NuHAG Numerical Harmonic Analysis Group

Numerical Issues in Time-Frequency Analysis

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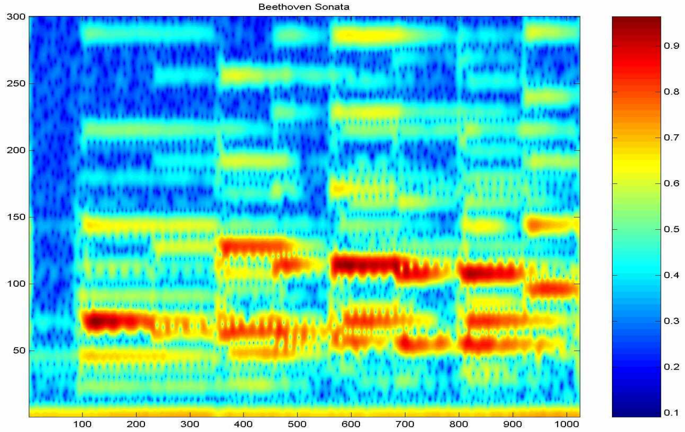
Gabor Analysis: Beethoven Piano Sonata



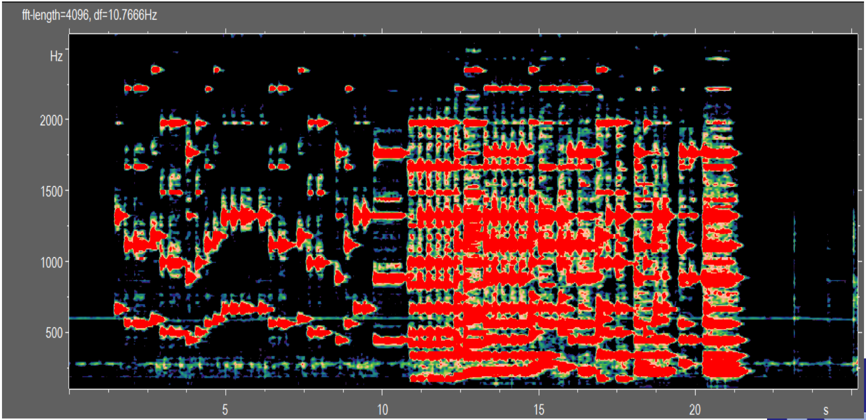
Background on Gabor Analysis

http://www.univie.ac.at/nuhag-php/home/db.php

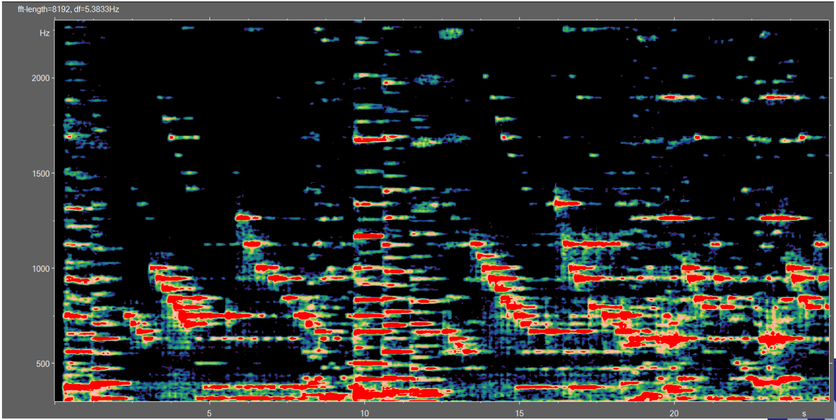
The BIBTEX section contains all our papers, including the ones with the code `feluwe07` (from Linear Algebra to Gabor Analysis) or `fekolu09` (Gabor Analysis over finite Abelian Groups).

MATLAB code for all this can also be obtained from **hgfei**. THe LTFAT toolbox is highly recommended (running on Windows or Linux, and Octave or MATLAB).

Guess the Children's Song Played!



From a Mozart Piano Trio



Aspects of Gabor Analysis

Gabor analysis is concerned with a very intuitive way of representing signals, also allowing to realize time-variant filtering (the computational analogue of the action of an audio engineer).

The classical literature emphasizes the functional-analytic subtleties of such non-orthogonal expansions. Describing Gabor Analysis from a Numerical Linear Algebra and Harmonic Analysis point of view helps to provide a constructive approximation theoretic view-point to the "continuous case".

Linear Signal Representations

Let us compare various classical settings where signal representations occur in a natural way:

- Fourier Series;
- Fourier transforms (on $\mathbb{R}^d$);
- FFT resp. DFT (Discrete Fourier Transform);

In the first case we have learned that the correct view-point is to put oneself into the *Hilbert space* $\mathbf{L}^2(\mathbb{U})$ and consider the sequence of pure frequencies as a CONB for this space.
In the last (FFT) case the sum is even a finite one and we just have a unitary change of basis (up to the normalization factor $\sqrt{N}$).

Linear Signal Representations

Good **wavelet bases** provide CONBs for $\mathbf{L}^2(\mathbb{R}^d)$ (complete orthonormal bases), obtained as double indexed families $(\psi_{k,l})$. Hence every $f \in \mathbf{L}^2(\mathbb{R}^d)$ has an (unconditional) convergence of the form

$$f = \sum_{k,l} \langle f, \psi_{k,l} \rangle \psi_{k,l}.$$

Good wavelets are well concentrated, smooth wavelets satisfying a few moment conditions allow to completely characterize the membership of f in the classical function spaces (e.g. Besov spaces) as unconditional bases!.
Such orthonormal bases are **difficult to build**, but **easy to use**.
For other (non-orthogonal, redundant) systems the task of obtaining (in a linear and constructive way) coefficients is of course more challenging.

Gabor versus Wavelet Theory: Analogies

There are many aspects that make wavelet theory and Gabor analysis comparable.

- In both case there is a group acting on the Hilbert space $\mathbf{L}^2(\mathbb{R}^d)$ (affine group vs. Weyl-Heisenberg-group);
- There is a continuous transform $f \mapsto V_g(f)$ given by $V_g(f)(x) := \langle f, \pi(x)g \rangle$, and a "continuous reconstruction formula".
- There are natural discretizations of this transform and recovery from sufficiently dense and (quasi-) regular sampling sets can be granted (for nice atoms g);

YET, there are however some significant DIFFERENCES:

Gabor versus Wavelet Theory: Differences

- The natural subsets for discretization are NOT a sub-group in the wavelet case (beware of inverse elements), but they form a group in the Gabor case;
- the "ax+b"-group is non-unimodular ($>$ technical difficulties, admissibility condition), while the Heisenberg group is "nice";
- Balian-Low prohibits the existence of Gaborian Riesz Basis for $\mathbf{L}^2(\mathbb{R}^d)$ (against D. Gabor's hope) while GOOD orthonormal wavelets exist.
- the *Schrödinger representation* of the Weyl-Heisenberg-Group corresponds to a *projective* representation of phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ only.

General Aspects of Gabor Analysis

Gabor Analysis is concerned with discrete expansions of functions or (tempered) distributions in the form of sums of building blocks, which are obtained from a single *Gabor atom*, typically a Gaussian window, by applying TF-shifts along some lattice in phase space. It is thus (even for functions of a *continuous variable*) a discrete expansions, often viewed as *discretizations* of the *coherent states expansion* used in theoretical physics.

In contrast to wavelet analysis, where sophisticated constructions allow to discretize the continuous wavelet transform and work with discrete *orthogonal expansions* of wavelet type, the *Balian-Low principle* prohibits the existence of orthonormal systems obtained by TF-shifts of a single atom which is well-concentrated in the TF-sense.

Frames defined by a Pair of Inequalities

Definition

A family $(f_i)_{i \in I}$ in a Hilbert space $\mathbf{H}$ is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathbf{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2$$

It is well known that condition (1) is satisfied if and only if the associated frame operator is positive definite:

Frames and Frame Operators II

Definition

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for} \quad f \in \mathbf{H},$$

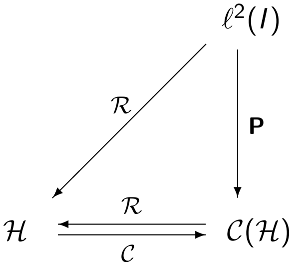
is invertible. The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\tilde{f}_i)_{i \in I}$, defined by $\tilde{f}_i := S^{-1}(f_i)$ has the property that one has for $f \in \mathbf{H}$:

$$f = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i \qquad (1)$$

These formulas emphasize either the reconstruction (sampling) point of view or the *atomic composition* aspect of frames.

The frame diagram for Hilbert spaces:

If we consider **A** as a collection of column vectors, then the role of **A**$'$ is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities.

Riesz Basic Sequences versus Frames

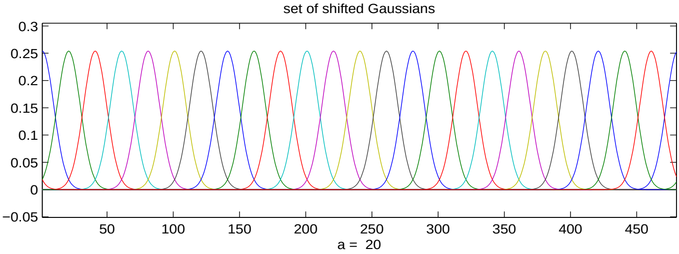
The collection of all integer-shifted boxcar-functions, multiplied with all pure frequencies (local Fourier expansions) provides an orthonormal basis for $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, but it is not suitable for the analysis of smooth functions (bad approximations, many coefficients).

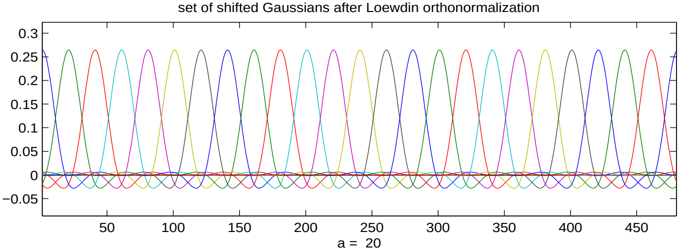
Unfortunately, the Balian-Low principle prohibits the existence of CONBs for $\mathbf{L}^2(\mathbb{R}^d)$ with good TF-localization (good decay in time and in frequency, i.e. with additional smoothness).

Therefore we have a splitting into two questions:

1. How can one build good Gaborian Riesz sequences (spanning proper subspaces of $\mathbf{L}^2(\mathbb{R}^d)$)?
   $>$ mobile communication!
2. How can one build low redundant Gaborian Frames?
   $>$ digital signal processing!

Illustration of Loewdin orthonormalization





Invariance Properties of Frame operator

SIMILARLY we can prove that a *Gabor frame operator*

$$Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$$

commutes with the whole family of TF-shifts used from Λ:

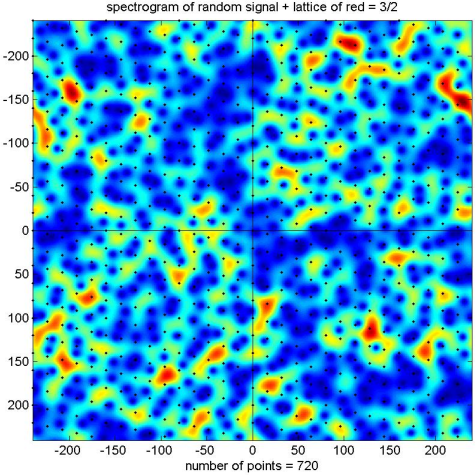
$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

This has many striking consequences, others:

Lemma

Given a regular Gabor system (induced by a lattice Λ) which is a frame or a Riesz basis, then the corresponding dual system is also a Gabor system.

Sampling point of view for Gabor



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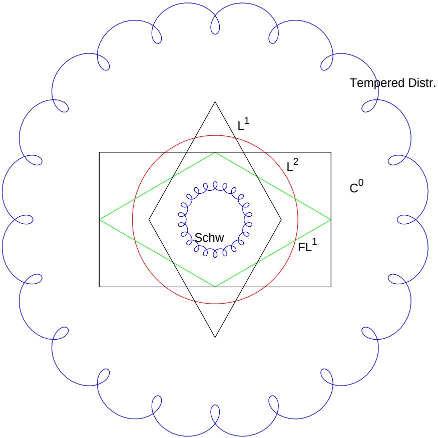

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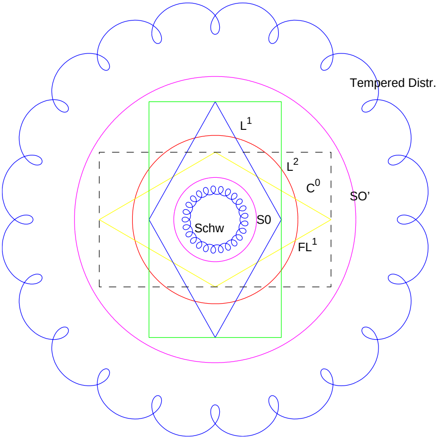


The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

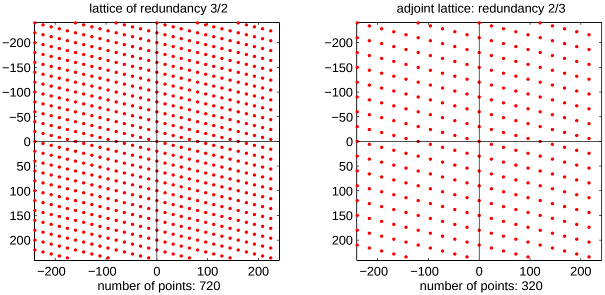
$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

The Gabor Frame Operator for (g, Λ)

Main properties of the Gabor frame operator

$$Sf = \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \pi(\lambda) g = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, f \in \mathbf{L}^2(\mathbb{R}^d).$$

A typical example: every point of the left lattice below (Λ) corresponds to one "atom centered at $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$":



Commutation Rules in a Non-Commutative Setting

The commutation relation

$$S \circ \pi(\lambda) = \pi(\lambda) \circ S, \quad \forall \lambda \in \Lambda.$$

implies that the matrix/operator can be written as a superposition of TF-shift operators from the adjoint lattice. This is called the Janssen representation of the Gabor frame operator.

$$S_{g,\gamma,\Lambda} = red(\Lambda) \cdot \sum_{\lambda^\circ \in \Lambda^\circ} V_\gamma g(\lambda^\circ) \pi(\lambda^\circ).$$

Note the explicit form of the coefficients. Good decay and smoothness imply that for $\gamma = g$ the invertibility of $S_{g,\Lambda}$ follows from concentration of $V_g(g)$ around zero.

The Ron-Shen Principle

From the Janssen criterion one finds that (g, Λ) generates a Gabor frame (i.e. S is invertible on $\mathbf{L}^2(\mathbb{R}^d)$) if and only if there exists $\gamma \in \mathbf{L}^2(\mathbb{R}^d)$ such that $V_g\gamma(\lambda^\circ) = \delta_{0,\lambda^\circ}$. In fact, if g is normalized with $\|g\|_2 = 1$ the zero-element $\pi(0,0) = Id$ takes a dominant role within the Janssen expansions and guarantees invertibility (not only over $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$).
In particular, invertibility is granted if Λ° is coarse enough or equivalently if Λ is dense enough.

Theorem

$G(g, \Lambda)$ is a frame if and only if the Gabor system $G(g, \Lambda^\circ)$ is a Riesz basis for its linear span. Moreover, the condition number of the frame operator for $G(g, \Lambda)$ coincides with the condition number for the Gramian matrix for the system $G(g, \Lambda^\circ)$.

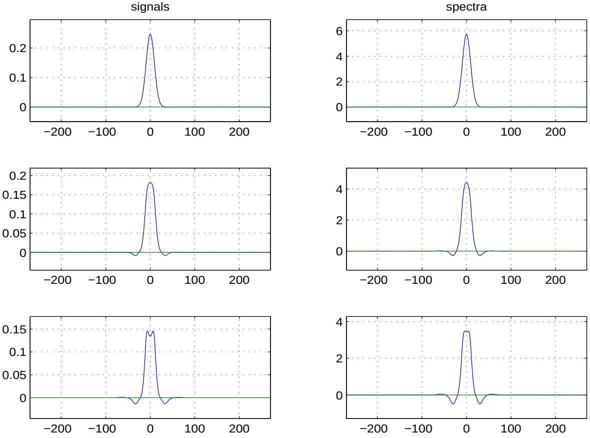
Solving the Biorthogonality Problem

The Ron-Shen principle shows that one can replace the inversion of the frame operator S by the inversion of the Gram matrix for the system $(g_{\lambda^\circ})_{\lambda^\circ \in \Lambda^\circ}$, which is smaller.
For the finite setting, e.g. $n = 480$, $red = 3/2$ we have 720 Gabor atoms for the space $\mathbb{C}^n$, and the Gram-matrix has only size 320×320.
The invariance properties mentioned allow to solve the problem to solve the equation

$$S(h) = g$$

for $h \in \mathbf{L}^2(\mathbb{R}^d)$. In fact one obtains the canonical dual atom by inverting the positive definite and sparse matrix.

Dual Systems generated by Dual Atoms”



Signal Representation and Mobile Communication

The Ron-Shen principle also says that the stability of the two related families, namely the Gabor frame $(g_\lambda)_{\lambda\in\Lambda}$, expressed by the condition number of the Gabor frame operator S is exactly the same as the quality of the (linear independent) Riesz basic sequence $(g_{\lambda^\circ})_{\lambda^\circ\in\Lambda^\circ}$ (for its closed linear span), i.e. the condition number of the corresponding Gram matrix.
While frames are good for the representation of "arbitrary signals" (functions or even tempered distributions) the good stability of Gaborian Riesz bases, which provide approximate eigenvectors to slowly variant channels (linear operators).
Our patents concern efficient algorithms to identify such operators (from the received pilot tones) and to do a fast approximate inversion (*channel identification and decoding).*

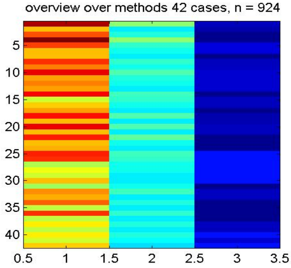
Further Numerical Issues

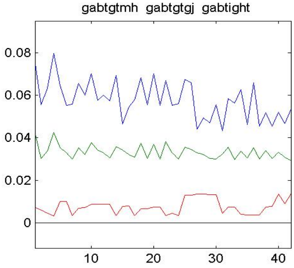
In addition to the general structural properties of Gaborian families (frame resp. Riesz basic sequences) we have studied and implemented methods considering:

1. preconditioners, double preconditioners (obtained by inverting e.g. the diagonal or circulant "component" of S, resp. commutative subalgebras!)
2. functional analytic (spectral - Banach algebra methods) allow to show good properties of the *atom* g (decay at infinity and smoothness) imply corresponding properties for the dual atom $\tilde{g} = h$ (as above), which indicates that a *local biorthogonality problem* will/can give good approximate dual window;
3. Locality allows to go for a theory where regularity is only valid locally (but not globally).

Consumer Reports

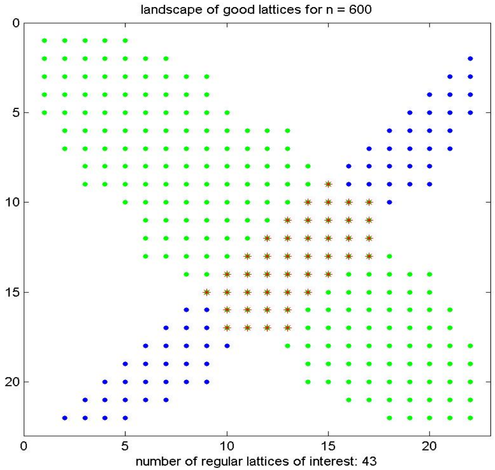
I would also like to mention that nowadays there is a large variety of algorithms which allow to compute the dual or the *tight* Gabor atoms efficiently. The plot below indicates that in a competition (over a family of separable lattice) the LTFAT tight algorithms (red) performs best.





Relevant subfamilies of regular lattices

Since we are not interested in general (bad) lattices one may want to

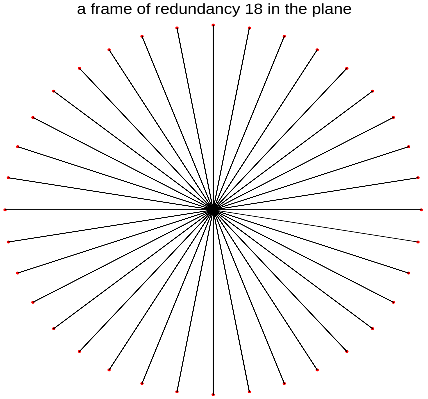


Finally some Applications

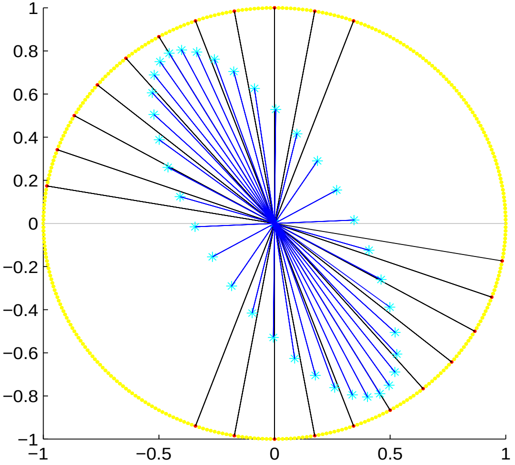
Gabor multipliers are just time-variant filterbanks:

A Geometric Picture: Frames

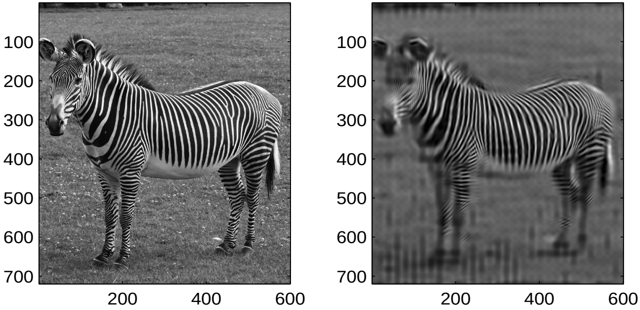
Tight Frames (to be used *like orthonormal systems*), i.e. satisfying $f = \sum_{i \in I} \langle f, f_i \rangle f_i$. In the plane $\mathcal{H} = \mathbb{R}^2$ such a frame of redundancy 18 $(2 \cdot 18)$ may look like this:



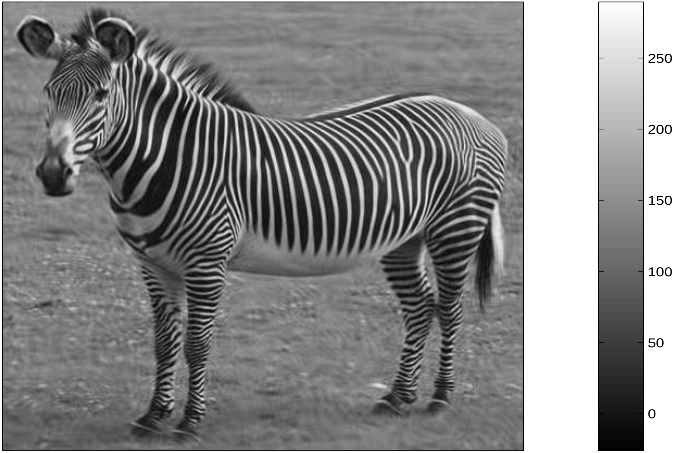
A Geometric Picture: Frame Multipliers



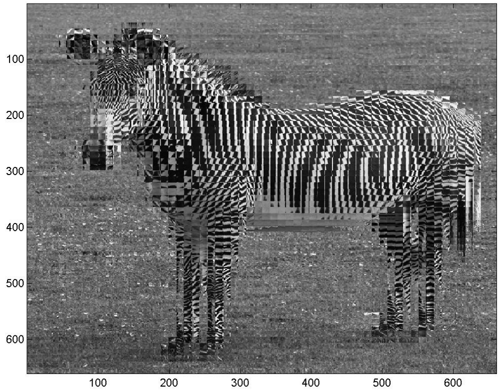
Applications to Image Processing



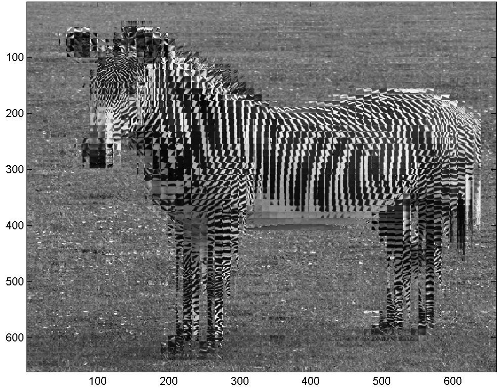
Applications to Image Processing 2



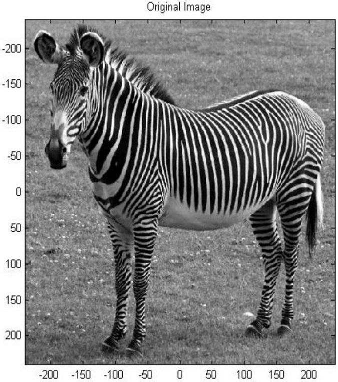
Applications to Image Processing 3

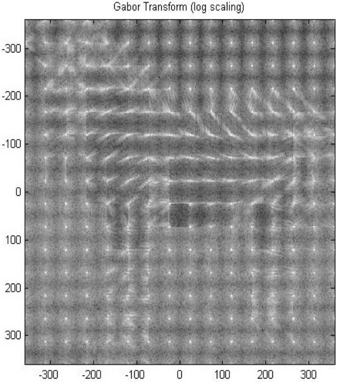


Applications to Image Processing 3

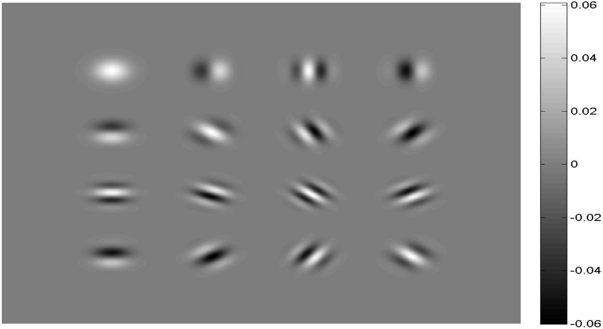


The Gabor Coefficients of the Zebra



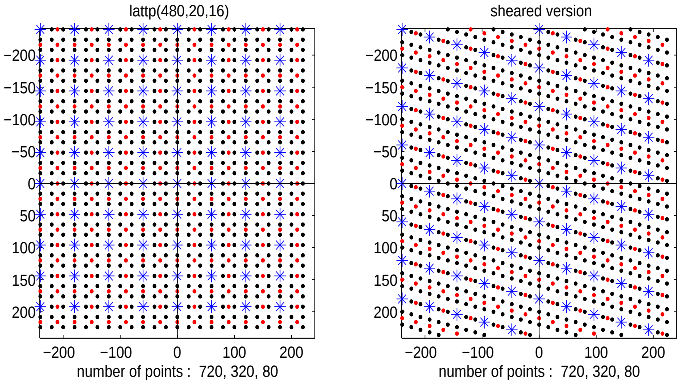


The Gabor Atoms in Image Processing

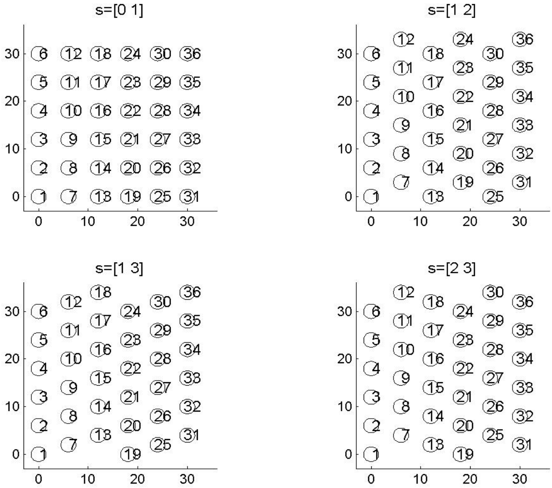


Some Group Theoretical Questions

The lattice, the adjoint lattice and their common, *commutative* subgroup.



Some Group Theoretical Questions: Labeling



Gabor and Spline-type Spaces

Due to the fact that efficient Gabor expansions also allow to realize Gabor multipliers one may ask, whether a given operator can be optimally approximated by a Gabor multiplier, resp. whether a given matrix can be best approximated by the action of a Gabor multiplier for a given Gabor frame generated by (g, Λ), measured in the Frobenius norm.

For that purpose it is of course optimal if the trivial multiplier by $m(\lambda) \equiv 1$ provides the identity. Gabor atoms h with $S_{h,\Lambda} = Id$ are called *tight* Gabor atoms, and they can be obtained from a general Gabor atom by computing $S^{-1/2}g$.

Using the so-called Kohn-Nirenberg symbol for general operators this problem can be equivalently expressed as a best approximation of a given $\mathbf{L}^2(\mathbb{R}^{2d})$-function by a spline-like function.

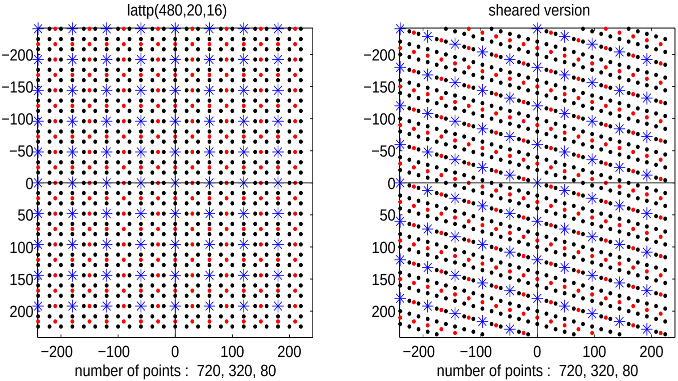
Applications of Gabor Analysis

There is a variety of application areas of Gabor Analysis (similar and sometimes in competition with *wavelets* or *shearlets,curvelets*). Let us mention project related topics:

- audio signal processing (e.g. for electro-cars);
- image processing (see our contribution to the hand-book of image processing, [1], based on S. Paukner's master thesis);
- mobile communication (Gabor Riesz bases, ADSL, OFDM,...).

A short survey of the subject is given in Encyclopedia Applied Mathematics (see [2]).

Some Group Theoretical Questions



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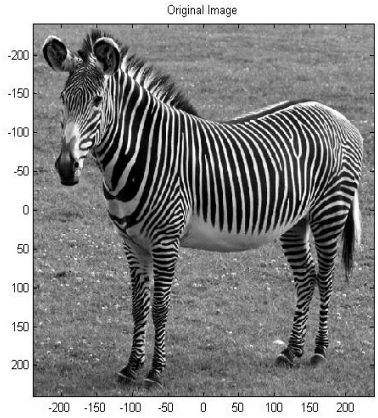
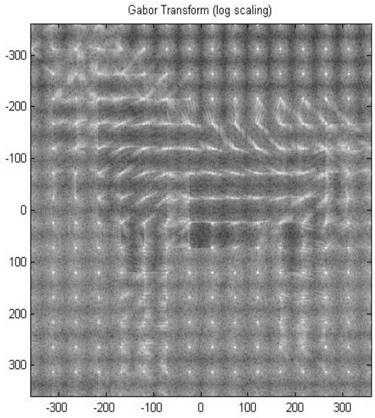
Applications to Image Processing 4





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O. Christensen, H. G. Feichtinger, and S. Paukner.
Gabor Analysis for Imaging, volume 3, pages 1271–1307.
Springer Berlin, 2011.

H. Feichtinger and F. Luef.
Gabor analysis and time-frequency methods.
Encyclopedia of Applied and Computational Mathematics, 2012.