NuHAG Numerical Harmonic Analysis Group

# Universal usefulness of BUPUs in Harmonic Analysis

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# Abstract

The usual system of B-splines, obtained as convolution powers of the standard *box-function* ($\mathbf{1}_{[-1/2,1/2]}$) are known to form a uniform partition of unity (by taking its translates along $\mathbb{Z}$) of increasing smoothness. By taking tensor products, by dilating them or applying transformations one obtains prototypical *partitions of unity* of given smoothness (and possibly support size), obtained by moving a standard atom along a suitable lattice.

The purpose of this talk is to demonstrate the universal role of such BUPUs (regular or not!) for many tasks in (abstract and computational) harmonic analysis.

# Defining BUPUs

Bounded Uniform Partitions of Unity (**A-BUPUs**):

## Definition

A bounded family $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ in a Banach algebra $(\mathbf{A}, \|\cdot\|_A)$ is a regular **A-Bounded Uniform Partition of Unity if**

$$\sum_{n\in\mathbb{Z}^d} \psi(x-n) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d$$

Typically **A** is a translation invariant space, with

$$\|T_x g\|_{\mathbf{A}} = \|f\|_{\mathbf{A}}, \quad \forall x \in \mathbb{R}^d, f \in \mathbf{A},$$

and $\psi = \psi_0$ has compact support, but such requirements can be relaxed, as we will explain below.

# General Λ-BUPUs

In a more general context we have the following definition for BUPUs over general LC groups $\mathscr{G}$ (regular A-BUPUs):

**Definition**

A *coherent family* $\Psi = (\psi_\lambda)_{\lambda\in\Lambda} = (T_\lambda \psi)_{\lambda\in\Lambda}$ in a Banach algebra $(\mathbf{A}, \|\cdot\|_A)$ is forms a regular **A-Bounded Uniform Partition of Unity** generated by the pair $(\psi, \Lambda)$ if $\psi$ has compact support, if the the family is bounded in $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ and

$$\sum_{\lambda\in\Lambda} T_\lambda \psi(x) \equiv 1 \quad \text{for all} \quad x \in \mathscr{G}$$

For many examples Λ can be a lattice (discrete subgroup, e.g. $\Lambda = A * \mathbb{Z}^d$ in $\mathbb{R}^d$), but this is not a requirement.

# Defining general BUPUs

## Definition

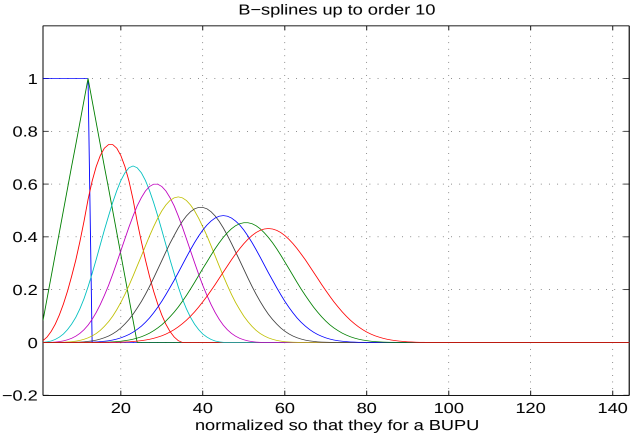
A bounded family $\Psi = (\psi_i)_{i \in I}$ in some normed algebra $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ is a called an **A- Uniform Partition of Unity of size** $\delta > 0$ if:

1. $\sup_{i \in I} \|\psi_i\|_{\mathbf{A}} = C_A < \infty$;
2. there is a family of points $(x_i)_{i \in I}$ such that
$$\mathrm{supp}(\psi_i) \subseteq B_\delta(x_i) \quad \forall i \in I;$$
3. there is limited overlap of the supports, i.e.
$$\sup_{i \in I} \#\{j \mid B_\delta(x_j) \cap B_\delta(x_i) \neq \Phi\} = C_X < \infty;$$
4. $$\sum_{i \in I} \psi_i(x) \equiv 1 \quad \mathrm{over} \mathbb{R}^d$$

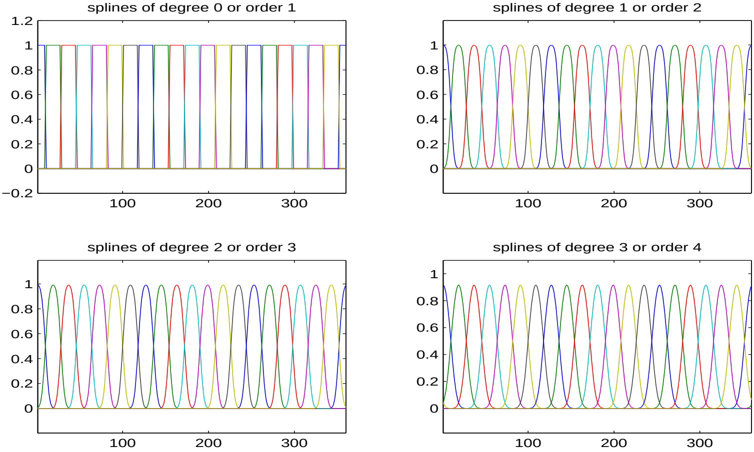
# Defining general BUPUs: Comments

- There are various reasons for this definition and depending on the context one of the other may be relaxed. E.g. typically we assume that **A** consists of continuous functions, but a collection of shifted box-functions serves the purpose in many cases equally well.
- sometimes it is natural to require that $\psi(x) \leq 0$, and then uniform boundedness in the sup-norm sense follows from the partition of unity property;
- one may not have compact support in the strict sense, but just inform concentration with uniformly small tails, e.g. regular BUPUs with $\psi_0 \in \mathbf{W}(\mathbf{C}_0, \ell^1)(\mathbb{R}^d)$.
- It is not hard to create on any LCA group arbitrary fine BUPUs, even on many non-commutative LC groups.

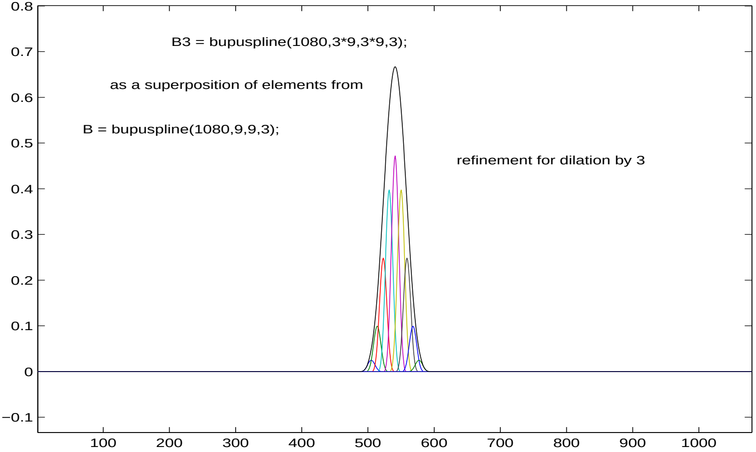
# Different types of B-splines



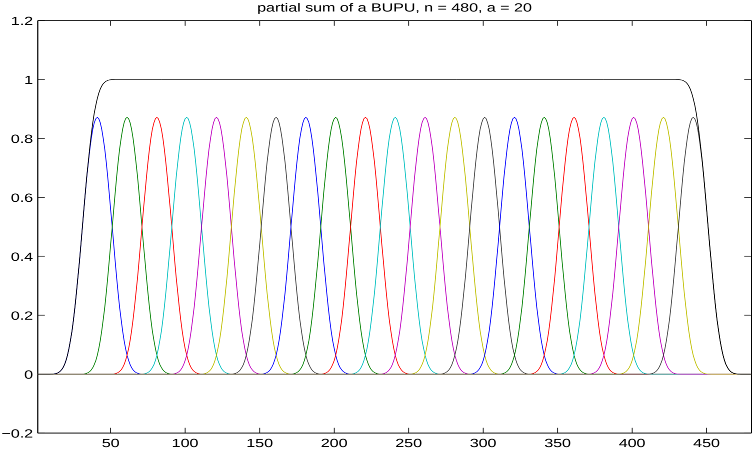
# Different types of B-splines



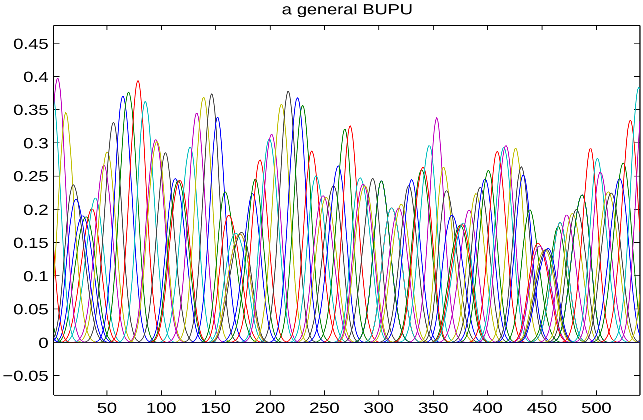
# Different types of B-splines



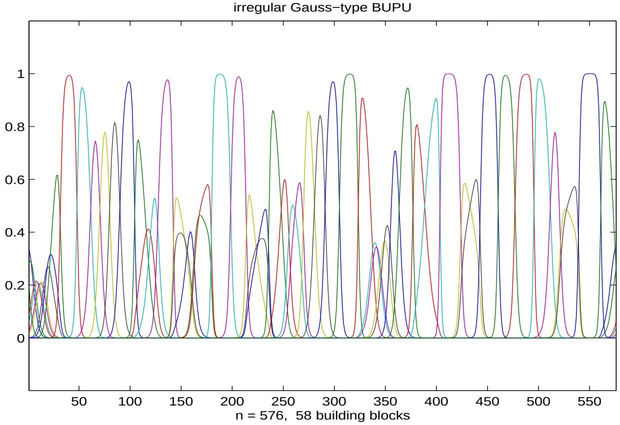
# Different types of B-splines



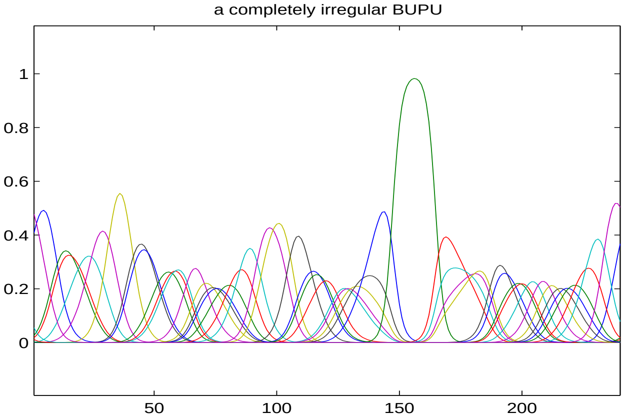
# A general BUPU (for a given irregular set)



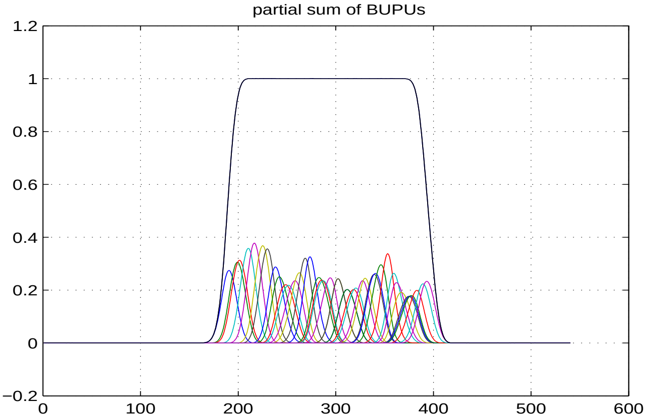
# An approx. BUPU arising from shifted Gaussians



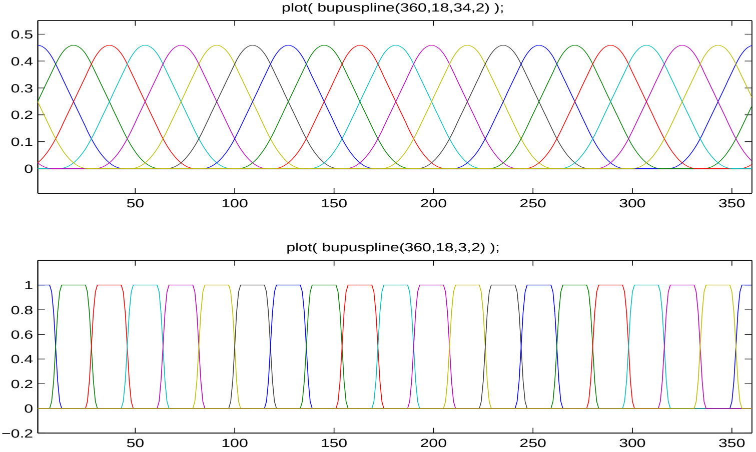
# A general BUPU (for a given irregular set)



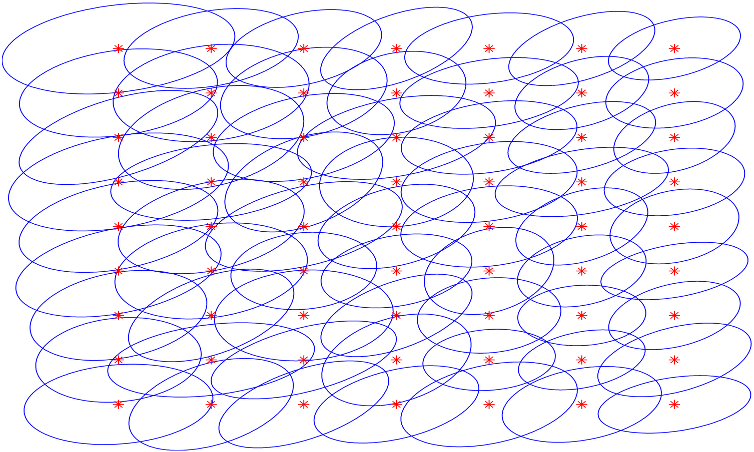
# A general BUPU (partial sums)



# Different types of BUPUS



# A general 2D-BAPU can be based on a covering



# Historical Notes:

BUPUs have been introduced formally in [2, 1] (presented at two conferences in 1980) in order to characterize Wiener-type spaces (meanwhile called Wiener amalgam spaces, following a suggestion by J.J.Benedetto in 1989). This construction was aiming at a generalization (suitable for the context of LCA Groups) of Besov spaces, which had been using dyadic partitions of unity, and thus they where an important ingredient for the construction (an then atomic decomposition) for *modulation spaces*.
But as we want to show they have their role in many other (sometimes elementary questions) as well!

# Bounded Measures over LC groups

Obviously functions on groups, measures, convolution and the Fourier transform are basic notions in Fourier analysis and abstract harmonic analysis. Despite general belief it is not true that Lebesgue integration and measure theory are required to properly introduce these basic concepts (for details see my ongoing course on "Applied Analysis" at `www.nuhag.eu`).
What is needed instead is a bit ore functional analysis, and the elementary theory of Banach algebras and Banach modules (useful anyway later on).
This somewhat cynical view-point is of course based on the author's experience and a number of application oriented projects which helped to identify the important from the technical arguments.

# Bounded Measures over LC groups

1. Let us start with an arbitrary LCA (for simplicity) group $\mathscr{G}$. Then it is clear that we have a non-trivial translation invariant space $\mathbf{C}_c(\mathscr{G})$ of compactly supported, complex-valued functions on $\mathscr{G}$.
2. It is natural to endow it with the sup-norm $\|\cdot\|_\infty$ and complete it. In this way we obtain the isometrically translation invariant Banach algebra $\left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)$.
3. With any Banach space the dual space is of interest, and motivated by the Riesz-representation theorem we call it the *space of bounded measures on* $\mathscr{G}$:

$$(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b}) = \left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)';$$

which carries both the norm and the $w^*$-topology.

# Bounded Measures over LC Groups

The Banach algebra properties of $\left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)$ (regular action of elements by pointwise multiplication) can be turned into a Banach module action of $\left(\mathbf{C}_0(G), \|\cdot\|_\infty\right)$ on $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$ in the usual way:

$$h\mu(f) := \mu(h \cdot f) \quad \text{for} \quad \mu \in \mathbf{M}_b(G), f \in \mathbf{C}_0(G); \qquad (1)$$

which makes even sense for $h \in \mathbf{C}_b(G)$, the bounded and continuous functions (pointwise multipliers of $\mathbf{C}_0(G)$).

Since we have $f = \sum_{i \in I} f\psi_i$ (in norm) we clearly have $\mu = \sum_{i \in I} \psi_i \mu$ for any $\mu \in \mathbf{M}_b(G)$, but much more is true:

# Measures with Compact Support

The following lemma (relying only on simple properties of the sup-norm) is a bit more than an exercise:

**Lemma**

*For any non-negative BUPU in* $\mathbf{C}_0(G)$ *one has:*

$$\|\mu\|_{\mathbf{M}} = \sum_{i \in I} \|\psi_i \mu\|_{\mathbf{M}}.$$

Consequently the compactly supported measures are *norm dense* in $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$.

# $w^*$-Approximation by Discrete Measures

Clearly the Dirac measures $\delta_y : f \mapsto f(y)$ are the most elementary linear functionals on $\mathbf{C}_0(G)$. In order to show that they are $w^*$-dense we define the **Spline-type Approximation Operator:**

$$Sp_\Psi(f) = \sum_{i \in I} f(x_i)\psi_i;$$

has the adjoint

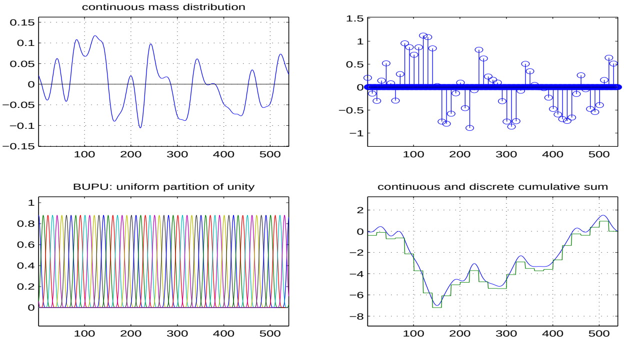
$$D_\Psi(\mu) = \sum_i \mu(\psi_i)\delta_{x_i}$$

Since

$$\mu = w^* - \lim_{|\Psi| \to 0} D_\Psi(\mu)$$

the (finite) discrete measures (finite sums of Dirac measures) are $w^*$-dense in $\mathbf{M}_b(\mathbb{R}^d)$. Clearly this family is also (uniformly) bounded and uniformly concentrated (tight).

# Adjoint Action on Distributions: Discretization of Mass



# Compactness in $(\mathbf{C}_0(G), \|\cdot\|_\infty)$

The above claim is of course the consequence of the following lemma, which can be based on an oscillation estimate (pointwise):

$$|f(x) - \mathrm{Sp}_\Psi(f)(x)| \leq osc_\delta f(x) \quad \text{for} \quad x \in \mathbb{R}^d;$$

implying by a uniform continuity argument:

**Lemma**

$$\|f - \mathrm{Sp}_\Psi f\|_\infty \to 0 \quad \textit{for} \quad |\Psi| \to 0.$$

From this (and using the tightess preserving property of the $\mathrm{Sp}_\Psi$-operator) it is not difficult to derive the characterization of relatively compact subsets:

**Lemma**

*A bounded and closed subset $M \subset \mathbf{C}_0(G)$ is compact in $(\mathbf{C}_0(G), \|\cdot\|_\infty)$ if and only if it is equicontinuous and tight.*

# The Fourier Stieltjes Transform

The decomposition trick given by Lemma 4 is not only a nice trick but is also the basis to extend the action of $\mu \in \mathbf{M}_b(G)$ to all of $\mathbf{C}_b(G)$ in a natural way (respecting the density of $\mathbf{C}_0(G)$ in $\mathbf{C}_b(G)$ with respect to uniform convergence over compact subsets).
The goal is of course the *convolution theorem*, telling us that convolution (of measures) is turned into pointwise multiplication of their Fourier transforms.
This is obvious in the case of the FFT (when interpreted as the conversion of coefficients of polynomials into their values over the unit-roots of order $N$), because there it is clear that the Cauchy product at the coefficient level simply corresponds to pointwise multiplication of polynomials.

# How to Define Convolution

In order to define convolution of bounded measures we may keep in mind that in the case of finite groups we simply have to take the group multiplication law $z = x + y$ and turn it into a convolution relation $\delta_x * \delta_y = \delta_z$.
An elegant approach is this identification theorem:

**Theorem**

*There is an isometric identification between the translation invariant operators* $T : \mathbf{C}_0(G) \to \mathbf{C}_0(G)$ *commuting with translation and the bounded measures, via*

$$C_\mu f(x) = \mu(T_x f^{\checkmark}), \quad \text{with} \quad f^{\checkmark}(x) = f(-x),$$

*with* $|||T|||_{\mathbf{C}_0} = \|\mu\|_{\mathbf{M}_b(G)}$.

Rf Engineers would call such system BIBOS-TLIS!

# How to Define Convolution II

- There is also an inverse mapping. Given the operator $T$ the corresponding measure is obtained via $\mu_T(f) := T(f^{\checkmark})(0)$.
- In the proof an important ingredient is the fact that $C_\mu(\mathbf{C}_c(\mathscr{G})) \subset \mathbf{C}_c(\mathscr{G})$ for compactly supported measures, hence one has not only uniform continuity of $C_\mu(f)$ for $f \in \mathbf{C}_0(G)$ (this is relatively easy to obtain) but $C_\mu(f) \in \mathbf{C}_0(G)$ for all $f \in \mathbf{C}_0$.
- The Dirac measure $\delta_y$ corresponds to the translation operator, i.e. $C_{\delta_y} = T_y$ in this correspondence.

# How to Define Convolution III

Since clearly the TLIS (operators commuting with translations) form a Banach algebra under composition it is clearly by *TRANSFER OF STRUCTURE* that there is a corresponding multiplication at the level of bounded measures.
In fact (as a special case of an even more general claim) one finds:

**Lemma**

*For every $f \in \mathbf{C}_0(G)$ one has*

$$\mu * f = \lim_{|\Psi| \to 0} D_\Psi \mu * f \quad in(\mathbf{C}_0(G), \|\cdot\|_\infty).$$

Consequently the defined convolution is the only possible multiplication respecting the law $\delta_x * \delta_y = \delta_{x+y}$ ,
and is thus automatically *convolution is commutative*!

# Consequences for Gabor Multipliers

A very similar principle can be used to show that so-called Anti-Wick operators (we call them STFT-multipliers), resp. Toeplitz operators which are obtained by multiplying the STFT with a continuous function and then applying $V_g^*$ (the synthesis operator), resp. apply a twisted convolution operator (realizing the projection onto the range of $V_g$) after the multiplication.
A typical concrete result in this direction may look like this:

**Theorem**

*Assume that an STFT-multiplier is given, with upper symbol on some Sobolev space $\mathbf{H}^s(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, with $s > d$, and analysis as well as synthesis window $g, \gamma \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$. Then the Gabor multipliers obtained by using lattices of the form $a\mathbb{Z} \times b\mathbb{Z}$, with $(a, b) \to (0, 0)$ (suitably normalized by the factor $ab$) tend to the STFT-multiplier in the Hilbert-Schmidt sense.*

# The Integrated Group Action

It is normally consider a triumph of integration theory to show that one can raise the action of a group to the whole measure algebra (equipped with convolution).

The typical setting is that of a *homogeneous Banach space* such as $\left(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p < \infty$, but the fact is more general:

**Theorem**

*Let $\rho$ be an isometric and strongly continuous action of a LC Group $\mathscr{G}$ on some (homogeneous) Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$. Then $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ becomes in a natural way a Banach module over $\mathbf{M}_b(G)$ (with convolution) by a suitable action, defined by $\mu \bullet_\rho f = \lim_{|\Psi| \to 0} D_\Psi \mu \bullet f$ (starting from $\delta_x \bullet_\rho f := \rho(x) f$), and with $\|\mu \bullet_\rho f\|_{\mathbf{B}} \leq \|\mu\|_{\mathbf{M}_b(G)} \|f\|_{\mathbf{B}}$ and*

$$(\mu_1 * \mu_2) \bullet_\rho f = \mu_1 \bullet_\rho (\mu_2 \bullet_\rho f).$$

# Wiener Amalgam Spaces

The idea of **Wiener amalgam space** (`Wasp`, initially called *Wiener-type spaces*) was to have a tool to describe the *global behavior* of *local properties*. As with wavelet theory the method used to *localize* should not influence the measurement (or at most by some constant), which typically means that the algebra $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ comes into play, and the assumption is thus that $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ is embedded into the multiplier algebra of the *local component* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ which is used to define the `Wasp` $\mathbf{W}(\mathbf{B}, \mathbf{C})$. BUPUs are then heavily used to verify convolution properties for `Wasps` or duality results allowing to quickly establish a number of good properties.

# Modulation Spaces

Modulation spaces are Wiener amalgam spaces of the form $\mathbf{W}(\mathcal{F}\mathbf{L}^p, \mathbf{L}^q_{v_s})$ on the Fourier transform size. Since one has $\mathbf{L}^1 * \mathbf{L}^p \subset \mathbf{L}^p$ for any $p \in [1, \infty]$ the natural algebra $(\mathbf{A}, \|\cdot\|_{\mathbf{A}})$ is the Fourier Algebra $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$.
Of course one could start from weighted *Lpsp*-spaces, then $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ would be replaced by $\mathcal{F}\mathbf{L}^1{}_w(\mathbb{R}^d)$, or if this is more convenient by sufficient classical smoothness in the sense of $\mathbf{C}^{(k)}$, as it is used in the characterization of Besov spaces.
The finite overlap condition is necessary for any $q > 1$ (e.g. for $q = 2$ in order to assure a form of weak orthogonality, because otherwise the natural (atomic) norms would not work anymore.

# The Segal algebra $\mathbf{S}_0(\mathscr{G}) = \mathbf{M}^1$

The Segal algebra $\mathbf{S}_0(\mathscr{G})$ was introduced as a special (minimal) Banach space and has turned to be the ideal replacement for $\mathcal{S}(\mathscr{G})$ for any non-PDE context.

It was originally defined as $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$, and later characterized as Coorbit space (for the Schrödinger representation) of $\mathbf{L}^1$, which is one way to understand why has so many good properties, including atomic decompositions.

It has been shown (from the very beginning) that it is the smallest isometrically TF-invariant Banach space containing (one or any) non-trivial Schwartz-function or band-limited $\mathbf{L}^1$-function. In addition it is Fourier invariant, i.e. $\mathcal{F}_{\mathscr{G}}(\mathbf{S}_0(\mathscr{G})) = \mathbf{S}_0(\widehat{\mathscr{G}})$.

# BUPUs are useful for Irregular Sampling

Non-regular BUPUs are easily constructed, e.g. for $\delta-$dense and relatively separated point sets $(x_i)_{i \in I}$ and quite useful for the recovery of smooth functions from their (sufficiently dense) irregular samples. In this context it is important that the points are given (and will not be chosen by the user), and reconstruction can be guaranteed based on their density only.
There are several contexts for this:

- band-limited functions;
- spline-type functions;
- voice transforms (such as STFT's, CWT, etc.);

# BUPUs are useful for Irregular Sampling II

Surprisingly the technical realization is less involved (at least at an abstract level) in the last context.
Using, for a fixed atom $g$ the notation $F := V_g f, G = V_g g$ we have (as a consequence of the square integrability) the convolution-relationship for $\mathbf{L}^2(\mathscr{G})$ functions:

$$F = F * G.$$

In fact, $F \mapsto F * G$ is an orthogonal projection operator from all of $\mathbf{L}^2(\mathscr{G})$ onto the range of $V_g$ (for fixed $g \neq 0$). The good properties of $g$ also imply smoothness and concentration of $G$ on $\mathscr{G}$.
Given the samples $(F(x_i))_{i \in I}$ of $F$ one can build as a first approximation the spline-function and then project it, i.e. define $A_\Psi(F) = Sp_\Psi(F) * G$, with $|||Id - A_\Psi|||_{\mathbf{L}^2} < 1$.

# BUPUs are useful for Irregular Sampling III

For the context of band-limited functions such smooth and well-concentrated projection operators $F \mapsto F * G$ do not exist (the SINC-function defines a projection operator, but is not well concentrated, and the other functions do not provide projection operators via convolution!).
Nevertheless a way to circumvent the problem and still argue quite similarly to the setting described above is to use two different functions which are kind of smooth plateau-functions on the Fourier transform side, satisfying the two equations

$$F = F * G; G = G * H; \quad \text{hence} \quad F = F * H.$$

By first discretizing the last equation to $A(F) = Sp_{\Psi} F * H$ to obtain a good approximation, and then convolving with $G$ allows still to apply a contraction principle.

# BUPUs are useful for Irregular Sampling IV

The use of Wiener amalgam space, their convolution relations etc. allow to carry out these iterative reconstruction algorithms in a quite general setting, proving rates of convergence in various norms, and show robustness e.g. against jitter errors (or aliasing errors).

For the spline-type theory one can still use a certain amount of smoothness of their elements, and the explicit description (and uniform boundedness over families of function spaces) of projection operators, which map the approximations $Sp_{\Psi}(f)$ (which may be piecewise linear functions) back to the spline-type space (e.g. a space of cubic spline functions with integer nodes, in an $\mathbf{L}^p$-setting).

# Approximating Continuous Problems by Finite ones

The prototypical questions here are: We treat many problems in Harmonic Analysis in a variety of contexts. We can define *the Fourier transform* for general LCA groups, hence we can also to TF-analysis an general groups.

Engineers work with discrete or continuous variables, with periodic or non-periodic signals, and somehow one can approximate one by the other, usually in a hand-waving manner (letting the period tend to infinity we get the Fourier transform ( plus inversion formula?) from the periodic setting of Fourier transforms? BUT

HOW reliable or mathematically sound are such transitions. In fact, from a distributional view-point one should see these transitions as $w^*$-limits, then it would be correct, but we want to do something more *constructive*.

# Approximating Continuous Problems by Finite ones, II

Let us consider two simple questions in TF-analysis:
Given a decent function (e.g. $f \in \mathbf{S}_0(\mathbb{R}^d)$), where we know that it also has a decent Fourier transform (namely $\hat{f} \in \mathbf{S}_0(\mathbb{R}^d)$). And we have are supposed to *approximately compute* the Fourier transform $\hat{f}$ in order to do e.g. a curve discussion. In this case on will net to find an explicit (computationally *realizable*) version of $\hat{f}$ obtained by using (probably) the FFT and some other tricks that help to come back from sequences (FFT output) to functions, in such a way that the error can be measured (in appropriate function space norms, here naturally in the $\mathbf{S}_0$-norm.
Such questions have been considered (by N. Kaiblinger and the author), qualitatively, a couple of years ago.

# Approximating Continuous Problems by Finite ones, III

The main point is, to bring the FT into the game, is to observe that it is (of course) NOT true that the FFT of a (finite sequence of) signal samples is the same as a a related sampled sequence of values of the (continuous) FT $\hat{f}$, even if $f \in \mathcal{S}(\mathbb{R}^d)$. One rather has to observe that for the case that $p = Na$ (the periodization of $f$ with period $p$ is an integer multiple of the sampling period) the following relation holds true:

$$\mathcal{F}[\sqcup\!\sqcup_p * (\sqcup\!\sqcup_a \cdot f)] = \sqcup\!\sqcup_q * (\sqcup\!\sqcup_b \cdot \hat{f}),$$

for $b = 1/p, q = 1/a$ and hence $q = Nb$, and furthermore that the finite sequence representing the discrete, periodic signal on the right hand side can be obtained (up to a known normalizing factor) via $FFT_N$ applied to the sequence characterizing the left hand side (which is of equal length $N$).

# BUPUs and Lagrange Interpolators

Let us consider the most simple case, i.e. BUPUs with respect to $\Lambda = \mathbb{Z}^d$ in $\mathbb{R}^d$. Then they can be characterized by the following simple criterion:

**Lemma**

*A function $\psi \in \mathbf{S}_0(\mathbb{R}^d)$ with compact support (i.e. a continuous function with $\widehat{\psi} \in \mathbf{L}^1(\mathbb{R}^d)$) defines a BUPU if and only if its FT $h := \widehat{\psi}$ satisfies the Lagrange interpolating property*

$$h(k) = \delta_{0,k} \quad \textit{[Kronecker Delta condition].}$$

The proof is easy, using $\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup$ and the fact that

$$\sqcup\!\sqcup * \psi = 1 \quad \text{if and only if} \quad \sqcup\!\sqcup \cdot h = \delta_0 .$$

# BUPUs and Lagrange Interpolators II

Since it is easy to generate a BUPU from a family $(\varphi_i)$ with uniform size of their support, say $\mathrm{supp}(\varphi_i) \subseteq B_\delta(x_i$ for all $i \in I$, simply by defining $\Phi(x) := \sum_{i \in I} \varphi_i(x)$ and then setting $\psi_i = \varphi_i/\Phi$ it is not surprising that one obtains the (uniquely determined) Lagrange interpolator in a spline-type space

$$\mathbf{V}_{\varphi,\Lambda} := \{\sum_{\lambda \in \Lambda} c_\lambda T_\lambda \varphi, \mathbf{c} \in \ell^2(\mathbf{\Lambda})\}$$

by just doing that on the Fourier transform side, i.e. one obtains $L \in \mathbf{V}_{\varphi,\Lambda}$ by setting

$$\widehat{L} := \frac{\widehat{\varphi}}{\sum_{\lambda \in \Lambda^\perp} T_{\lambda^\perp} \widehat{\varphi}}.$$

# WARNING: USING ENDBIBL!!!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