of the BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)$. Equivalently, when $g \in \mathbf{S}_0$ generates a Gabor frame (g_λ) , then the dual frame (of the form (\tilde{g}_λ)) is also generated by the element $\tilde{g} = S^{-1}(g) \in \mathbf{S}_0$.

The first version of this result has been based on matrix-valued versions of Wiener's inversion theorem, while the final result (due to Gröchenig and Leinert, see [5]) makes use of the symmetry in Banach algebras and Hulanicki's Lemma.

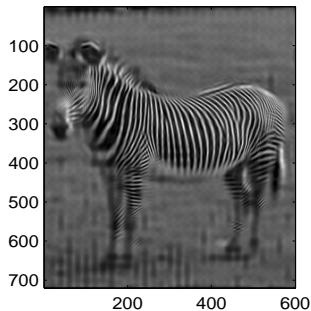
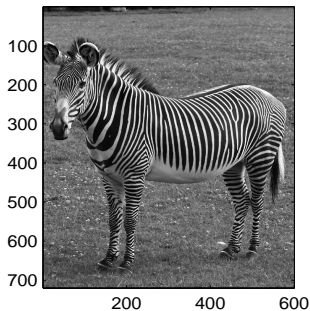


Finally some Applications

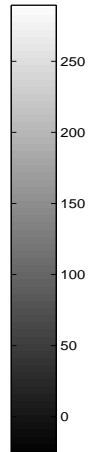
Gabor multipliers are just time-variant filterbanks:



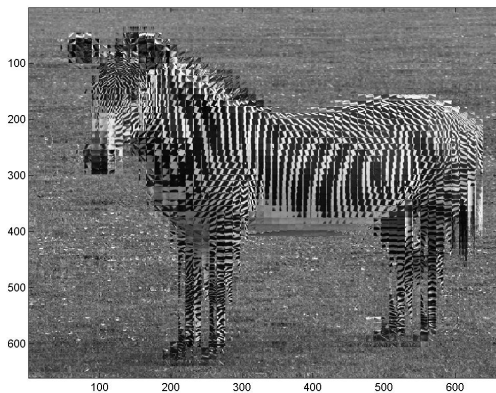
Applications to Image Processing



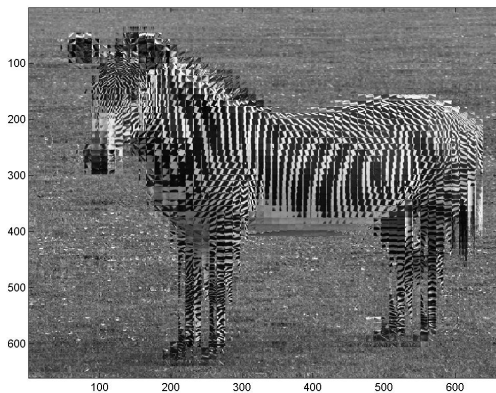
Applications to Image Processing 2



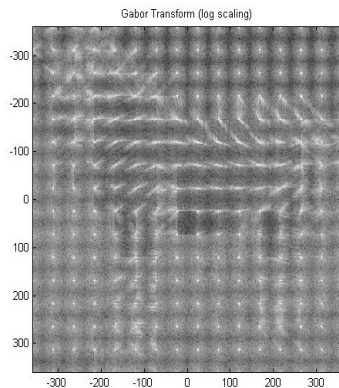
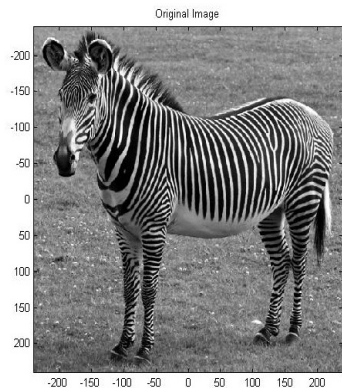
Applications to Image Processing 3



Applications to Image Processing 3

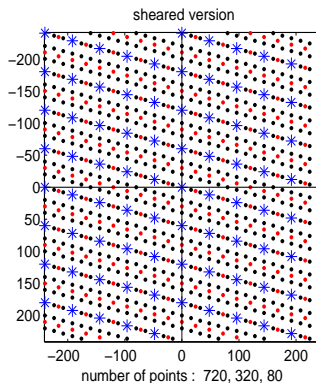
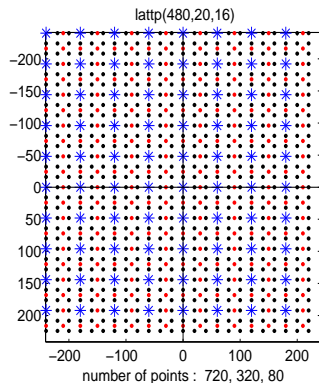


The Gabor Coefficients of the Zebra



Some Group Theoretical Questions

The lattice, the adjoint lattice and their common, *commutative* subgroup.



Gabor and Spline-type Spaces

Due to the fact that efficient Gabor expansions also allow to realize **Gabor multipliers** one may ask, whether a given operator can be optimally approximated by a Gabor multiplier, resp. whether a given matrix can be best approximated by the action of a Gabor multiplier for a given Gabor frame generated by (g, Λ) , measured in the Frobenius norm.

For that purpose it is of course optimal if the trivial multiplier by $m(\lambda) \equiv 1$ provides the identity. Gabor atoms h with $S_{h, \Lambda} = Id$ are called *tight* Gabor atoms, and they can be obtained from a general Gabor atom by computing $S^{-1/2}g$.

Using the so-called Kohn-Nirenberg symbol for general operators this problem can be equivalently expressed as a best approximation of a given $L^2(\mathbb{R}^{2d})$ -function by a spline-like function.





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