NuHAG Numerical Harmonic Analysis Group

Gabor Analysis: Foundations and Recent Progress

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

Norbert Wiener Center (College Park), May 7th, 2013

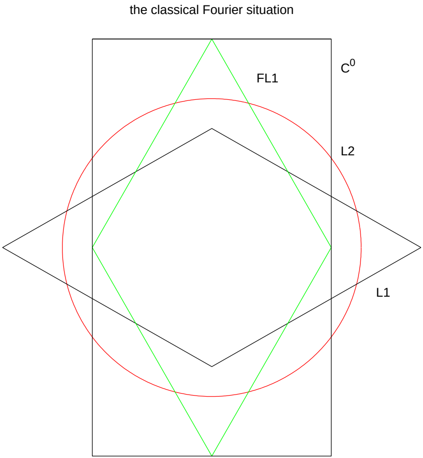
SUMMARY

One can argue (and A.J.E.M. Janssen saw it that way) that by 1998 ([8]) or at least by 2003 ([11, 9]) the field of Gabor analysis was established.
Starting from the questions, when the Gabor analysis resp. synthesis operators are bounded up to the Janssen representation for the Gabor frame operator most facts had been established in natural generality and basic principles are known since that time.
Neverthess the last 10 years have seen some significant progress in the field, and part of it took place at NuHAG and at the Norbert Wiener Center (NWC, here at UMD). (I don't have to report that part here!).

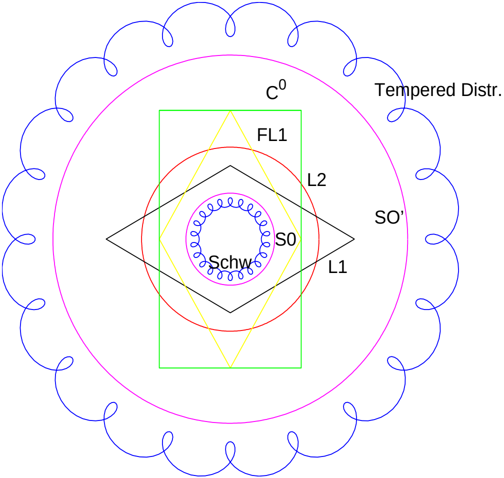
Overview

- Moyal's formula (STFT is isometric!);
- Boundedness of analysis resp. synthesis operator;
- Invariance properties of frame(-like) operators;
- Janssen representation of frame operator ([6]);
- the Walnut representation; sufficient conditions;
- irregular Gabor families (via coorbit theory, [4]);
- jitter error analysis (via coorbit theory);
- the Ron-Shen (resp. Wexler-Raz) principle ([10]);
- the use of Banach Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(G)$.
- there was some MATLAB code around as well (Sigang Qiu, etc., early 90th, Qian-Chen, Shidong Li);

Classical Results in Compact Terminology



Classical Results in Compact Terminology

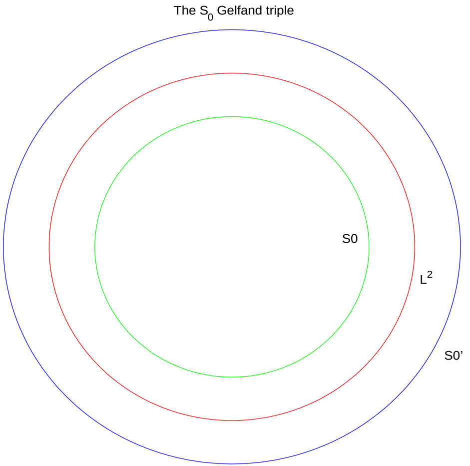


Classical Results in Compact Terminology

By the appearance of our book ([8], 1998) and subsequently Charly Gröchenig's "Foundations" ([11], 2001) it had become clear that modulation spaces (introduced in 1983, published 2003, [2, 3]) are the appropriate family of function spaces in order to describe questions in time-frequency analysis.
Meanwhile this perspective has been confirmed many times and in different settings, from classical Fourier analysis (through the work of Ferencz Weisz,e.g. [13]) to the theory of pseudo-differential operators or the Sjoestrand algebra of operators (work of Gröchenig and Strohmer: [12]).

Classical Results in Compact Terminology

Many facts can now be described using the **Banach Gelfand Triple** $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(G)$.



Achievements in the last decade!

The main topics that can be mentioned are

- Localization and Wiener Pairs
  (Banach algebra methods: Gröchenig)
- Excess and thinning (Balan - Casazza - Landau);
- Isomorphism Properties for modulation spaces (Gröchenig - Cordero - Toft);
- Applications to Schrödinger equation and Fourier integral operators (Cordero, Rodino, ...)
- Totally positive functions ($ab < 1$ suffices!, Gröchenig and Stöckler);

Robustness results

Robustness results deal with the question, to which extent the Gabor frame expansions are robust against modification of the ingredients. Typically the are requiring "some good quality" of atoms and apply uniformly to large families of function spaces (namely modulation spaces showing a certain for of control on the TF-shift operators on them). For the case of $\mathbf{M}^s_{p,q}(\mathbb{R}^d)$ one may typically require that $|s| \leq s_0 > 0$ and that the Gabor atom $g \in \mathbf{M}^{s_0}_{1,1}(\mathbb{R}^d)$.

Theorem

$\forall g \in \mathbf{S}_0(\mathbb{R}^d)$, *such that* (g, Λ) *induces a Gabor frame,* $\exists \varepsilon > 0 : \forall \varrho \in [1 - \varepsilon, 1 + \varepsilon]$ *the pair* $(g, \varrho\Lambda)$ *induces a Gabor frame (of uniform quality).*

Robustness results II

In fact, not only are the frames generated in this way all of the same quality (expressed e.g. in terms of the condition number of the corresponding frame operator), but even the canonical dual window $\tilde{g}$ depends continuously (in the sense of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$) on the dilation parameter ϱ.
Moreover, the result (as described in [5]) is not restricted to isotropic dilations, but also allows for rotations or anisotropic dilations, as long as they are close to the identity matrix.

The proof relies on a number of properties of modulation spaces (specifically $\mathbf{S}_0(\mathbb{R}^d) = M_0^{1,1}(\mathbb{R}^d)$) and its pointwise multiplier algebra $\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^\infty)(\mathbb{R}^d)$) and Janssen's representation of the frame operator, involving the adjoint lattice Λ°.

Robustness results III

Although the methods (in particular Poisson's formula for the symplectic Fourier transform, which can be used to establish the Fundamental Identity of Gabor Analysis [FIGA], [7]) do not generalize to the irregular settings it still has been possible to show (upcoming Trans. Amer. Math. Soc. paper ([1]) with G. Ascensi and N. Kaiblinger), that we have:

Theorem

Assume that the family $(\pi(\lambda_i)g)_{i\in I}$ *is a Gabor frame, for a discrete family* $(\lambda_i)_{i\in I}$ *in* $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. *Then* $\exists \varepsilon > 0 : \forall \varrho \in [1-\varepsilon, 1+\varepsilon]$ *the family* $(\pi(\varrho\lambda_i)g)_{i\in I}$ *is a Gabor frame.*

Note that such a result implies that individual points (already far away from the center) may move arbitrarily far, in contrast to *jitter error analysis*, was was known already for a long time (as a part of coorbit theory, [4]).

Robustness results IV

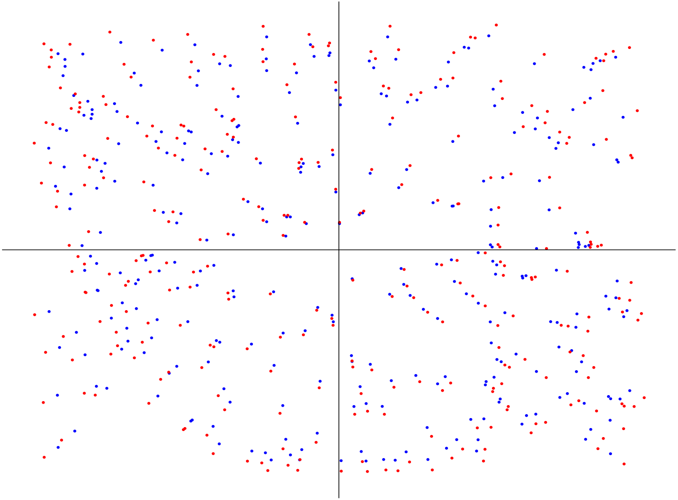
The results concerning jitter error reads as following:

Theorem

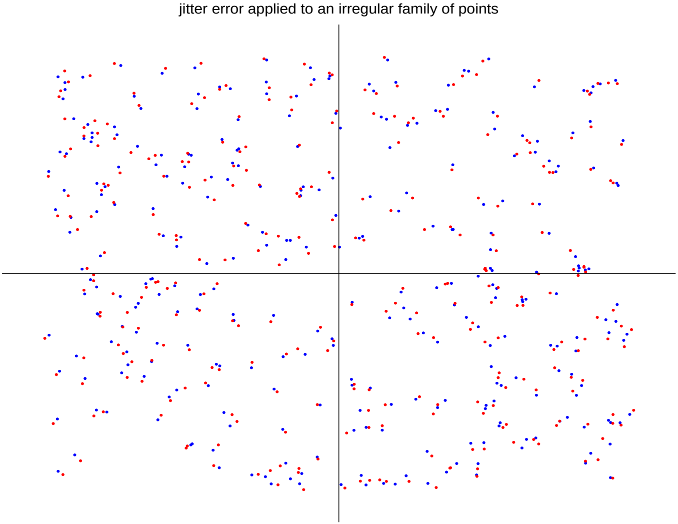
Assume that the family $(\pi(\lambda_i)g)_{i \in I}$ is a Gabor frame, for a discrete family $(\lambda_i)_{i \in I}$ in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Then $\exists \delta_0 > 0 : \forall$ families (y_i) with $|y_i - \lambda_i| \leq \delta_0$ one has $(\pi(y_i)g)_{i \in I}$ is a Gabor frame, and in fact the frame bounds for all these irregular Gabor families obtained in this way is uniformly controlled (for fixed δ_0).

Robustness results V

stretching an irregular family of points



Robustness results VI



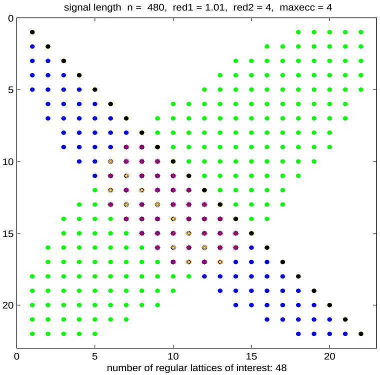
What are good Gabor families?

There is a multitude of criteria which can be applied to Gabor families, and one may ask (in the spirit of a consumer report) which of those systems should be considered good?

But it is like real life. The question is NOT: "What is the best computer that I can find on the market?", but rather, given my budget (or my needs), what are the best computers for the given price. Or how high is the price compared to the performance? *THERE is no absolute ranking*, and in a first approximation we may view the redundancy of the Gabor family as the "price label".

The world of separable lattices

For the case of periodic signals of length n the possible separable lattices are simply characterized by pairs of divisors (a, b). The redundancy is then computed as n/ab. The green points describe the "not-too-excentric" lattices.



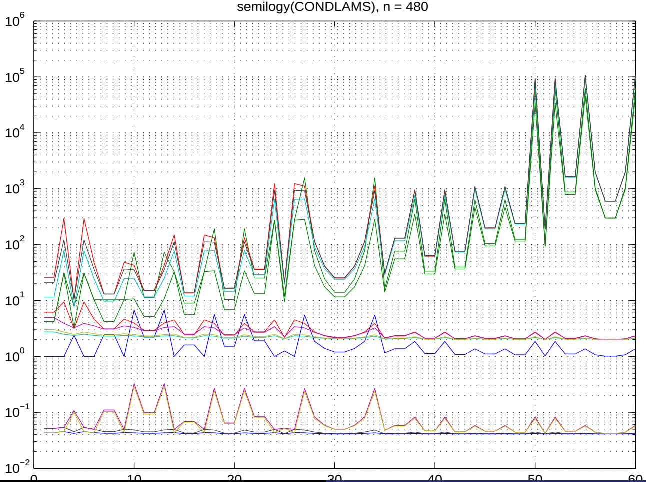
CONSUMER RECOMMENDATIONS

Among the various criteria for the quality of Gabor systems we have used in particular the following ones:

- the condition number of the frame operator, which equals the Riesz condition number of the adjoint Gabor system;
- the $\mathbf{S}_0(\mathbb{R}^d) = \mathbf{M}^1(\mathbb{R}^d)$-norm of the dual window;
- the spreading of $V_{\tilde{g}}\tilde{g}$ (by inspection);
- the covering properties of the of lattices;

Creating consumer reports

Typical consumer reports tell the potential user, which kind of Gabor system s/he might want to use for which application.



Janssen representation

The commutation relation $\pi(\lambda) \circ S_{g,\Lambda} = S_{g,\Lambda} \circ \pi(\lambda)$ is not only relevant for the fact, that the *dual* of a *Gabor frame* is again a Gabor frame, generated from what is called the *dual Gabor atom* (for the given atom g and lattice Λ), it has also other useful consequences, most of which directly related to the so-called Janssen representation of the Gabor frame operator.
Recall, that the spreading representation of an operator corresponds - in the case of finite groups - the representation of an $N \times N$ matrix as a superposition of N^2 TF-shift matrices. In fact, these matrices generate (up to the scaling factor $\sqrt(N)$) an orthonormal system within the $N \times N$-matrices, endowed with the natural Euclidean structure of $\mathbb{C}^{N^2}$ (Frobenius norm).

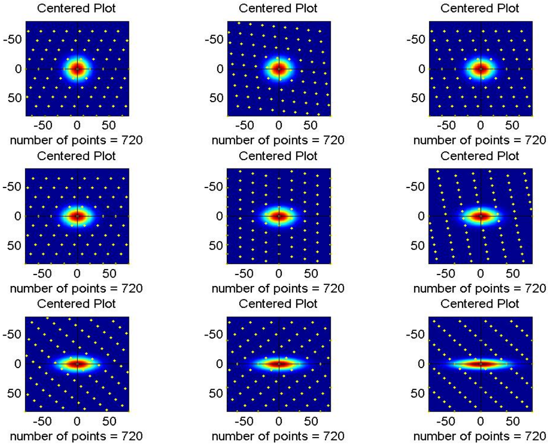
The general Janssen representation

Since this transition has many good properties analogous to the ordinary Fourier transform (e.g. periodization corresponds to sampling in the spreading domain!) it may not come as a complete surprise that the spreading operator
$S : f \mapsto S(f) := \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ can be viewed as $S = \sum_{\lambda \in \Lambda} P_\lambda$ resp. as the $\Lambda-$ periodization of the rank one operator $P_o : f \mapsto \langle f, g_0 \rangle g_0$, which is nothing but the orthogonal projection onto the 1d-subspace generated by the atom (assuming from now on that $\|g_0\|_2 = 1$). Using that the spreading function of P_0 coincides with $V_{g_0} g_0$, the short-time Fourier transform of the atom g_0 with respect itself (also ambiguity) function, on ends up (using the symplectic Poisson formula) with

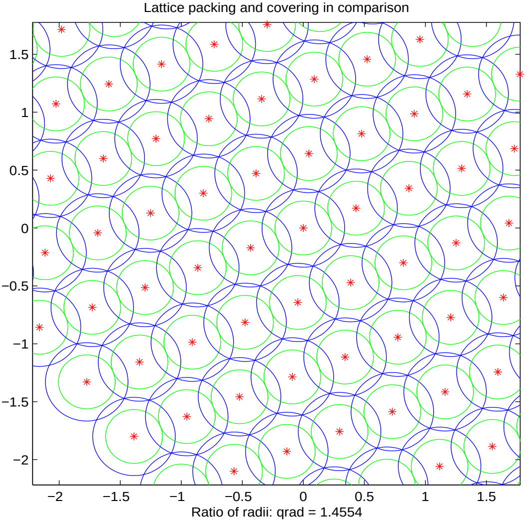
$$S_{g,\Lambda} = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ).$$

Optimal Matching of Atom and Lattice

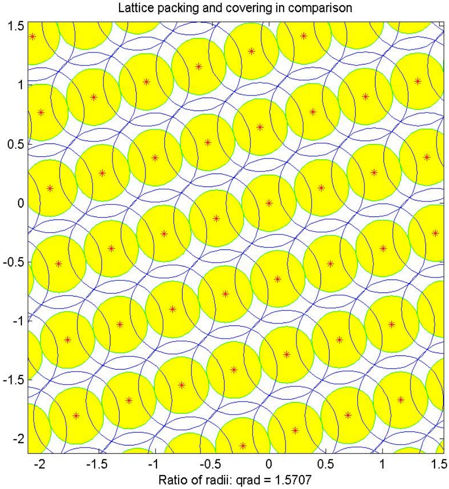
A collection of optimal **non-separable lattices** adapted to Gauss atoms of different stretching (within a discrete setting, fixed n).



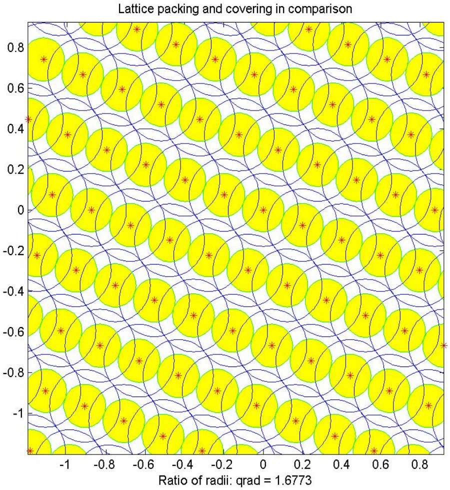
Lattice Visualization



Lattice Visualization



Lattice Visualization



CONCLUSION

Further material can be found from the NuHAG web pages: in particular:

- references and PDF-files at `http://www.nuhag.eu/bibtex`;
- MATLAB code and toolboxes (e.g. LTFAT);
- Talks and course notes; `http://www.nuhag.eu/talks` `http://www.nuhag.eu/db`

G. Ascensi, H. G. Feichtinger, and N. Kaiblinger.
Dilation of the Weyl symbol and Balian-Low theorem.
Trans. Amer. Math. Soc., pages 01–15, 2013.

H. G. Feichtinger.
Modulation spaces of locally compact Abelian groups.
In R. Radha, M. Krishna, and S. Thangavelu, editors, *Proc. Internat. Conf. on Wavelets and Applications*, pages 1–56, Chennai, January 2002, 2003. New Delhi Allied Publishers.

H. G. Feichtinger.
Modulation Spaces: Looking Back and Ahead.
Sampl. Theory Signal Image Process., 5(2):109–140, 2006.

H. G. Feichtinger and K. Gröchenig.
Banach spaces related to integrable group representations and their atomic decompositions, I.
J. Funct. Anal., 86(2):307–340, 1989.

H. G. Feichtinger and N. Kaiblinger.
Varying the time-frequency lattice of Gabor frames.
Trans. Amer. Math. Soc., 356(5):2001–2023, 2004.

H. G. Feichtinger and W. Kozek.
Quantization of TF lattice-invariant operators on elementary LCA groups.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor analysis and algorithms*, Appl. Numer. Harmon. Anal., pages 233–266. Birkhäuser Boston, Boston, MA, 1998.

H. G. Feichtinger and F. Luef.
Wiener amalgam spaces for the Fundamental Identity of Gabor Analysis.
Collect. Math., 57(Extra Volume (2006)):233–253, 2006.

H. G. Feichtinger and T. Strohmer.
Gabor Analysis and Algorithms. Theory and Applications.
Birkhäuser, Boston, 1998.

H. G. Feichtinger and T. Strohmer.
Advances in Gabor Analysis.
Birkhäuser, Basel, 2003.

H. G. Feichtinger and G. Zimmermann.
A Banach space of test functions for Gabor analysis.
In H. G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser Boston, Boston, MA, 2001.

K. Gröchenig and T. Strohmer.
Pseudodifferential operators on locally compact abelian groups and Sjöstrand's symbol class.
J. Reine Angew. Math., 613:121–146, 2007.

F. Weisz.
ℓ_1 -summability of higher-dimensional Fourier series.
J. Approx. Theory, 163(2):99 – 116, February 2011.