Function Spaces in Harmonic Analysis and Coorbit Theory

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Let us take a historical perspective (with Fourier Analysis in the center of our considerations):

Fourier Series have been introduced around 1822

1822 Jean Baptiste Fourier published his famous *Theorie analytique de la Chaleur* in 1922.

The problems arising from the theory of Fourier series caused discussions about a suitable form of convergence (exceptional sets, measure theory, etc.), even the concept of "functions" had to be clarified.

After a period of using Riemann integrable functions and pointwise convergence for piecewise continuous functions it was about 100 years later that Lebesgue introduced what is now known as the proper form of integration (Thesis of 1902).

Function Spaces and early Functional Analysis

He achieved convergence theorems which were general enough to allow to prove the completeness of the spaces $\left(\mathbf{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ (as prototypical examples), which turned quickly in to the basic examples in S. Banach's theory of what is now called *Banach spaces*. The results of F. Riesz identifying the dual space of $\mathbf{L}^2$ with the space itself (Riesz representation theorem) as well as the general duality theory form nowadays the basis for *functional analysis*, which itself is indispensable for any kind of analysis (PDE, Fourier Analysis, etc.).
Among the Banach spaces the family of *Hilbert spaces* plays a distinguished role.
Later on the relevance of *Frechet spaces* and *topological vector spaces* (of functions and operators) was recognized.

What are Banach Space of Functions?

The terminology is not quite uniform.
I myself was very much influenced by the work of Hans Triebel whose **Theory of Function Spaces** tried to encompass all the then (around 1970) "classical spaces" from the point of view of interpolation theory, that was developed very systematically by himself and Jaak Peetre (who was in fact also interested in Banach spaces of analytic functions).
Most of the spaces, including the Besov and the Triebel-Lizorkin spaces or Bessel potential spaces can be found in E. Stein's book on "Differentiability properties of Functions".
This development was almost independent of the path pursued A. Zaanen and W.A.J. Luxemburg, who used the word **Banach function spaces** for certain complete lattices of measurable functions.

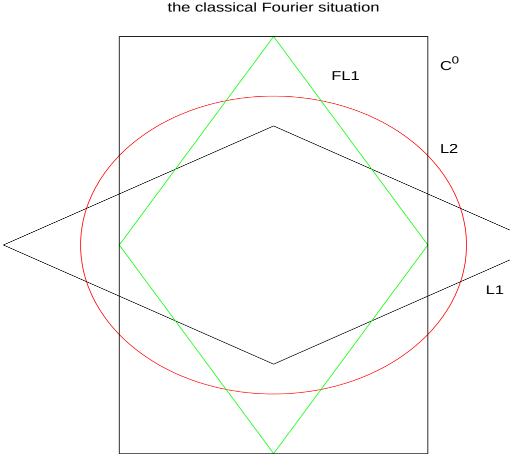
Function Spaces and their Usefulness

When one asks for which purpose different function spaces have been introduced, one finds a variety of different answers:

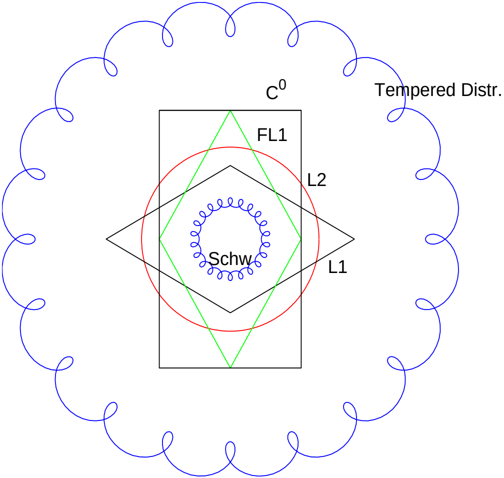
1. in order to describe smoothness
(Lipschitz spaces, Besov spaces, Bessel potentials)
2. in order to provide good domains of operators
$\left((\mathbf{L}^1(\mathbb{R}^d), \| \cdot \|_1 \right)$ for the Fourier transform, Sobolev spaces of differential operators, Hardy space for the Hilbert transform);
3. $BV(\mathbb{R})$ is the right space in order to describe general Riemann-Stieltjes integrals;

OVERALL (citation *Yves Meyer*): Function spaces are good in order to describe the mapping properties of operators.
So let us take a look at the Fourier transform: $f \mapsto \hat{f}$.

The Classical Setting: L^p-spaces



Function spaces and Schwartz spaces



Function Spaces of Generalized Functions

This last example already indicates, that one should not only consider Banach spaces of "ordinary" functions (i.e. in reality *equivalence classes of measurable functions, modulo null-functions*) but also spaces of generalized functions or distributions. These "objects" appear only as rather abstract and maybe difficult objects if the concept of a function as a pointwise mapping has been settled strongly (as it was historically), while - got example - physics student don't view it as a very abstract thing to consider *forces*, and to add up e.g. the gravitational force and the centrifugal force in order to determine the position of a rotating object.

Function Spaces of Generalized Functions

What I wanted to say, that already in the setting promoted by Hans Triebel (or Jaak Peetre) the role of *families of Banach spaces* of functions (in the work of A. Pietsch one finds Banach spaces of operators) is clear.

Such families are typically *closed under duality* (if one excludes the non-separable cases, typically associated with parameters $p, q = \infty$) and (real and complex) interpolation (of Banach spaces). In each such family a Hilbert space, very often $(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2)$, is at the center of the family.

E.g. we have the Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$, or the Triebel-Lizorkin $\mathbf{F}^s_{p,q}(\mathbb{R}^d)$ spaces, where typically such results can be easily described in terms of the relevant parameters.

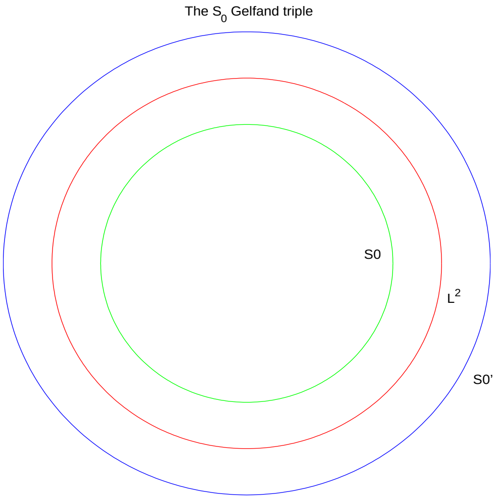
Function Spaces of Generalized Functions

Even during my thesis my interest was in such *families of function spaces*, in construction principles which allow to obtain new families from given ones and to understand the connection (sometimes commutation relations) between construction principles.
For example, one my produce from ordinary $\mathbf{L}^p$-spaces also weighed spaces $\mathbf{L}^p_w(\mathbb{R}^d)$, or one applied (say) complex interpolation to pairs of such spaces. But can one interchange the order of these construction steps and still get the same spaces (and equivalent norms)? The answer in this case is of course YES, if suitable weights are used.

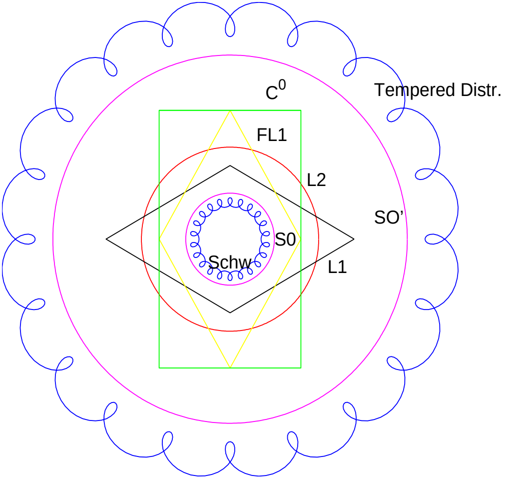
Modulation Spaces (over Euclidean spaces)

One of my early contributions to the theory of function spaces was the introduction of what became known as **modulation spaces**. This is a family of Banach spaces (of tempered distributions, in the "classical setting"), demoted by $\left(\mathbf{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^s_{p,q}}\right)$. They are characterized by the behaviour (in the sense of decay or summability) of a *uniform* decomposition of their spectrum. In other words, one can obtain them by replacing the *dyadic decomposition* of a function in $\mathbf{L}^p(\mathbb{R}^d)$ by a uniform decomposition (of some smoothness). They show decent behaviour (e.g. translation and dilation invariance), and a smoothness parameter $s \in \mathbb{R}$ which is playing almost the same role as in the definition of Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$. Especially, the spaces $\mathbf{M}^{p,p}(\mathbb{R}^d)$ are Fourier invariant spaces for $p \in [1, \infty]$.
I use to write $\mathbf{S}_0(\mathbb{R}^d) = M^{1,1}_0(\mathbb{R}^d)$ and $\mathbf{S}_0'(\mathbb{R}^d) = \mathbf{M}^\infty(\mathbb{R}^d)$.

The Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$



The FULL picture



Function Spaces and Coorbit Theory

One of the motivations for the establishment of *coorbit theory* (joint work with K. Gröchenig, late 80th) was the wish to unify the approach to different families of Banach spaces, among them the

- Besov-Triebel-Lizorkin spaces (over $\mathbb{R}^d$);
- Modulation Spaces (over LCA groups);
- Moebius invariant Banach spaces of analytic functions on the disc;

by understanding them through the fact that the are all forming a *family of Banach spaces* sharing joint invariance properties under certain groups of operators. Although differen when it comes to the concrete computations there is still a lot of common facts which can be established at a fairly general level.

Wavelets, Marseille and Jean Morlet

As most of you are aware Jean Morlet (the person to which the named chair I am presently holding is dedicated) was the inventor of **wavelets**. He had the idea of expanding signals into building blocks of constant shape, which translates into the questions, how one can expand arbitrary "signals" (i.e. elements from $\mathbf{L}^2(\mathbb{R})$) as a superpositions (series) of functions arising from a given shape (often called *mother wavelet*) by affine transformations of the form $f(x) \mapsto f(ax+b), a > 0, b \in \mathbb{R}$.
It was certainly Alex Grossmann who emphasized the role of group theory in this context and probably was the person to introduce what we then also called the *voice transform*.
The analogy between results in wavelet theory (continuous wavelet transforms, atomic decompositions, etc.) and modulation space theory (the sliding window Fourier transform or STFT) and Gabor expansions the motivated the search for a general structure.

THE END of the SLIDE SHOW

The rest is material from an earlier presentation and will only be used to a small extent during the opening talk of the CIRM conference.
http://www.univie.ac.at/nuhag-php/dateien/talks/2453-SPIETALK13fei.pdf
HGFei (26. October 2014)

Abstract

The purpose of this talk is to give a *historical perspective on some aspects of the theory of function spaces*, i.e. Banach spaces of functions (or distributions, when one looks at the dual spaces). The first approach to smoothness resulting in the definition of Sobolev spaces and Besov spaces (Besov, Taibleson, Stein) came from the idea of generalized smoothness, expressed by (higher order) difference expression and the corresponding *moduli of continuity*, e.g. describing smoothness by the decay of the modulus of continuity (via the membership in certain weighted Lq-spaces on (0,1]). Alternatively there is the line described in the book of S.Nikolksii characterizing smoothness (equivalently) by the degree of approximation using band-limited functions (S. M. Nikol'skij [9]). Fractional order Sobolev spaces can be expressed in terms of weighted Fourier transforms.

The second and third age

The second age is characterized by the Paley-Littlewood characterizations of Besov or Triebel-Lizorkin spaces using dyadic decompositions on the Fourier transform side, as used in the work of J.Peetre ([10]) and H.Triebel ([17, 18, 16, 12, 19]) , the masters of interpolation theory. Their contribution was to show that these families of function spaces are stable under (real and complex) interpolation methods.

The third age is - from our point of view - the characterization of function spaces in the context of coorbit spaces, using irreducible integrable group representations of locally compact groups.

Let us also remind that the concept of retracts plays an important role in the context of interpolation theory (see the book of Bergh-Loefstroem), and can be used to characterize Banach frames and Riesz projection bases.

Modulus of continuity

Definition

Assume that $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is an isometrically translation invariant Banach space of locally integrable functions (i.e. $(\mathbf{B}, \|\cdot\|_{\mathbf{B}}) \hookrightarrow \mathbf{L}^1_{loc}(\mathbb{R}^d)$) with

$$\|T_x f\|_{\mathbf{B}} = \|f\|_{\mathbf{B}} \quad \forall f \in \mathbf{B}.$$

In this situation we can define for every $f \in \mathbf{B}$ its *modulus of continuity with respect to* $\|\cdot\|_{\mathbf{B}}$ via

$$\omega_\delta(f) = sup_{|x|\leq\delta} \{\|T_x f - f\|_{\mathbf{B}}\}.$$

In most cases ω is considered a function of δ for fixed $f \in \mathbf{B}$, but the notation is following the traditional one.

Modulus of continuity 2

For each such space it is easy to show that the elements with $\lim_{\delta \to 0} \omega_\delta(f) = 0$ are those for which $x \to T_x f$ is (uniformly) continuous from $\mathbb{R}^d$ into $\left(\mathbf{B}, \|\cdot\|_{\mathbf{B}}\right)$. They form a *closed subspace* of $\left(\mathbf{B}, \|\cdot\|_{\mathbf{B}}\right)$, which we denote by $\mathbf{B}_{cs}$ [1].

Within this class we can identify functions which have a higher degree of "smoothness", i.e. which are not just uniformly continuous, but behave better than the general function in $\mathbf{C}_{ub}(\mathbb{R}^d)$, because ω_δ tends to zero for $\delta \to 0$ at a given rate.

The so-called Lipschitz spaces $Lip(\alpha)$ are characterized by the property that there exists some constant $C > 0$ such that

$$\sup_{\delta > 0} \delta^{-\alpha} \omega_\delta(f) = C < \infty. \tag{1}$$

There are also so-called "small Lipschitz spaces" characterized by

$$\lim_{\delta \to 0} \delta^{-\alpha} \omega_\delta(f) = 0. \tag{2}$$

[1] "cs" standing for "continuous shift". for the case $(\mathbf{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ we

Classical Lipschitz spaces

It is an easy exercise to show that $Lip(\alpha)$ is a Banach space with the norm

$$\|f\|_{Lip(\alpha)} := \|f\|_\infty + \sup_{\delta>0} \delta^{-\alpha}\omega_\delta(f), \qquad (3)$$

and that $(lipa, \|\cdot\|_{Lip(\alpha)})$ is a *closed subspace* of $Lip(\alpha)$, in fact $Lip(\alpha)_{cs} = lipa$.

This construction makes only sense for $\alpha \in (0,1]$, because the class becomes trivial for $\alpha > 1$[2].

There are two ways out, which turn out to be equivalent: Either one assumes that f is continuously differentiable and f' satisfies a Lipschitz condition, *or* one makes use of higher order difference operators e.g. for $k = 2$ one expects decay of the sup-norm of the function $f(x-h) - 2f(x) + f(x+h)$ as $h \to 0$, with some order of h, up to order < 2 (also higher order differences).

[2]Only constant functions: because the assumption implies that the function *is differentiable everywhere* and that $f'(x) \equiv 0$.

Generalized Lipschitz spaces $\mathbf{Lip}(p, \alpha)$

Replacing in this traditional the sup-norm by an $\mathbf{L}^p$-norms and the corresponding modulus of continuity one arrives at the concept of the Lipschitz spaces $\mathbf{Lip}(p, \alpha)$ arise.
The next step towards a general theory of smoothness spaces was taken by **Besov**. Instead of considering just decay of a given order for the modulus of continuity (as a function on $(0, 1]$ or $\mathbb{R}^+$) he was making use of weighted $\mathbf{L}^q$-spaces with respect to the (natural $=$ Haar) measure dt/t on $\mathbb{R}^+$?
The corresponding norms (on $\mathbb{R}^+$ or $(0, 1]$) are of the form

$$\left[\int_0^1 (|H(t)| t^{-s})^q dt/t\right]^{1/q}. \tag{4}$$

Besov spaces

Note that the natural (say exponential function) isomorphism of $(\mathbb{R},+)$ with $(\mathbb{R}^+,\cdot)$ via the exponential function transports functions H on $\mathbb{R}^+$ back into functions $h(t) := H(exp(t))$, so that the condition (4) is equivalent to the membership of h in the usual (polynomial) weighted $\mathbf{L}^q$-space,

$$\mathbf{L}^q_{w_s}(\mathbb{R}) := \{f \mid fw_s \in \mathbf{L}^q(\mathbb{R}^d)\}, \quad \text{with} \quad w_s(t) := (1+|t|)^s. \quad (5)$$

which is a Banach space with its natural norm $\|f\|_{q,w_s} := \|fw_s\|_q$.
The resulting family of spaces is then just the family of Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$.
In the work of S. Nikolskij (still alive!? at age of 102?) the Besov spaces have been characterized by their approximation behaviour with respect to band-limited functions (in his work: entire functions of exponential type,).

Sobolev spaces, fractional derivatives

On the other hand there was the idea of describing smoothness in the sense of differentiability in terms of the Fourier transform. The classical Sobolev spaces $\mathbf{W}^k(\mathbb{R}^d)$ or $\mathcal{H}_s(\mathbb{R}^d)$ or $\mathcal{L}^2_s$ are defined as the function having a derivative up to order k in $\mathbf{L}^2(\mathbb{R}^d)$.
Of course it requires some care to explain in which sense this existence is to be interpreted. There are various natural options:

- assuming the existence of the classical (partial) derivatives a.e. and assuming that they define $\mathbf{L}^2$-functions;
- taking the derivative in the distributional sense and assume that those derivatives are *regular* distributions, i.e. can be represented by $\mathbf{L}^2$-functions;
- use Plancherel's theorem and make use of the fact that the differentiation corresponds to pointwise multiplication with polynomials on the Fourier transform side;

Fortunately these conditions are all *equivalent*!!

The Fourier and Littlewood-Paley age

To my knowledge it have been mostly the two pioneers in interpolation theory, namely Jaak Peetre and Hans Triebel.
The most important alternative description of Besov (and also Bessel potential spaces $\mathcal{H}_s(\mathbb{R}^d)$, which are special cases of the more general Triebel-Lizorkin spaces) is through *dyadic partitions of unity*, typically in the form of dilation of a fixed function ψ which is assumed to be such that one can control all of its derivatives.
The classical description of Besov spaces in the books of Triebel makes use of terms such as

$$\| \mathcal{F}^{-1}[\widehat{f} \cdot \psi(2^k \cdot)]\|_p \qquad (6)$$

Since we are working with Banach spaces (such as $\mathbf{L}^p(\mathbb{R}^d)$ etc.) within the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ anyway, I prefer to rather *take the* $\mathbf{L}^p$*-norm over to the Fourier transform side*, rather than jumping between time- and frequency side all the time.

This means, that I prefer to use dilation operators

$$[D_\rho h](z) = h(\rho z), \quad \rho > 0 \tag{7}$$

and define for $h = \widehat{f}$, with $f \in \mathbf{L}^p(\mathbb{R}^d)$:

$$\|h\|_{\mathcal{F}\mathbf{L}^p} := \|f\|_p. \tag{8}$$

Dilation on the Fourier transform side using D_ρ corresponds to $\mathbf{L}^1$-norm preserving dilation on the time side using:

$$\mathrm{St}_\rho f(z) = \rho^{-d} f(z/\rho), \quad \text{for} \rho \neq 0, \tag{9}$$

we find that $\|\mathrm{D}_\rho f\|_{\mathcal{F}\mathbf{L}^1} = \|f\|_{\mathcal{F}\mathbf{L}^1} \quad \text{for} \quad \rho \neq 0.$
Consequently (6) is equivalent to

$$\|\widehat{f} \cdot D_{2^k}\psi\|_{\mathcal{F}\mathbf{L}^p} \tag{10}$$

with the side condition that $\sum_{k \in \mathbb{Z}} D_{2^k}\psi(x) \equiv 1$ on $\mathbb{R}^d \setminus \{0\}$. This is what we call a dyadic decomposition of unity.

NEXT

In fact, the smoothness assumptions on ψ can easily be translated into an uniform boundedness condition of the family $(\psi_k) := (D_{2^k}\psi)_{k\in\mathbb{Z}}$ in $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$.
There is a deep result from analysis which helps to also characterize the Triebel-Lizorkin spaces (recall, you have among them the $\mathbf{L}^p$-potential spaces, obtained by applying the *Fourier multiplier* $w_{-s}(\xi) = (1+|\xi|^2)^{-s/2}$ to $\mathcal{F}\mathbf{L}^p$, are among them) using also the sequence of functions $(\mathcal{F}^{-1}\widehat{f}\cdot D_{2^k}\psi)_{k\in\mathbb{Z}}$, using the so-called Paley-Littlewood decomposition. It allows to express the $\mathbf{L}^p$-norm equivalently by the $\mathbf{L}^p$-norm of the function

$$h(t) = \left(\sum_{k\in\mathbb{Z}} |\mathcal{F}^{-1}(\widehat{f}\cdot\psi_k)(t)|^2\right)^{-1/2} \qquad (11)$$

continued ..

Putting weights into the sum, i.e. using the functions

$$h_s(t) = \left(\sum_{k \in \mathbb{Z}} |w_s(2^{-k}) \cdot \mathcal{F}^{-1}(f \cdot \psi_k)(t)|^2 \right)^{-1/2} \qquad (12)$$

we find (cf. work of E.Stein, Triebel etc.) that the p-Bessel potential norm or order s of f is equivalent to $\|h_s\|_p$.
At first sight it looks that the difference between the two types (Sobolev or Besov spaces) consists in the order in which the continuous $\mathbf{L}^p$-norm resp. the discrete ℓ^q-norm are applied.
However, there are also other mixtures, e.g. a *completely continuous* characterization, where finally only the order in which the summation is realized is relevant. For $p = 2 = q$ we just have the classical $\mathbf{L}^2$ Sobolev spaces.

The method of Frazier-Jawerth: atomic decompositions

The approach taken by Frazier and Jawerth (certainly heavily influenced by the work of Jaak Peetre) established a connection between the characterization of the different *function spaces* (to use Triebel's terminology) with dyadic decompositions in order to arrive at atomic decomposition of these spaces resp. characterizations of function spaces by the coefficients.
In a nutshell the dyadic decompositions allow to decompose a function (or tempered distribution) into contributions sitting in dyadic *frequency bands* which in turn can expanded into series of shifted atoms (suitably chosen) making use of (dilated versions) of Shannon's sampling theorem (for each of the blocks).

Comments on those early atomic decompositions

The atomic decompositions proposed in the work of Frazier-Jawerth claim that there are function spaces (in fact pairs of functions, matching well to each other, but different from each other) such that one could be used for analysis, i.e. in order to generate a set of coefficients, while the other is used for synthesis. An important point is the fact that these atoms (used for analysis and synthesis) are transformed jointly (using dyadic dilations and essentially integer translations), and make sure that for each of the classical function spaces there is an appropriate (solid) Banach space of sequences, allowing to characterize the distributions by the coefficients arising in the decomposition.

Connection to Wavelet Theory

With the advent of wavelet theory it was found, that all those *function spaces* (Besov-Triebel-Lizorkin spaces) have a characterization in terms of the CWT (continuous wavelet transform), which is defined over the upper half-plan (parameterized by the parameters $a > 0, b \in \mathbb{R}$), better viewed as the "ax+b"-group G, which is a locally compact group with left (and different from it) right Haar measure.
The correct characterization of function spaces is in terms of mixed norm spaces (mixed $\mathbf{L}^p - \mathbf{L}^q$-norms over G), with a weight depending only on the scale variable $a > 0$ in a natural way.
Anisotropic and weighted spaces can be characterized by alternative weight functions depending on *a and b* as well.

Calderon's reproducing formula

Reinterpretation of older results in the light of wavelet theory shows that the characterization of function spaces by *higher order differences* is more or less using a wavelet transform with respect to some very "rough" wavelet, namely a weighted sum of Dirac-measures (e.g. $\delta_{-1} - 2\delta_0 + \delta_1$ or its convolution powers), which are however satisfying the admissibility by having the *correct behaviour of their Fourier transform* near the origin.
The role of the partition of unity property (only valid for specific Schwartz functions) for dyadic partitions on the FT-side is taken by the more flexible continuous analogue, the so-called Calderon Reproducing formula, which can be seen as a direct consequence of the fact that the CWT is isometric from $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ into $\mathbf{L}^2(G)$. Hence the inverse operator on the range of the CWT is just its adjoint. This allows to characterize all those function spaces using arbitrary *admissible* wavelets in $\mathcal{S}(\mathbb{R}^d)$.

Orthonormal Wavelet Bases

One of the important developments in wavelet theory has been the construction of orthonormal wavelet basis due to Yves Meyer, Lemarie, and above all Ingrid Daubechies, who was the first to construct orthonormal wavelet bases with compact support and a given degree of smoothness. They cannot be used to characterize all the function spaces, but e.g. Besov spaces $\mathbf{B}^s_{p,q}(\mathbb{R}^d)$ up to some order $|s| \leq s_0$.

It was certainly an important property of wavelets (aside from the fact that they came early on together with efficient algorithms) that they could be used to *characterize* most of the important function spaces known at that time, using the wavelet coefficients. Again, the quality of the atoms g (typically a combination of decay and smoothness conditions) are relevant for the range of parameters they could handle.

Coorbit Theory: the third age

Coorbit theory gives a group theoretical framework to all those statements, using a group theoretical point of view.
It started out as an attempt to understand the similarities between known results in the theory of function spaces, wavelet transforms, including orthonormal expansions.
The analogy between Besov spaces and modulation spaces (introduced in the early 80s, imitating the definition of $\mathbf{B}^s_{p,q}$ by replacing the dyadic BAPUs by uniform partitions of unity (BUPUs) in order to get to the $\mathbf{M}^s_{p,q}$-family) was quite obvious and motivated the search for their common properties and analogies.

Coorbit Theory, group representation theory

The insight was, that one only needs an integrable group representation of some locally compact group (such as the "ax+b" or the reduced Heisenberg group), say $\pi(x)$ on some Hilbert space $\mathcal{H}$, in order to come up with a continuous *voice transform*

$$V_g f(x) = \langle f, \pi(x) g \rangle_{\mathcal{H}}, \quad x \in G. \tag{13}$$

Then one can use Moyal's formula (a kind of Plancherel theorem for non-commutative groups) in order to come up with (the weak form) of a reproducing formula, allowing to write any element $f \in \mathcal{H}$ as a *"continuous" superposition* of elements of the form $\pi(x)g$, for suitable (admissible) atoms $g \in \mathcal{H}$. There is an abundance of such situations, shearlet theory being the most recent one. Margit Pap is studying the Moebius group.

Function spaces from the Coorbit point of view

Already a first step towards a continuous characterization is the reinterpretation of the Calderon reproducing formula which - in a modern interpretation - shows that the family $\pi(x)g, x \in G$ defines a continuous frame (at least for *admissible atoms* $g \in \mathcal{H}$).
Coorbit theory unifies various aspects and exhibits analogies between different families of spaces, such as modulation spaces (linked to the *Schrödinger representation of the (reduced) Heisenberg group*) or Besov-Triebel-Lizorkin spaces, linked to the affine group ("ax+b"-group).
While it is possible to have wavelet orthonormal bases (i.e. orthonormal bases of the form $(\pi(\lambda_i)g)_{i\in I}$, where (λ_i) is a discrete set in $''ax + b''$ nothing similar is possible in the case of modulation spaces (despite D. Gabor's original hope and suggestion).

Literature and relevant citations

Taibleson: Fourier analysis on local fields ([15])
Stein-Weiss [14] Introduction to Fourier Analysis on Euclidean Spaces.
E. Stein:[13] Singular Integrals and Differentiability Properties of Functions.
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H.G. Feichtinger and K. Gröchenig: coorbit papers [2, 3]
I. Daubechies, A. Grossmann and Y. Meyer [1] Painless nonorthogonal expansions
M. W. Frazier, B. D. Jawerth and G. Weiss [7] Littlewood-Paley Theory and the Study of Function Spaces.
M. Frazier and B. Jawerth [6] A discrete transform and decompositions of distribution spaces.
M. Frazier and B. Jawerth [5] Decomposition of Besov spaces.

END OF THE PRESENTATION in Marburg

Question (A) from the audience: Where did the name modulation space come from:
Answer: While Besov spaces and other function spaces can be characterized by the rate of convergence by which the solution of the heat equation approaches the initial value f, i.e. by

$$\|(\mathrm{St}_\rho h) * f - f\|_{\mathbf{L}^p} = \|[\mathrm{St}_\rho(h - \delta_0)] * f\|_{\mathbf{L}^p}$$

(where h is the Gauss function, with $\int_{\mathbb{R}^d} h(x)dx = 1$), we can reformulate the growth conditions of $V_g(f)$ over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ equivalently by looking at the decay of $\|M_t g * f\|_{\mathbf{L}^p}$ for $t \to \infty$ (which can be seen as a kind of quantitative variant of the *Riemann-Lebesgue Lemma*, according to which $\widehat{f} \in \mathbf{C}_0(\mathbb{R}^d)$ for $f \in \mathbf{L}^1(\mathbb{R}^d)$). The name is based on the fact that $M_t g$ is a modulated version of g.

END OF THE PRESENTATION in Marburg

Question B from the audience: Where does the name coorbit space come from.
Answer: This is related to terminology already used in a more general setting by Jaak Peetre in his paper [11]:
Jaak Peetre [pe85] Paracommutators and minimal spaces. In "Operators and Function Theory", Proc. NATO Adv Study Inst, Lancaster/Engl 1984, NATO ASI Ser, Ser C 153, 163-224, (1985)
There are certainly motivations coming from the two equivalent descriptions of the real interpolation method, namely the K- and the J-method, which are also kind of dual to each other.

THANK you very much for your attention!

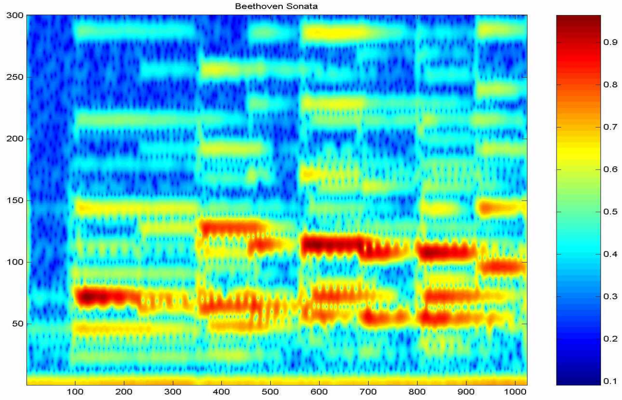
REST FROM OLD TALK: there is an implicit message:

Aside from the various technical terms coming up I hope to convey implicitly a few other messages:

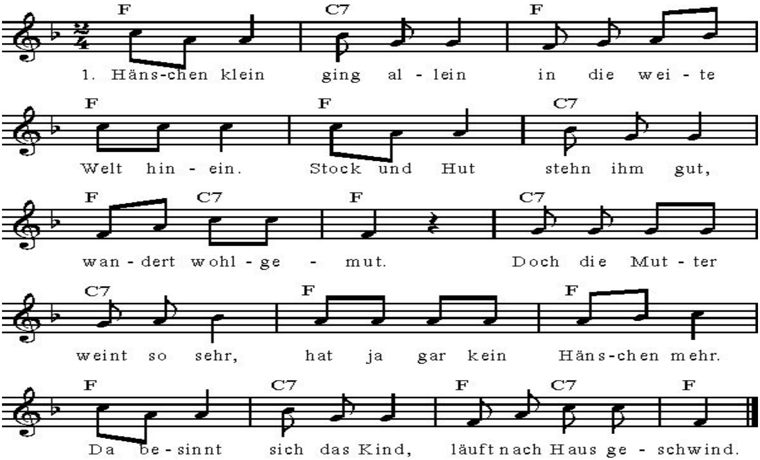
- staying with Banach spaces and their duals one can do amazing things (without touching the full theory of topological vector spaces, Lebesgue integration, or usual distribution theory);
- alongside with the norm topology just the very natural w^*-topology, just in the form of pointwise convergence of functionals, for the dual space has to be kept in mind (allowing thus among other to handle non-reflexive Banach spaces);
- diagrams and operator descriptions allow to naturally generalize concepts from finite dimensional theory up to the category of Banach Gelfand triples.

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) pictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



compared to musical score ...



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

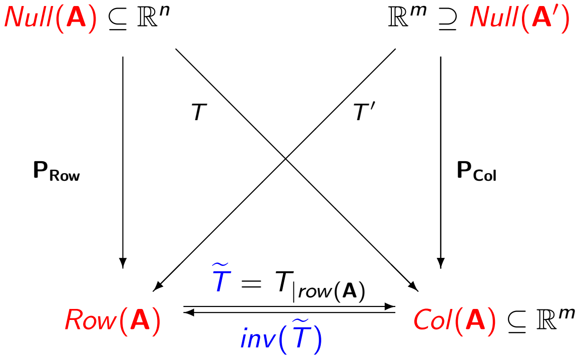
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

The Schrödinger Representation

For people in representation theory I could explain the spectrogram is just displaying to you a typical representation coefficient of the (projective) Schrödinger Representation of the (reduced) Heisenberg Group $\mathbb{H}^d$ (for $d = 1$).
According to Roger Howe this group has the phantastic "hinduistic multiplicity in one" property of allowing a variety of different looking but in fact mathematically equivalent representations (due to the von-Neumann uniqueness theorem), which indicates the connection to quantum mechanics, the theory of coherent states, and related topics (where e.g. rigged Hilbert spaces, the bras and kets appear already), where concepts as described below are in fact also helpful (to put expressions such as continuous integral representations on a firm mathematical ground); but we will start from known grounds...

Geometric interpretation of matrix multiplication



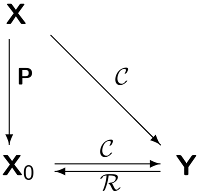
$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Matrices of maximal rank

We will be mostly interested (as models for Banach Frames and Riesz projection bases) in the situation of matrices of maximal ranks, i.e. in the situation where $r = \mathrm{rank}(A) = max(m, n)$, where $A = (a_1, \cdots, a_k)$.
Then either the synthesis mapping $x \mapsto A * x = \sum_k x_k a_k$ has trivial kernel (i.e. the column vectors of A are a linear independent set, spanning the column-space of which is of dimension $r = n$), or the analysis mapping $y \mapsto A' * y = (\langle y, a_k \rangle)$ has trivial kernel, hence the column spaces equals the target space (or $r = m$), or the column vectors are a spanning set for $\mathbb{R}^m$.

.... continued

For *Riesz basic sequences* we have the following diagram:



Definition

A sequence (h_k) in a separable Hilbert space $\mathcal{H}$ is a *Riesz basis* for its closed linear span (sometimes also called a Riesz basic sequence) if for two constants $0 < D_1 \le D_2 < \infty$,

$$D_1 \|c\|^2_{\ell^2} \le \Big\| \sum_k c_k h_k \Big\|^2_{\mathcal{H}} \le D_2 \|c\|^2_{\ell^2}, \qquad \forall c \in \ell^2 \tag{14}$$

A detail description of the concept of *Riesz basis* can be found in

Reflect also for a moment about daily actions:

We are calculating with all kind of numbers in our daily life. But just recall the most beautiful equation

$$e^{2\pi i} = 1.$$

It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

$$cos(2\pi) + i * sin(2\pi) = 1 \quad \text{in} \quad \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

Existing examples of Gelfand Triples

So-called *Gelfand Triples* are already widely used in various fields of analysis. The prototypical example in the theory of PDE is certainly the *Schwartz Gelfand triple*, consisting of the space of test functions $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions, densely sitting inside of $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, which in turn is embedded into the space of tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d). \tag{15}$$

Alternatively (e.g. for elliptic PDE) one used

$$\mathcal{H}_s(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{H}'_s(\mathbb{R}^d). \tag{16}$$

It is obtained via the Fourier transform form

$$\mathbf{L}^2_w(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2(\mathbb{R}^d) \hookrightarrow \mathbf{L}^2_w(\mathbb{R}^d)'. \tag{17}$$

What is a generating set in a Hilbert space

We teach in our linear algebra courses that the following properties are equivalent for a set of vectors $(f_i)_{i \in I}$ in $\mathbf{V}$:

1. The only vector perpendicular to a set of vectors is $\emptyset$;
2. Every $v \in \mathbf{V}$ is a linear combination of these vectors.

An attempt to transfer these ideas to the setting of Hilbert spaces one comes up with several different generalizations:

- a family is *total* if its linear combinations are dense;
- a family is a *frame* if there is a bounded linear mapping from $\mathcal{H}$ into $\ell^2(I)$ $f \mapsto \mathbf{c} = c(f) = (c_i)_{i \in I}$ such that

$$f = \sum_{i \in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{18}$$

The usual definition of frames

There is another, *equivalent* characterization of frames. First, it is an obvious consequence of the characterization given above, that

$$f = \sum_{i \in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{19}$$

implies that there exists $C, D > 0$ such that

$$C\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{20}$$

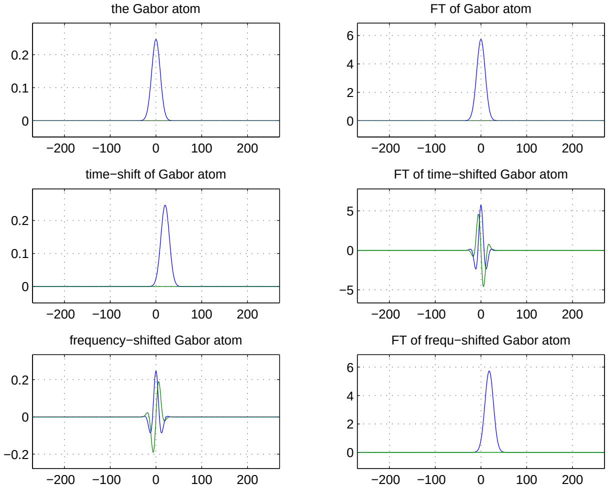
For the converse observe that $Sf := \sum_{i \in I} \langle f, f_i \rangle f_i$ is a strictly positive definite operator and the *dual frame* $(\tilde{f}_i)$ satisfies

$$f = \sum_{i \in I} \langle f, \widetilde{f_i} \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \widetilde{f_i}$$

Dennis Gabor's suggestion of 1946

There is one very interesting example (the prototypical problem going back to D. Gabor, 1946): Consider the family of all time-frequency shifted copies of a standard Gauss function $g_0(t) = e^{-\pi|t|^2}$ (which is invariant under the Fourier transform), and shifted along $\mathbb{Z}$ ($T_n f(z) = f(z-n)$) and shifted also in time along $\mathbb{Z}$ (the modulation operator is given by $M_k h(z) = \chi_k(z) \cdot h(z)$, where $\chi_k(z) = e^{2\pi i k z}$).
Although D. Gabor gave some heuristic arguments suggesting to expand every signal from $\mathbf{L}^2(\mathbb{R})$ in a unique way into a (double) series of such "Gabor atoms", a deeper mathematical analysis shows that we have the following problems (the basic analysis has been undertaken e.g. by A.J.E.M. Janssen in the early 80s):

TF-shifted Gaussians: Gabor families



Problems with the original suggestion

Even if one allows to replace the time shifts from along $\mathbb{Z}$ by time-shifts along $a\mathbb{Z}$ and accordingly frequency shifts along $b\mathbb{Z}$ one faces the following problems:

1. for $a \cdot b = 1$ (in particular $a = 1 = b$) one finds a *total* subset, which is not a frame nor Riesz-basis for $\mathbf{L}^2(\mathbb{R})$, which is redundant in the sense: after removing one element it is still total in $\mathbf{L}^2(\mathbb{R})$, while it is not total anymore after removal of more than one such element;
2. for $a \cdot b > 1$ one does not have anymore totalness, but a Riesz basic sequence for its closed linear span ($\subsetneq \mathbf{L}^2(\mathbb{R})$);
3. for $a \cdot b < 1$ one finds that the corresponding Gabor family is a *Gabor frame*: it is a redundant family allowing to expand $f \in \mathbf{L}^2(\mathbb{R})$ using ℓ^2-coefficients (but one can remove infinitely many elements and still have this property!);

Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?);
- It is computed using the FFT (what is the connection);
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency (FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \mathbf{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \qquad (21)$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \qquad (22)$$

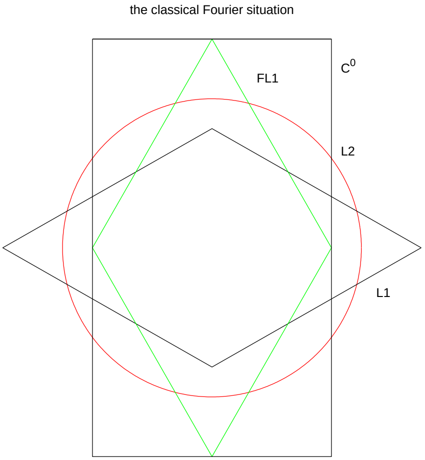
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\widehat{f} \in \mathbf{L}^1(\mathbb{R}^d)$. One often speaks of Fourier analysis followed by Fourier inversion as a method to build f from the pure frequencies (Fourier synthesis).

The classical situation with Fourier

Unfortunately the Fourier transform does not behave well with respect to $\mathbf{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \mathbf{L}^1(\mathbb{R}^d)$ we have $\widehat{f} \in \mathbf{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\widehat{f} \in \mathbf{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\mathbf{L}^1(\mathbb{R}^d) \cap \mathbf{L}^2(\mathbb{R}^d)$ is isometric in the $\mathbf{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\mathbf{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\widehat{f}$.

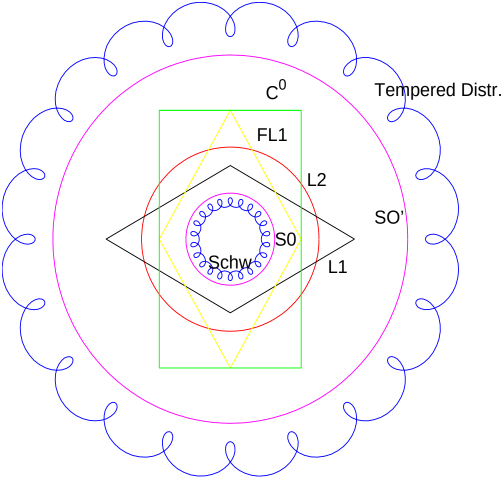
A schematic description of the situation



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

A schematic description of the situation

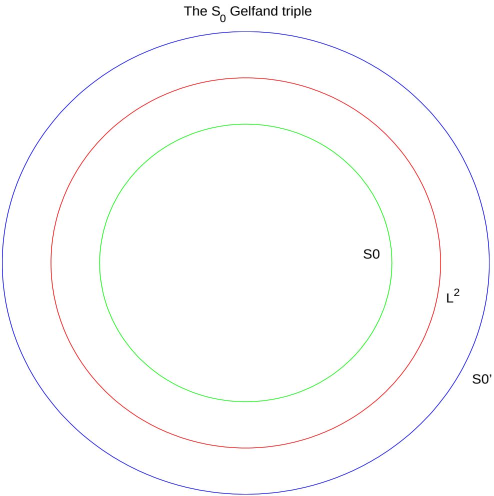


The Banach space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ or $(\mathbf{M}^1(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^1})$), which will serve our purpose. Its dual space $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The $\mathbf{S}_0$-Banach Gelfand Triple



The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

$$(T_x f)\widehat{} = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \mathbf{L}^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\mathbf{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $\mathbf{M}^1 = \mathbf{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathbf{S}_0(\mathbb{R}^d)$, *then the following holds:*

(1) $\pi(u,\eta)f \in \mathbf{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

(2) $\widehat{f} \in \mathbf{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\mathbf{S}_0} = \|f\|_{\mathbf{S}_0}$.

In fact, $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\mathbf{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

OVERVIEW old

THE GOAL OF THIS PRESENTATION IS TO CONVEY THE CONCEPTS OF MODULATION SPACES, BANACH FRAMES AND BANACH GELFAND TRIPLES BY DESCRIBING THEM AND SHOW THEIR USEFULNESS IN THE CONTEXT OF MATHEMATICAL ANALYSIS, IN PARTICULAR TIME-FREQUENCY ANALYSIS

- Recall some concepts from linear algebra, especially that of a *generating system*, a *linear independent* set of vectors, and that of the dual vector space;
- already in the context of Hilbert spaces the question arises: *what is a correct generalization of these concepts?*
- Banach Gelfand Triple (comparable to rigged Hilbert spaces) are one way out;

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\mathbf{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{U}) = \mathbf{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{23}$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

FACTS

Gröchenig and Leinert have shown (J. Amer. Math. Soc., 2004):

Theorem

Assume that for $g \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor frame operator

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

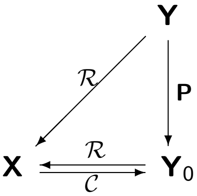
is invertible as an operator on $\mathbf{L}^2(\mathbb{R}^d)$, then it is also invertible on $\mathbf{S}_0(\mathbb{R}^d)$ and in fact on $\mathbf{S}_0'(\mathbb{R}^d)$.
In other words: Invertibility at the level of the Hilbert space automatically !! implies that S is (resp. extends to) an isomorphism of the Gelfand triple automorphism for $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

The w^*- topology: a natural alternative

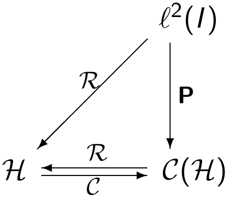
It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

Fames and Riesz Bases: the Diagram

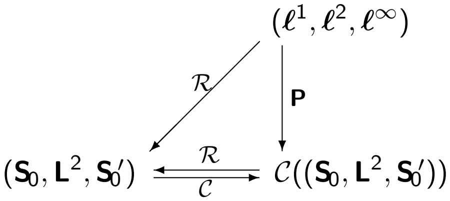
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\mathbf{Y}$ onto the range $\mathbf{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ. Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping $\mathcal{C}$ from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse $\mathcal{R}$ (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.
In practice it means, that the dual Gabor atom $\tilde{g}$ is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathbf{S}_0(\mathbb{R}^d)$ (typically one will take g and $\tilde{g}$ respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten classe $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity (> Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without loosing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ)*, with* $g \in \mathbf{S}_0(\mathbb{R}^d)$ *defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these to situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the* $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$*-sense on both parameters.*

Invertibility, Surjectivity and Injectivity

In another, very recent paper, Charly Groechenig has discovered that there is another analogy to the finite dimensional case: There one has: A square matrix is invertible if and only if it is surjective or injective (the other property then follows automatically).
We have a similar situation here (systematically describe in Charly's paper):
`K.Groechenig: Gabor frames without inequalities, Int. Math. Res. Not. IMRN, No.23, (2007).`

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\mathbf{B} = \mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$.

Theorem

There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ and $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$. This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators $\pi(\lambda)$ and the corresponding point masses δ_λ.

The w^*- topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.
Therefore it is interesting to check what the w^*-convergence looks like:

Lemma

For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ a sequence σ_n is w^*-convergent to σ_0 if and only the spectrograms $V_g(\sigma_n)$ converge uniformly over compact sets to the spectrogram $V_g(\sigma_0)$.

The FOURIER transform, viewed as a BGT-automorphism is uniquely determined by the fact that it maps pure frequencies onto the corresponding point measures δ_ω.

The w^*- topology: dense subfamilies

From the practical point of view this means, that one has to look at the spectrograms of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution $V_g(\sigma_0)$ over compact sets.
The approximation of elements from $\mathbf{S}_0'(\mathbb{R}^d)$ takes place by a bounded sequence.
Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^*-dense subsets of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.
Let us look at some concrete examples: Test-functions, finite discrete measures $\mu = \sum_i c_i \delta_{t_i}$, trigonometric polynomials $q(t) = \sum_i a_i e^{2\pi i \omega_i t}$, or discrete AND periodic measures (this class is invariant under the generalized Fourier transform and can be realized computationally using the FFT).

The w^*- topology: approximation strategies

- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \to \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an $\mathbf{L}^1$-Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite disrete sequence using quasi-interpolation (N. Kaiblinger): $Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x)$.

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