Numerical Aspects of Gabor Analysis

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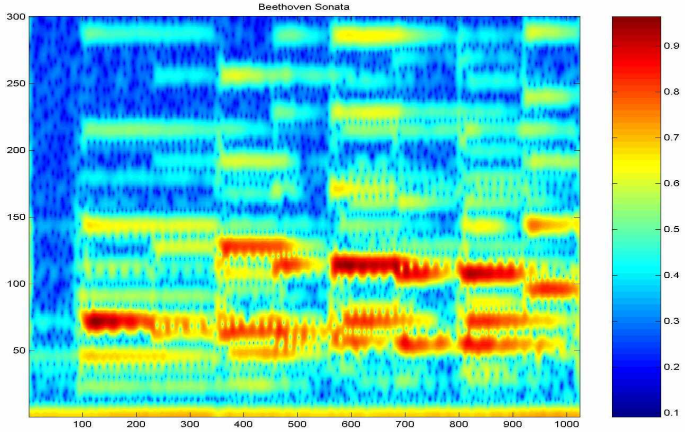
Gabor Analysis: Beethoven Piano Sonata



Gabor analysis is concerned with a very intuitive way of representing signals, also allowing to realize time-variant filtering (i.e. to do the computational analogue of the action of an audio engineer).
The classical literature emphasizes the functional-analytic subtleties of such non-orthogonal expansions. Describing Gabor Analysis from a Numerical Linear Algebra and Harmonic Analysis point of view however helps to separate the points to be observed and allows to also explain the right view-point to the "continuous case".

Aspects of Gabor Analysis II

There is a variety of application areas of Gabor Analysis (similar and sometimes in competition with *wavelets* or *shearlets*, *curvelets*, *etc.*.

Let us mention project related topics:

- audio signal processing (e.g. for electro-cars);
- image processing (see [2]);
- mobile communication (Gabor Riesz bases, ADSL, OFDM,...).

A short survey in Encyclopedia Applied Mathematics is given in [3].

AGENDA (for home-reading)

We would like to address the following questions:

- What is Gabor analysis (motivation, problems, applications)?
- What are the numerical challenges arising from this theory?
- In which sense provides a combination of arguments from numerical linear algebra combined with concepts from (abstract) harmonic analysis the foundation for the DEVELOPMENT OF EFFICIENT ALGORITHMS

Harmonic Analysis and Algorithms

?? Abstract versus (?) computational harmonic analysis [1] ??

1. Abstract Harmonic Analysis provides a unified terminology for Fourier Analysis over general LCA[2] groups **G**, be they continuous or discrete, compact or non-compact, finite or infinite, one- or high-dimensional;
2. Not only Fourier Analysis has its natural analogue over finite Abelian groups, but even all the ingredients of *time-frequency analysis* have their natural meaning for finite groups;
3. In contrast to the *valid analogy of concepts* harmonic analysis provides only little support from an approximation theoretic view-point, e.g. quantitative error estimations.

[1] I like to call their combination *conceptual harmonic analysis*.

[2] Locally Compact Abelian!

Functional Analytic Aspects

From a modeling point of view real world signals are analogue while their representation in the computer are digital. Sound signals are sampled at $44.1 kHz$, digital cameras turn images in the optical lens into (stacks of 3) matrices (R-G-B).

Ignoring the (non-linear) problem of appropriate quantization a good recording device (and then a system to perform digital signal processing on the recorded signal) we realize that we are facing an approximation theoretical problem, which in turn brings us to functional analysis (measuring the errors by some norms) and function spaces.

Functional Analytic Aspects II

Analyzing more carefully what the typical situation is we are facing various steps:

1. Describe according to which measure (norm) the result should be "optimal" (e.g. forming a simulation routine should provide good approximation of the "real output" up to a given error, in some norm, and e.g. stochastically);
2. *Approximation theory* provides general possibilities, constructive approximation theory is outlining a concrete method, but at the end realization on a given computer has to be carried out!
3. Ideally one should try to demonstrate that the chosen strategy is close to optimal.

Functional Analytic Aspects III

Without going into details let us mention that the classical repertoire of function spaces is by no means satisfactory. Within the large zoo of possible function space norms the most popular ones in "hard analysis", namely the spaces $\left(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ are not really important for applications, except of course $p = 1, 2, \infty$.

While there is a variety of norms which describe the smoothness of functions only the classical Sobolev spaces $\mathcal{H}^s(\mathbb{R}^d)$ are really important for PDE applications. For $s \in \mathbb{N}$ they can be described as $\mathbf{L}^2$-functions with s (distributional) derivatives in $\mathbf{L}^2(\mathbb{R}^d)$.

Function spaces based on TF-analysis

Function spaces resulting from TF-analysis[3] turn out to be more useful for a variety of applications, among them the Segal algebra $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ (functions from $\mathbf{L}^1 \cap L^\infty(\mathbb{R}^d)$ with a STFT $V_g(f) \in \mathbf{L}^1(\mathbb{R}^{2d})$), which is the smallest Banach space of functions with an isometrically translation invariant norm and also Fourier invariant.

Together with the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ and its dual space $\mathbf{S}_0'(\mathbb{R}^d)$, which can be characterized as the space of all (tempered) distributions with uniformly bounded spectrogram one obtains the Banach Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$, which is naturally isomorphic (via Wilson bases, to the BGTriple $(\ell^1, \ell^2, \ell^\infty)$, endowed with three types of norm convergence plus also $w^* =$ coordinatewise convergence in ℓ^∞.

[3] i.e. analyzing a distribution by looking at its Short-time Fourier transform or spectrogram, which is a continuous function over phase-space anyway!

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t,\omega);$$

LINEAR ALGEBRA: Gilbert Strang's FOUR SPACES

Let us now take a LINEAR ALGEBRA POINT OF VIEW!
We recall the *standard linear algebra situation*. We view a given $m \times n$ matrix $\mathbf{A}$ either as a collection of *column* or as a collection of *row vectors*, generating $Col(\mathbf{A})$ and $Row(\mathbf{A})$. We have:

$$\textbf{row-rank}(\mathbf{A}) = \textbf{column-rank}(\mathbf{A})$$

Each *homogeneous linear system of equations* can be expressed in the form of scalar products[4] we find that

$$\mathbf{Null}(\mathbf{A}) = \quad \mathbf{Rowspace}(\mathbf{A})^{\perp}$$

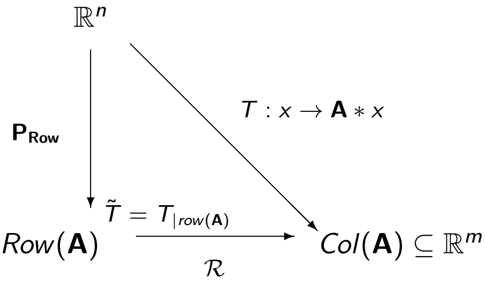
and of course (by reasons of symmetry) for $\mathbf{A}' := conj(A^t)$:

$$\mathbf{Null}(\mathbf{A}') = \quad \mathbf{Colspace}(\mathbf{A})^{\perp}$$

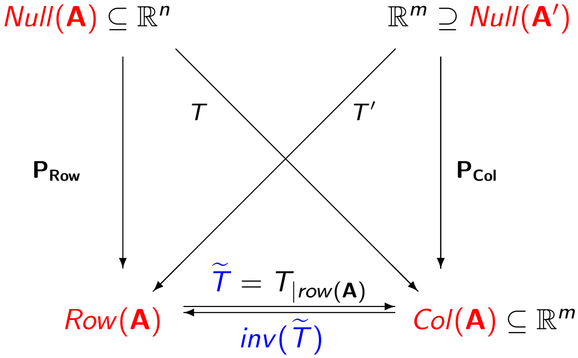
[4] Think of $3x + 4y + 5z = 0$ is just another way to say that the vector $\mathbf{x} = [x, y, z]$ satisfies $\langle \mathbf{x}, [3, 4, 5] \rangle = 0$.

Geometric interpretation of matrix multiplication

Since *clearly* the restriction of the linear mapping $x \mapsto \mathbf{A} * x$ is injective we get an isomorphism $\tilde{T}$ between $Row(\mathbf{A})$ and $Col(\mathbf{A})$.



Geometric interpretation of matrix multiplication



$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Four spaces and the SVD

The **SVD** (the so-called Singular Value Decomposition) of a matrix, described in the MATLAB helpful as a way to write A as

$$A = U * S * V'$$

, where the columns of U form an ON-Basis in $\mathbb{R}^m$ and the columns of V form an ON-basis for $\mathbb{R}^n$, and S is a (rectangular) diagonal matrix containing the non-negative *singular values* (σ_k) of A. We have $\sigma_1 \geq \sigma_2 \ldots \sigma_r > 0$, for $r = rank(A)$, while $\sigma_s = 0$ for $s > r$. In standard description we have for A and $pinv(A) = A^+$:

$$A * x = \sum_{k=1}^{r} \sigma_k \langle x, v_k \rangle u_k, \quad A^+ * y = \sum_{k=1}^{r} \frac{1}{\sigma_k} \langle y, u_k \rangle v_k.$$

Generally known facts in this situation

The Four Spaces are well known from LINEAR ALGEBRA, e.g. in the **dimension formulas**:

ROW-Rank of **A** *equals COLUMN-Rank of* **A**.

The defect (i.e. the dimension of the Null-space of **A***) plus the dimension of the range space of* **A** *(i.e. the column space of* **A***) equals the dimension of the domain space* $\mathbb{R}^n$. Or in terms of linear, homogeneous equations: The dimension of set of all solution to the homogeneous linear equations equals the number of variables minus the dimension of the column space of **A**.

The SVD also shows, that the *isomorphism* $\widetilde{T}$ *between the Row-space and the Column-space* can be described by a diagonal matrix, if suitable orthonormal bases are used.

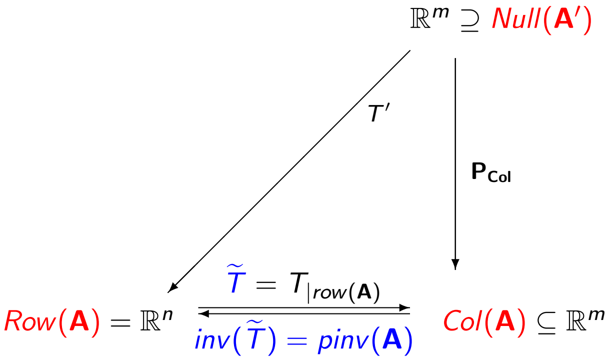
Consequences of the SVD

We can describe the quality of the isomorphism $\widetilde{T}$ by looking at its condition number, which is σ_1/σ_r, the so-called **Kato-condition number** of T.
It is not surprising that for **normal matrices** with $A' * A = A * A'$ one can even have diagonalization, i.e. one can choose $U = V$, using to following simple argument:

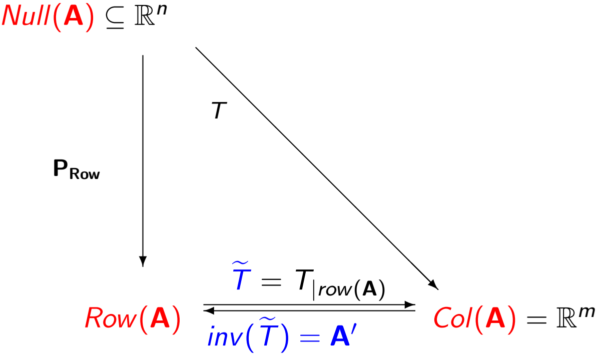
$$Null(A) =_{always} \; Null(A' * A) = Null(A * A') = Null(A').$$

The most interesting cases appear if a matrix has maximal rank, i.e. if $rank(\mathbf{A}) = min(m, n)$, or equivalently if one of the two Null-spaces is trivial. Then we have either linear independent columns of $\mathbf{A}$ (injectivity of $T >>$ RIESZ BASIS for subspaces) or the columns of $\mathbf{A}$ span all of $\mathbb{R}^m$ (i.e. surjectivity, resp. $Null(A') = \{0\}$): $>>$ FRAME SETTING!

Geometric interpretation: linear independent set > R.B.

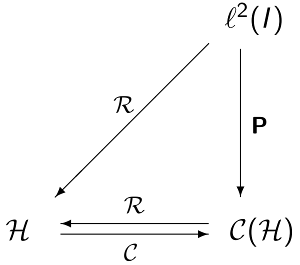


Geometric interpretation: generating set > FRAME



The frame diagram for Hilbert spaces:

If we consider **A** as a collection of column vectors, then the role of **A**′ is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (**??**).

Riesz basic sequences in Hilbert spaces:

The diagram for a Riesz basis (for a subspace), nowadays called a Riesz basic sequence (RBS) looks quite the same ([1]).
In fact, from an abstract sequence there is no! difference, just like there is no difference (from an abstract viewpoint) between a matrix $\mathbf{A}$ and the transpose matrix $\mathbf{A}'$.
In this way in the RBS case one has the synthesis mapping $\mathbf{c} \mapsto \sum_i c_i g_i$ from $\ell^2(I)$ into the Hilbert space $\mathcal{H}$ is *injective*, while in the frame case the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ from $\mathcal{H}$ into $\ell^2(I)$ is injective (with bounded inverse).
Of course one can consider a RBS as a Riesz basis for the closed linear span of its elements, establishing an isomorphism between $\ell^2(I)$ and $\mathcal{H}$.

Algebraic Properties of Gabor Frames

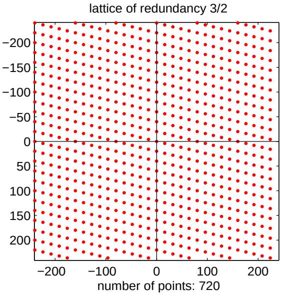
The *algebraic properties* allowing to use appropriate (non-commutative) Banach algebras of sparse matrices can thus be explained in the context of linear algebra and finite groups.
In a second step the transition to the infinite-dimensional situation can be done using functional analytic arguments (some new questions arise) and the proper function space setting (namely modulation spaces, which contain ordinary Sobolev spaces and Shubin classes, arising in the study of the harmonic oscillator).

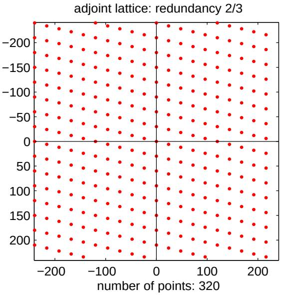
The Gabor Frame Operator for (g, Λ)

Main properties of the Gabor frame operator

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, f \in \mathbf{L}^2(\mathbb{R}^d).$$

A typical example: every point of the left lattice below (Λ) corresponds to one "atom centered at $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$":





Commutation Rules in a Non-Comm. Setting

The commutation relation

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

implies that the matrix/operator can be written as a superposition of TF-shift operators from the adjoint lattice. This is called the Janssen representation of the Gabor frame operator.

$$S_{g,\gamma,\Lambda} = red(\Lambda) \cdot \sum_{\lambda^\circ \in \Lambda^\circ} V_\gamma g(\lambda^\circ) \pi(\lambda^\circ).$$

Note the explicit form of the coefficients. Good decay and smoothness imply that for $\gamma = g$ the invertibility of $S_{g,\Lambda}$ follows from concentration of $V_g(g)$ around zero.

The Ron-Shen Principle

From the Janssen criterion one finds that (g, Λ) generates a Gabor frame (i.e. S is invertible on $\mathbf{L}^2(\mathbb{R}^d)$) if and only if there exists $\gamma \in \mathbf{L}^2(\mathbb{R}^d)$ such that $V_g\gamma(\lambda^\circ) = \delta_{0,\lambda^\circ}$. In fact, if g is normalized with $\|g\|_2 = 1$ the zero-element $\pi(0,0) = Id$ takes a dominant role within the Janssen expansions and guarantees invertibility (not only over $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$).
In particular, invertibility is granted if Λ° is coarse enough or equivalently if Λ is dense enough.

Theorem

$G(g, \Lambda)$ is a frame if and only if the Gabor system $G(g, \Lambda^\circ)$ is a Riesz basis for its linear span. Moreover, the condition number of the frame operator for $G(g, \Lambda)$ coincides with the condition number for the Gramian matrix for the system $G(g, \Lambda^\circ)$.

Solving the Biorthogonality Problem

The Ron-Shen principle shows that one can replace the inversion of the frame operator S by the inversion of the Gram matrix for the system $(g_{\lambda^\circ})_{\lambda^\circ \in \Lambda^\circ}$, which is smaller.

For the finite setting, e.g. $n = 480$, $red = 3/2$ we have 720 Gabor atoms for the space $\mathbb{C}^n$, and the Gram-matrix has only size 320×320.

The invariance properties mentioned allow to solve the problem to solve the equation

$$S(h) = g$$

for $h \in \mathbf{L}^2(\mathbb{R}^d)$. In fact one obtains the canonical dual atom by inverting the positive definite and sparse matrix.

Signal Representation and Mobile Communication

The Ron-Shen principle also says that the stability of the two related families, namely the Gabor frame $(g_\lambda)_{\lambda\in\Lambda}$, expressed by the condition number of the Gabor frame operator S is exactly the same as the quality of the (linear independent) Riesz basic sequence $(g_{\lambda^\circ})_{\lambda^\circ\in\Lambda^\circ}$ (for its closed linear span), i.e. the condition number of the corresponding Gram matrix.

While frames are good for the representation of "arbitrary signals" (functions or even tempered distributions) the good stability of Gaborian Riesz bases, which provide approximate eigenvectors to slowly variant channels (linear operators).

Our patents concern efficient algorithms to identify such operators (from the received pilot tones) and to do a fast approximate inversion (*channel identification and decoding).*

Further Numerical Issues

In addition to the general structural properties of Gaborian families (frame resp. Riesz basic sequences) we have studied and implemented methods considering:

1. preconditioners, double preconditioners (obtained by inverting e.g. the diagonal or circulant "component" of S, resp. commutative subalgebras!)
2. functional analytic (spectral - Banach algebra methods) allow to show good properties of the *atom* g (decay at infinity and smoothness) imply corresponding properties for the dual atom $\tilde{g} = h$ (as above), which indicates that a local biorthogonality problem will/can give good approximate dual window;
3. Locality allows to go for a theory where regularity is only valid locally (but not globally).

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

SOGTr-results in Banach Triple terminology

In the setting of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$ a quite similar results is due to Gröchenig and coauthors:

Theorem

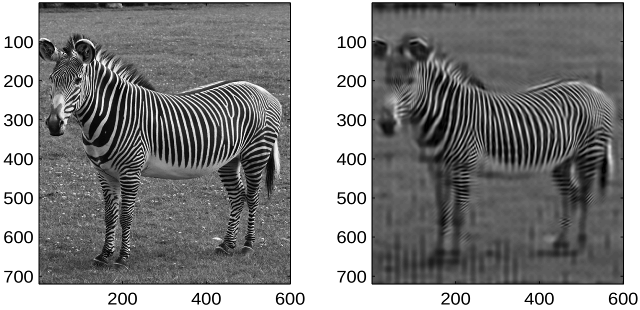
Assume that for some $g \in \mathbf{S}_0$ the Gabor frame operator $S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ is invertible at the Hilbert space level, then S defines automatically an automorphism of the BGT $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$. Equivalently, when $g \in \mathbf{S}_0$ generates a Gabor frame (g_λ), then the dual frame (of the form $(\widetilde{g}_\lambda)$) is also generated by the element $\widetilde{g} = S^{-1}(g) \in \mathbf{S}_0$.

The first version of this result has been based on matrix-valued versions of Wiener's inversion theorem, while the final result (due to Gröchenig and Leinert, see [4]) makes use of the symmetry in Banach algebras and Hulanicki's Lemma.

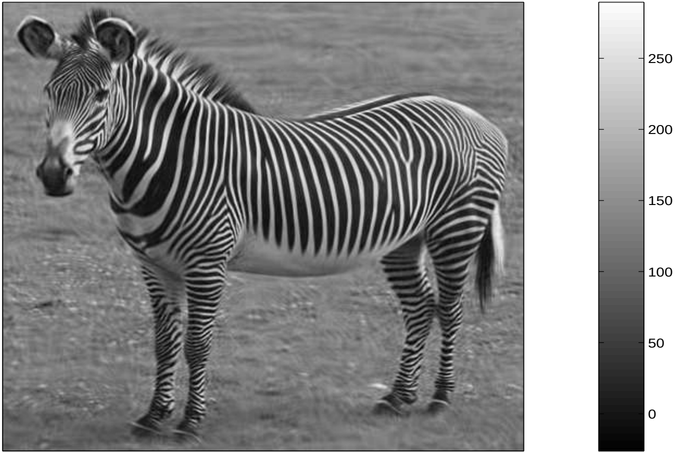
Finally some Applications

Gabor multipliers are just time-variant filterbanks:

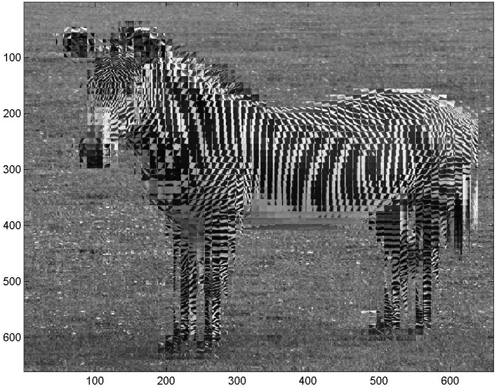
Applications to Image Processing



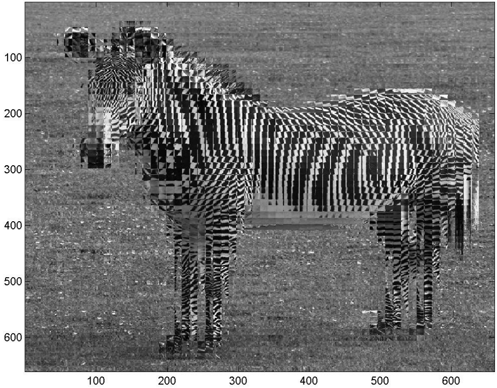
Applications to Image Processing 2



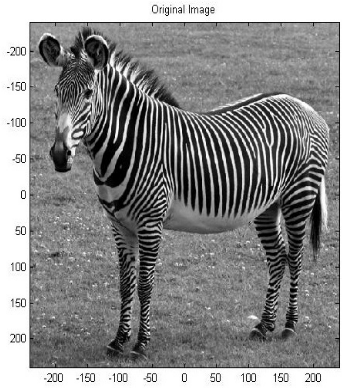
Applications to Image Processing 3

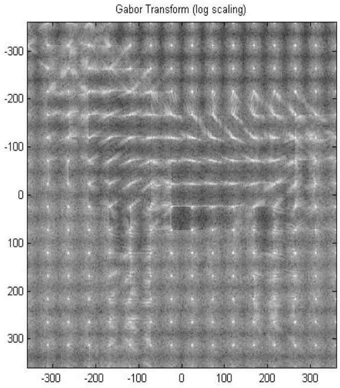


Applications to Image Processing 3

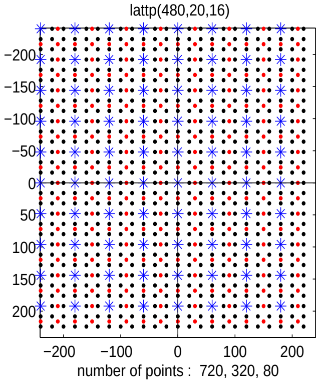


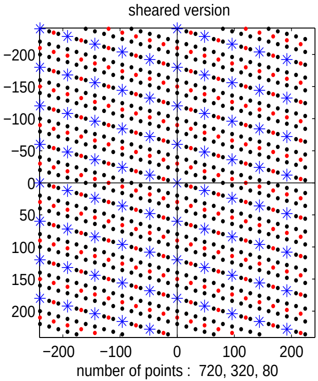
The Gabor Coefficients of the Zebra





Some Group Theoretical Questions





Gabor and Spline-type Spaces

Due to the fact that efficient Gabor expansions also allow to realize Gabor multipliers one may ask, whether a given operator can be optimally approximated by a Gabor multiplier, resp. whether a given matrix can be best approximated by the action of a Gabor multiplier for a given Gabor frame generated by (g, Λ), measured in the Frobenius norm.

For that purpose it is of course optimal if the trivial multiplier by $m(\lambda) \equiv 1$ provides the identity. Gabor atoms h with $S_{h,\Lambda} = Id$ are called *tight* Gabor atoms, and they can be obtained from a general Gabor atom by computing $S^{-1/2}g$.

Using the so-called Kohn-Nirenberg symbol for general operators this problem can be equivalently expressed as a best approximation of a given $\mathbf{L}^2(\mathbb{R}^{2d})$-function by a spline-like function.

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