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WIENER AMALGAMS and GABOR ANALYSIS

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Time-Frequency Workshop 2015

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Outline for TALK I

- Interpolation theory, families of compatible Banach spaces
- Real interpolation of Banach spaces: K and J-method;
- retracts "PAUSE! "
- Wiener Amalgam spaces;
- Modulation spaces; "PAUSE! "
- The Banach Gelfand Triples and their use;
- various unitary Gefand triple isomorphism involving $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$

What are function spaces good for?

- Describe the smoothness or variation/oscillation of functions;
- Describe (rate of) decay of functions, summability properties;
- Describe the mapping properties of linear operators, domains of unbounded operators;

"PAUSE! "

There is a huge zoo of Banach spaces of functions or distributions used in the literature:

- the classical $\boldsymbol{L}^p$-spaces, but also Lorentz or Orlicz spaces (typically defined by the distribution of their values, hence rearrangement invariant);
- Lipschitz spaces, Besov spaces, Bessel potential spaces, Triebel Lizorkin spaces (smoothness);
- weighted spaces, mixed norm spaces;
- spaces describing bounded variation, Morrey-Campanato spaces;
- Hardy spaces, characterized by atomic decompositions;
- Herz spaces, defined by decompositions;

Real Interpolation Methods: the K and J methods

Assumptions: $\mathcal{X}$ Hausdorff topological vector space, and two Banach spaces X_1, X_2, both continuously embedded into $\mathcal{X}$ (a compatible pair of Banach spaces).
The natural norm on $X_1 + X_2$ is given by:

$$\|f\|_{X_1+X_2} := \inf_{f=f_1+f_2} \left(\|f_1\|_{X_1} + \|f_2\|_{X_2}\right), \; f_i \in X_i$$

"PAUSE! "The natural norm on $X_1 \cap X_2$ is given by:

$$\|f\|_{X_1 \cap X_2} := \max\left(\|f\|_{X_1}, \|f\|_{X_2}\right)$$

Remark: $X_1 + X_2$ and $X_1 \cap X_2$ are Banach spaces, continuously embedded into $\mathcal{X}$.
"PAUSE! "
An *intermediate sapce* of X_1, X_2 is any Banach space X with

$$X_1 \cap X_2 \hookrightarrow X \hookrightarrow X_1 + X_2$$

Definition (K-functional): In the situation above define (according to Butzer/Scherer):

$$K(t, f; X_1, X_2) = \inf_{f=f_1+f_2} \left(\|f_1\|_{X_1} + t\,\|f_2\|_{X_2} \right), \; f \in X_1 + X_2$$

for $0 < t < \infty$. (These are norms for fixed t).
"PAUSE! "*Definition (interpolation spaces, K-method, discrete version):*
For $a > 1$, $\theta \in \mathbf{R}$, and $1 \leq q < \infty$ define

$$[X_1, X_2]_{\theta,q;K} = \left\{ f \in X_1 + X_2 \,\middle|\, \|f\| = \left[\sum_{n \in \mathbf{Z}} a^{-n\theta q} K(a^n, f)^q \right]^{1/q} < \infty \right\}$$

"PAUSE! "Modifications for $q = \infty$ are obvious. .For $q < \infty$ these spaces are intermediate spaces for X_1 and X_2, and there are natural inclusion for different parameters (θ, q)
An equivalent (usually used!) norm is given by the following $continuous$ norm:

$$\left\{ \int_0^\infty t^{-\theta} K(t, f)^q \frac{d}{dt} \right\}^{1/q}$$

Definition (J-functional):

$$J(t,f;X_1,X_2) = \max\left(\|f_1\|_{X_1}, t\,\|f_2\|_{X_2}\right), \ f \in X_1 \cap X_2$$

Definition (interpolation spaces, J-method, discrete version):
Assumptions as above. $[X_1,X_2]_{\theta,q;K}$ is the space of all $f \in X_1 + X_2$ such that $f = \sum_{n\in\mathbf{Z}} f_n, \ f_n \in X_1 \cap X_2$ in the $X_1 + X_2$-norm, and

$$\|f\|_{\theta,p;J} = \left[\sum\nolimits_{n\in\mathbf{Z}} a^{n\theta q} J\left(a^{-n}, f_n\right)^q\right]^{1/q} < \infty$$

"PAUSE! "There are equivalence theorem showing that with the same parameters the interpolation spaces defined by means of the "selective K-method" and those by the "constructive J-methods" are - with equivalence of norms, the same spaces. This is the basis for the compatibility of duality with interpolation. The proof uses the fact that a dual for - say - a space obtained by J-interpolation is of the corresponding K-type, but altogether one obtains that in most cases real (K- or J) interpolation is compatible with duality.

Complex Interpolation of Weighted L^p-spaces

Complex interpolation methods (at least from a user's point of view) are quite well known. Let us give as an typical example the Hausdorff-Young inequality, showing that for $1 \leq p \leq 2$ one has (with the usual convention $1/p + 1/q = 1$):

$$\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^q(\mathbb{R}^d) \quad \text{and} \quad \|\hat{f}\|_q \leq \|f\|_p,$$

as a consequence of the validity of this estimate for the cases $p = 1$ (Riemann Lebesgue Lemma) and $p = 2$ (Plancherel). "PAUSE! "The "scale" of space obtained by complex interpolation of weighted L^p spaces are identified as weighted L^p-spaces themselves, according to the rule $1/p = (1-\theta)/p_1 + \theta/p_2$, and $w = w_1^{1-\theta} \cdot w_2^{\theta}$:

$$(L^{p1}_{w_1}, L^{p2}_{w_2})_{[\theta]} = L^p_w$$

"PAUSE! "The situation is quite different with respect to real interpolation, except for the case that the interpolation parameter q coincides with the parameter p of the spaces to be interpolated. For simplicity of notations let us consider the case where $w_1 = 1$ ($w = w_2$).

Real Interpolation of Weighted L^p-spaces

$$[\boldsymbol{L}^p, \boldsymbol{L}^p_w]_{\theta,p;K} = \{f \in \boldsymbol{L}^p{}_{loc} \mid \left(\sum_n 2^{nsq} \|f \cdot \mathbf{1}_{D_n}\|_p^q\right)^{1/q} < \infty\}$$

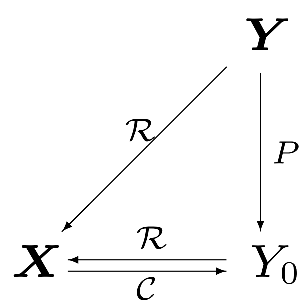
"PAUSE! "where the sets D_n are characterized as the sets where the weight function $w(x)$ is approximately equal to 2^n (i.e. one could think of 2^n as $w(x)$ for $x \in D_n$:

$$D_n = \{x \mid w(x) \in [2^n, 2^{n+1})\}$$

"PAUSE! "This fact can be seen as a good reason (aside of the well known characterizations of $\boldsymbol{L}^p$-spaces via Paley-Littlewood theorems) for the fact that the Besov spaces $\boldsymbol{B}^s_{2,q}$ are on the one hand characterized by wavelets resp. dycadic decompositions on the Fourier transform sides, but also arise as *real* interpolation spaces from the standard Sobolev spaces $\mathcal{L}^s = \mathcal{F}(\boldsymbol{L}^2{}_{w_s})$: the level set of the corresponding weights $w_s(\xi) = (1 + |\xi|^2)^{s/2}$ happen to be the dyadic intervals resp. rings (in $\mathbb{R}^2$), for any $s > 0$ (cf. Triebel).

Retracts in Interpolation Theory

Definition 1. *A linear mapping $\mathcal{C}$ is defining a retract from $\boldsymbol{X}$ into $\boldsymbol{Y}$ if there exists a left inverse to it, i.e. a mapping $\mathcal{R}$ from $\boldsymbol{Y}$ into $\boldsymbol{X}$ such that $\mathcal{R} \circ \mathcal{C} = Id_{\boldsymbol{X}}$.*
In other words, we have the following commutative diagram, with $\boldsymbol{Y}_0$ being the range of the analysis mapping $\mathcal{C}$. Moreover, it is clear that the mapping $\mathcal{C} \circ \mathcal{R}$ is an idempotent, and for this reason $\boldsymbol{Y}_0$ is a closed an complemented subspace of $\boldsymbol{Y}$. Moreover, T establishes and isomorphism from $\boldsymbol{X}$ onto $\boldsymbol{Y}_0$. Since $\mathcal{R}$ is also surjective, we find that $\boldsymbol{X}$ is not only identified with a closed subspace of $\boldsymbol{Y}$, but also isomorphic to the quotient $\boldsymbol{Y}/\boldsymbol{Y}_0$.
"PAUSE! "



"PAUSE! "The method of retracts is often used to push results known for vector-valued $\boldsymbol{L}^p$-spaces to the setting of Besov- or modulation spaces.

Wiener Amalgams: Wiener's Role

The first appearance of amalgam spaces can be traced to Norbert Wiener in his development of the theory of generalized harmonic analysis ("The Fourier Transform and Certain of its Applications") and Tauberian Theorems. In particular, Wiener defined already around 1929-1932 the spaces that we will call $W(L^1, L^2)$ and $W(L^2, L^1)$, $W(L^1, L^\infty)$ and a bit later $W(L^\infty, L^1)$, using what we will refer to as a discrete norm for these spaces, namely,

$$\|f\|_{W(L^p,\ell^q)} = \left(\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|^p \, dt \right)^{q/p} \right)^{1/q}, \tag{1}$$

with the usual adjustments if p or q is infinity. We will not attempt to provide a complete historical discussion of papers on amalgam spaces, but mention here only some specific instances in which amalgams were introduced or studied.

"PAUSE! "Advantage of these spaces compared to ordinary L^p-spaces: natural inclusions, in the local component as over the torus, while globally on has the natural inclusions between sequence spaces, with opposite orientation.

Hence $W(L^\infty, \ell^1)$ is the smallest within *this* family and $W(L^1, \ell^\infty)$ is the largest.

Wiener Amalgams: Basic Properties

The use of amalgam spaces (cf. e.g. the survey article by Fournier and Stewart, Bull. Amer. Math. Soc., 1980) shows their usability in a wide range of problems of analysis. In most cases one can just argue, that one has to think $coordinatewise$. "PAUSE! "

For example, with respect to duality, pointwise multiplication, or a Hausdorff-Young type statement for the Fourier transform:

$$W(L^p, \ell^q)' = W(L^{p'}, \ell^{q'}), \quad 1 \leq p, q, < \infty$$

$$\mathcal{F}W(L^p, \ell^q) \subseteq W(L^{q'}, \ell^{p'}), \quad 1 \leq p, q, \leq 2$$

Wiener Amalgam Convolution Theorem

Theorem 1. *Assume the indices* p_i, q_i *and the moderate weights* w_i *are such that there exist constants* C_1, $C_2 > 0$ *so that*

$$\forall\, h \in L^{p1}, \quad \forall\, k \in L^{p2}, \quad \|h * k\|_{L^{p3}} \;\le\; C_1 \,\|h\|_{L^{p1}}\, \|k\|_{L^{p2}}$$

and

$$\forall\, h \in L^{q1}_{w_1}, \quad \forall\, k \in L^{q2}_{w_2}, \quad \|h * k\|_{L^{q3}_{w_3}} \;\le\; C_2 \,\|h\|_{L^{q1}_{w_1}}\, \|k\|_{L^{q2}_{w_2}}.$$

Then there is a constant $C > 0$ *such that for all* $f \in W(L^{p1}, L^{q1}_{w_1})$ *and* $g \in W(L^{p2}, L^{q2}_{w_2})$ *we have*

$$\|f * g\|_{L^{q3}_{w_3}} \;\le\; C\, \|f\|_{W(L^{p1}, L^{q1}_{w_1})}\, \|g\|_{W(L^{p2}, L^{q2}_{w_2})}.$$

In other words, if $L^{p1} * L^{p2} \subseteq L^{p3}$ *and* $L^{q1}_{w_1} * L^{q2}_{w_2} \subseteq L^{q3}_{w_3}$*, then*

$$W(L^{p1}, L^{q1}_{w_1}) * W(L^{p2}, L^{q2}_{w_2}) \subseteq W(L^{p3}, L^{q3}_{w_3}).$$

Relatively Separated Sets

Definition 2. *A set* (y_j) *in a metric space is uniformly separated, if there is a positive* $\delta > 0$ *such that* $d(y_j, y_{j'}) \geq \delta$ *for all pairs* $j \neq j'$. *A family of points* $(x_i)_{i\in I}$ *in* $\mathbf{R}^d$ *is called relatively separated if it is the finite union of uniformly separated sets.*

Proposition 1. *The following properties of* $X = (x_i)_{i\in I}$ *are equivalent:*

1. $(x_i)_{i\in I}$ *is relatively separated in* $\mathbf{R}^d$;
2. *For some (hence for each)* $R > 0$, *the number of points within balls of radius* R *is uniformly bounded, i.e.*

$$sup_{z\in\mathbf{R}^d} \#\{i \mid x_i \in B_R(z)\} \leq C_X < \infty$$

3. $\sum_{i\in I} g(t - x_i)$ *is a bounded function on* $\mathbf{R}^d$ *for some (any) nonnegative Schwartz function;*
4. $\mu_X = \sum_{i\in I} \delta_{x_i}$ *is a translation bounded measure, i.e. it belongs to* $\boldsymbol{W}(\mathbf{M}, \ell^\infty)$.

Relatively Separated Sets: Applications

Proposition 2. *Let* $X = (x_i)_{i \in I}$ *be a relatively separated set in* $\mathbf{R}^d$ *and* $\mu_X = \sum \delta_{x_i}$. *Then we have:*

1. $\mathbf{c} = (c_i) \in \ell^p(I)$ *if and only if*
   $\sum_{i \in I} c_i \delta_{x_i} \in \boldsymbol{W}(\mathbf{M}, \ell^p)$ *and* $\|\sum_{i \in I} c_i \delta_{x_i}\|_{\boldsymbol{W}(\mathbf{M},\ell^p)} \leq \|\mathbf{c}\|_{\ell^{\mathbf{p}}} \cdot \|\mu\|_{\boldsymbol{W}(\mathbf{M},\ell^{\mathbf{p}})}$.
   "PAUSE! "

2. *For* $g \in \boldsymbol{W}(\mathbf{C}^0, \ell^1)$ *and* $\mathbf{c} = (c_i) \in \ell^p(I)$ *one has*

$$\sum_{i \in I} c_i T_{x_i} g = \left(\sum_{i \in I} c_i \delta_{x_i} \right) * g \in \boldsymbol{W}(\mathbf{M}, \ell^p) * \boldsymbol{W}(\mathbf{C}, \ell^1) \subseteq \boldsymbol{W}(\mathbf{C}, \ell^p) \, .$$

   "PAUSE! "

3. $h \in \boldsymbol{W}(\mathbf{C}, \ell^p)$ *implies*

$$h \cdot \mu = \sum_{i \in I} f(x_i) \delta_{x_i} \in \boldsymbol{W}(\mathbf{C}, \ell^p) \cdot \boldsymbol{W}(\mathbf{M}, \ell^\infty) \subseteq \boldsymbol{W}(M, \ell^p) \, .$$

Homogeneous Banach spaces

First recall the definition of *homogeneous Banach spaces [Katznelson]*.

Definition 3. *A Banach space* $\boldsymbol{B}$ *of locally integrable functions on* $\mathbf{R}^d$ *or more generally tempered distributions is called a homogeneous Banach space, if it is isometrically translation invariant, and translation is strongly continuous, in other words, for each* $h \in B$ *one has :*

$$\|h - T_x h\|_B \to 0 \quad as \quad x \to 0. \tag{2}$$

"PAUSE! "

Of course we require that the embedding into the tempered distributions is continuous. If $\boldsymbol{B}$ consists of locally integrable functions this follows e.g. from the closed graph theorem in case that norm convergence of a sequence in $\boldsymbol{B}$ implies pointwise convergence of a subsequence almost everywhere.

By vector-valued integration one verifies that $\boldsymbol{L}^1(\mathbb{R}^d)$ acts upon $\boldsymbol{B}$ via convolution:

$$g \in \boldsymbol{L}^1(\mathbb{R}^d), f \in \boldsymbol{B} \Rightarrow g * f \in \boldsymbol{B} \qquad \text{and} \qquad \|g * f\|_B \leq \|g\|_1 \|f\|_B.$$

Obvious examples are the L^p spaces for $1 \leq p < \infty$.

Definition 4. *A Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *of tempered distributions is called a* **standard space** *if it satisfies the following conditions:*

1. $\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$,
2. $\boldsymbol{B}$ *is translation and modulation invariant (not necessarily in the isometric sense):* $T_x\boldsymbol{B} = \boldsymbol{B}$ *and* $M_y\boldsymbol{B} = \boldsymbol{B}$ *for all* $x, y \in \mathbf{R}^d$. *"PAUSE! "*
3. *The Banach algebra of pointwise multipliers of* $\boldsymbol{B}$ *contains some regular (pointwise) Banach algebra* $\boldsymbol{A}$, *which is assumed to be a homogeneous Banach space as well; "PAUSE! "*
4. *There is some Beurling algebra* $L^1_w(\mathbf{R}^d)$ *(with some submultiplicative weight* w *of polynomial growth) which acts boundedly on* $\boldsymbol{B}$ *through convolution, i.e.*

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{1,w}\|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}, g \in L^1_w.$$

Selective, Continuous Description of Wiener Amalgam Spaces

Definition 5. *(Wiener Amalgam spaces) Let* $(\mathbf{B}, \|.\|_{\mathbf{B}})$ *be a standard space and* $(\mathbf{C}, \|.\|_{\mathbf{C}})$ *a solid and translation invariant Banach space of functions, i.e., a complete space of measurable functions, such that* $f \in \mathbf{C}$, g *measurable and* $|g(x)| \leq |f(x)|$ *for all* X, *implies* $g \in \mathbf{C}$ *and* $\|g\|_{\mathbf{C}} \leq \|f\|_{\mathbf{C}}$ *as well as* $T_x\mathbf{C} = \mathbf{C}$.
"PAUSE! "
Then we define for $f \in \mathbf{B}_{loc}$ *and some compactly supported "window"* $k \in A$ *the so-called control function with respect to the* $\mathbf{B}$*-norm:*

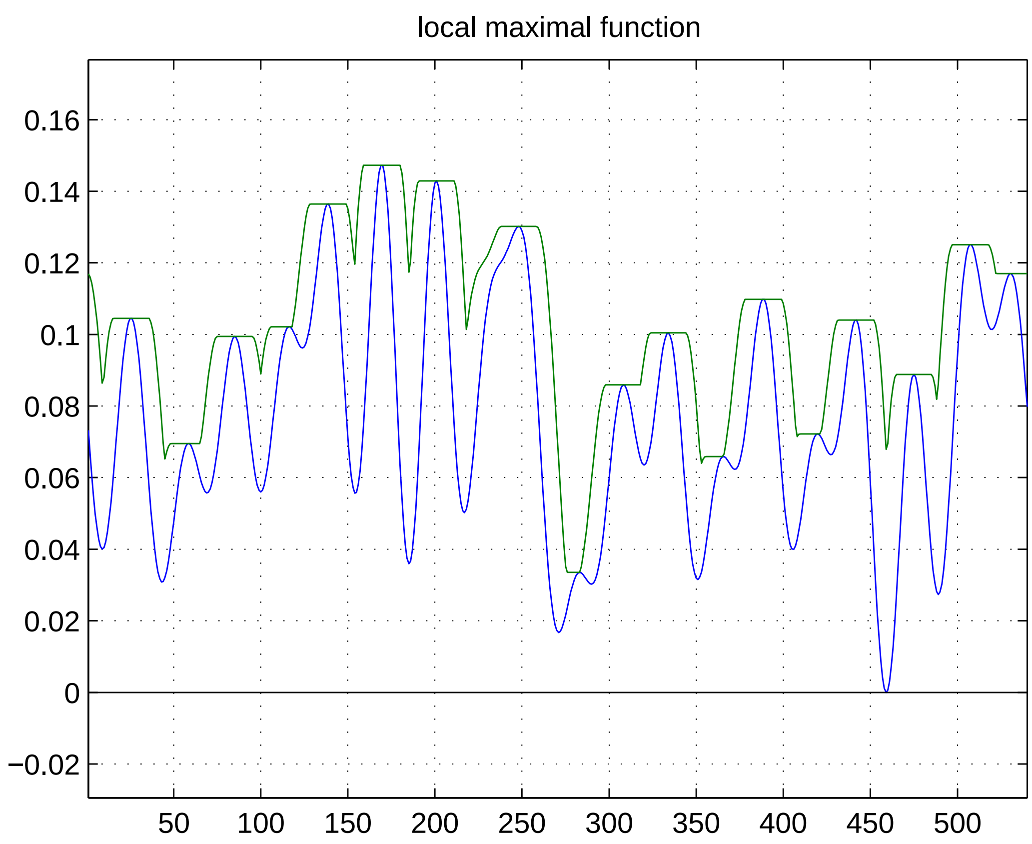
$$K(f, k) : x \mapsto \|(T_x k) \cdot f\|_{\mathbf{B}}.$$

"PAUSE! "On the basis of this control function a linear space, the Wiener amalgam space with local component $\mathbf{B}$ *and global component* $\mathbf{C}$, *denoted by* $\boldsymbol{W}(\mathbf{B}, \mathbf{C})$ *is defined as follows:*

$$\boldsymbol{W}(\mathbf{B}, \mathbf{C}) := \{f \in \mathbf{B}_{loc} | K(f, k) \in \mathbf{C}\}.$$

Different windows k *define the* same *space and equivalent norms.*

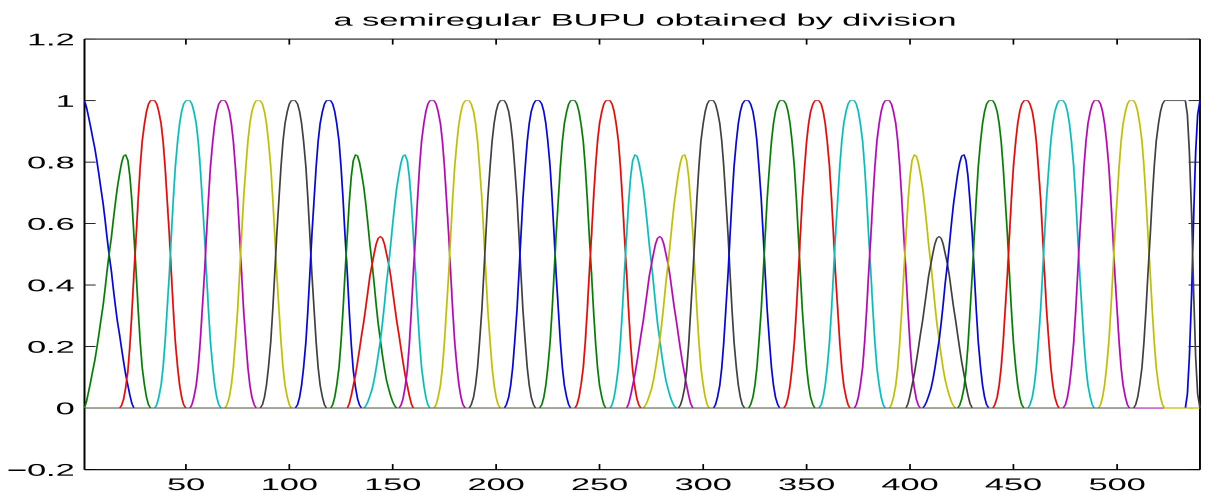
A Typical Control Function



Bounded Uniform Partitions of Unity

Definition 6. *A bounded family* $\Psi = (\psi_i)_{i \in I}$ *in a Banach algebra* $(\mathbf{A}, \| \|_A)$ *is called a* *Bounded Uniform Partition of Unity in* $(\mathbf{A}, \| \|_A)$ *(a BUPU in* $\mathbf{A}$*, for short), if there are a relatively separated set* $(x_i)_{i \in I}$ *and some* $R > 0$ *such that*

1. $supp(\psi_i) \subseteq B_R(x_i)$ *for each* $i \in I$*, and*
2. $\sum_{i \in I} \psi_i(x) = 1$ *for all* $x \in \mathbf{R}^d$



Selective, Discrete Description of Wiener Amalgam Spaces

Theorem 1. *Assume that* $\mathbf{A} \cdot \mathbf{B} \subseteq \mathbf{B}$, *with* $\|h \cdot f\|_B \leq \|h\|_A \|f\|_B$ *for all* $h \in \mathbf{A}$, $f \in \mathbf{B}$. *Then* $f \in W(\mathbf{B}, L^q_w)$, $1 \leq q < \infty$, *if and only if for each (or just for one individual)* $\mathbf{A}$-*BUPU* Ψ *one has*

$$\|f\|'_W = \left(\sum_{i \in I} \|f \psi_i\|_B^q w^q(x_i) \right)^{1/q} < \infty$$

Synthetic Description of Wiener Amalgam Spaces

Given $R > 0$ and $X = \{x_i\}$ relatively separated and R-dense, i.e., $\bigcup B_R(x_i) = \mathbf{R}^d$, and $1 \leq q < \infty$, then

$$f \in W(B, \ell^q_w) \Leftrightarrow f = \sum_{i \in I} f_i \text{ with } \left(\sum_{i \in I} \|f_i\|^q_B w^q(x_i) \right)^{1/q} < \infty$$

"PAUSE! "Even more generally (think of $f_i = T_{x_i} h_i$ or $h_i = T_{-x_i} f_i$),

$$f = \sum_{i \in I} T_{x_i} h_i \text{ with } \left(\sum_{i \in I} \|h_i\|^q_B w^q(x_i) \right)^{1/q} < \infty$$

Moreover, the corresponding quotient norm defines an equivalent norm.

Modulation Spaces (HF: around 1983)

Definition 7.

$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) = \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}L^p, \ell^q_s)\right)$$

"PAUSE! "

$$T_x f(t) = f(t-x)$$

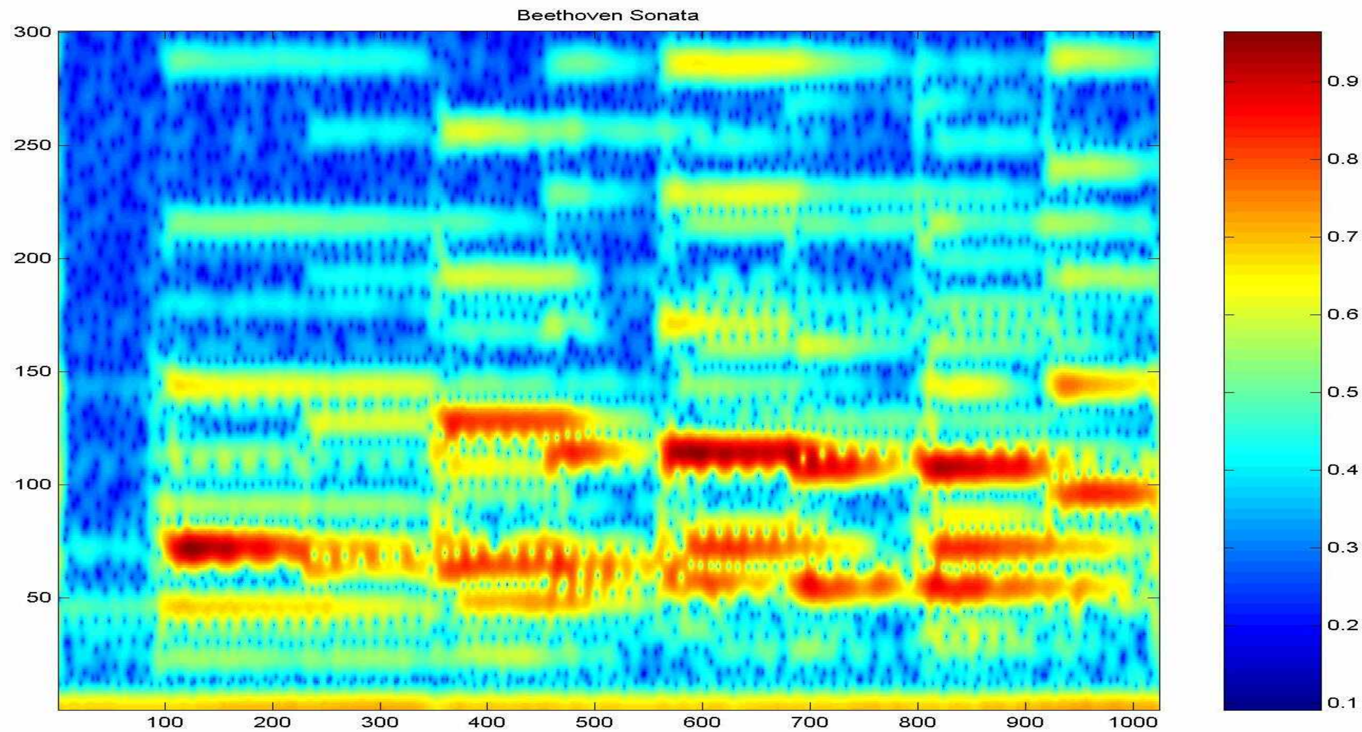
and $x, \omega, t \in \mathbf{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

A Typical Musical STFT



Modulation Spaces

The modulation spaces occur in the study of the concentration of a function in the time-frequency plane. They are defined in the following way:
Let $g \in \mathcal{S}$ be a Schwartz function, $1 \leq p, q < \infty$, $s \in R$, then

$$M^s_{p,q}(\mathbf{R}) = \{f \in \mathcal{S}' : \text{ with } \|f\| < \infty\},$$

"PAUSE! "where the norm $\|f\|$ is given as

$$\left(\int \left(\int |\langle f, M_y T_x g \rangle|^p dx \right)^{q/p} (1 + |y|)^{sq} dy \right)^{1/q},$$

"PAUSE! " i.e. for which $V_g f$ belongs to some weighted mixed norm space over phase space. In the "classical" case the weight depends only on frequency, hence the spaces are isometrically translation invariant. The only important facts about the constraint imposed on $V_g f$ is the membership in some *solid and translation invariant* Banach space of functions.

$M^s_{pq}(\mathbf{R})$ is a Banach space of tempered distributions, the definition is independent of the analyzing function g, and different g's yield equivalent norms on these spaces.
Among the modulation spaces are the following important function spaces:

(a) the Segal algebra $S_0(R)$ as $S_0 = M^0_{1,1}$.

(b) $L^2(R) = M^0_{2,2}$, and

(c) the Bessel potential spaces as $M^s_{2,2}$:

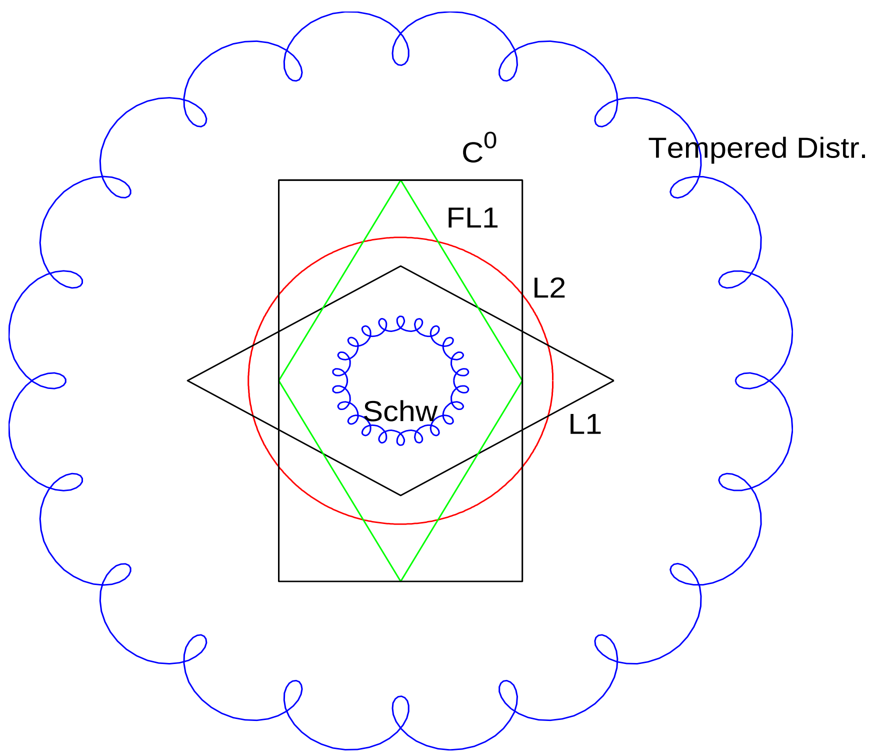
$$H^s(R) = \{f \in \mathcal{S}' : \int |\hat{f}(t)|^2 (1 + |t|)^{2s} dt < \infty\}.$$

The original description of the modulation spaces was in terms of Wiener amalgams, on the Fourier transform side:

$$\mathcal{F}M^s_{pq}(\mathbf{R}) = W(\mathcal{F}L^p, \ell^q_{w_s})$$

where $w_s(\omega) = (1 + |\omega|^2)^{s/2}$ denotes a typical polynomial weight.

The classical view on the Fourier Transform



$S_0(\mathbf{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$

A function in $f \in L^2$ is (by definition) in the subspace $S_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $S_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often it is convenient to use the Gaussian as a window.

Lemma 3. *Let $f \in S_0(\mathbb{R}^d)$, then the following holds:*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Basic properties of $S_0(\mathbf{R}^d)$ resp. $S_0(G)$

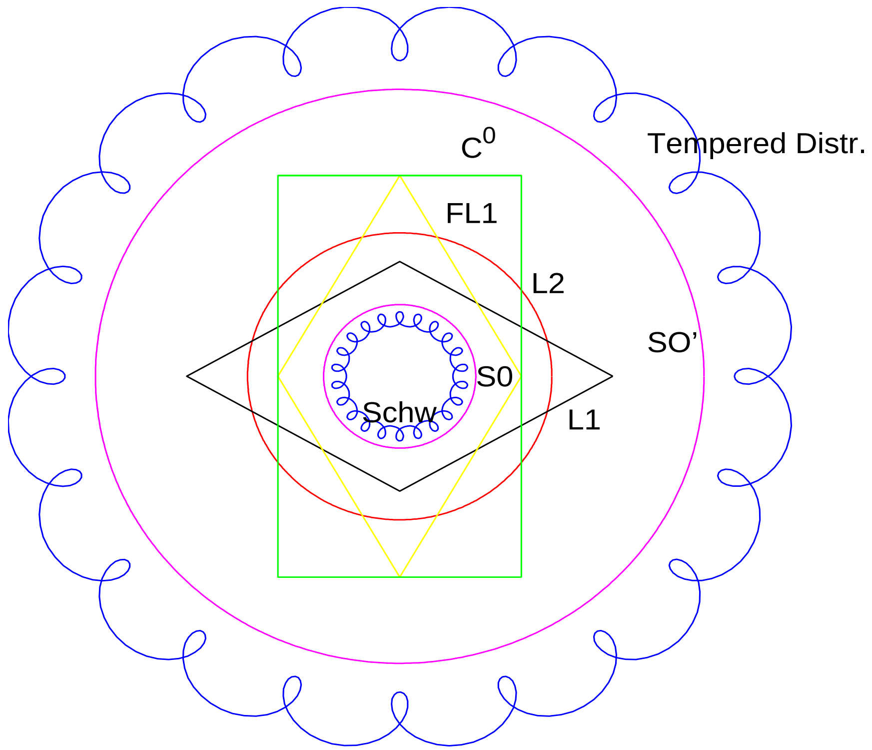
THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $S_o(G)$; $[with (\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F} S_o(G) = S_o(\hat{G})$; (Invariance under the Fourier Transform)
- $T_H S_o(G) = S_o(G/H)$; (Integration along subgroups)
- $R_H S_o(G) = S_o(H)$; (Restriction to subgroups)
- $S_o(G_1) \hat{\otimes} S_o(G_2) = S_o(G_1 \times G_2)$. (tensor product stability);

THEOREM: (Consequences for the dual space)

- $S_o'(G)$ is a Banach space with a translation invariant norm;
- $S_o'(G) \subseteq \mathcal{S}'(G)$, i.e. $S_o'(G)$ consists of tempered distributions;
- $P(G) \subseteq S_o'(G) \subseteq Q(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq S_o'(G)$; (contains translation bounded measures);
- $\mathcal{M}_T(G) \subseteq S_o'(G)$ (contains "transformable measures" by Gil-de-Lamadrid).

Schwartz space, S_0, L^2, S_0', tempered distributions



Basic properties of $S_0(\mathbf{R}^d)$ continued

THEOREM:

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in S_o(\hat{G}), \sigma \in S_o'(G)$, satisfies
$$\mathcal{F}(S_o'(G)) = S_o'(\hat{G}).$$
- $\sigma \in S_o'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in S_o'(G/H)$ such that
$$\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle \, .$$
- $S_o'(H)$ can be identified with a subspace of $S_o'(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in S_o'(G)$ one has $\sigma \in i_H(S_o'(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

The Usefulness of $S_0(\mathbf{R}^d)$

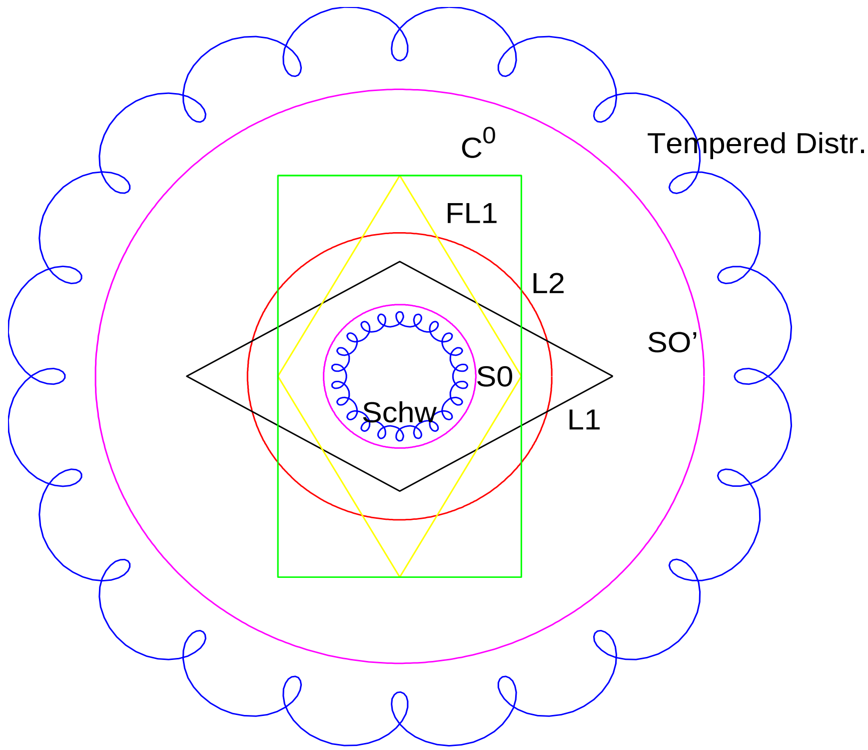
Theorem 2. Poisson's formula
For $f \in S_0(\mathbf{R}^d)$ and any discrete subgroup H of $\mathbf{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \tag{3}$$

with absolute convergence of the series on both sides.

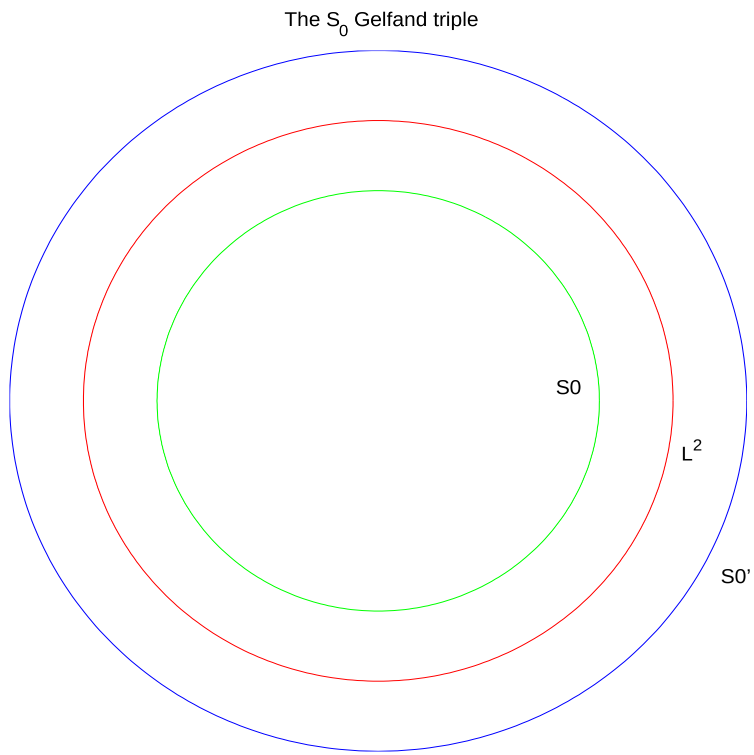
By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $S_0'(\mathbf{R}^d)$ is invariant under the (generalized) Fourier transform. This in turn gives a correct mathematical argument for the fact that the sampling over Z, which corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$ corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side, which in turn is nothing else than periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in S_0(\mathbf{R}^d)$ all this makes perfect sense.

Schwartz space, S_0, L^2, S_0', tempered distributions



Wiener Amalgams and Gabor Analysis (Talk II):

- Recall: Banach Gelfand triples, unconditional Banach frames,
- suggest: Gelfand frames and Gelfand Riesz bases
- kernel theorem, various symbols
- use of amalgam spaces for the basic questions of Gabor analysis
- sufficiently dense lattices Λ generate good Gabor frames
- robustness: perturbation of either the atom or the lattice is continuous in the $\boldsymbol{S}_0$-setting
- and much more . . . , e.g. $\alpha-$modulation spaces;



The Fourier transform is a prototype for the notion of a Gelfand triple isomorphism.

The Fourier transform as Gelfand Triple Automorphism

Theorem 3. *The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $S_0(\mathbb{R}^d)$ to $S_0(\widehat{\mathbb{R}^d})$,*

(2) *$\mathcal{F}$ is a unitary map between $L^2(\mathbb{R}^d)$ and $L^2(\widehat{\mathbb{R}^d})$,*

(3) *$\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $S_0'(\mathbb{R}^d)$ onto $S_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle \tag{4}$$

is valid for $(f, g) \in S_0(\mathbb{R}^d) \times S_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$.

One can characterize the Fourier transform as the *uniquely determined* Gelfand triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$ which maps "pure frequencies" into the corresponding Dirac measures (as one would expect in the case of a finite Abelian group).

The Kernel Theorem

Theorem 4. *If K is a bounded operator from $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in S_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in S_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.*

"PAUSE! "

In the description of G. Pfander one would write

$$Kf(x) = \int_{\mathbf{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in S_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} k(x,y)f(y)dyg(x)dx = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} k(x,y)g(x)f(y)dxdy.$$

This result is the "outer shall of the Gelfand triple isomorphism, which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The complete picture can again be expressed by a unitary Gelfand triple isomorphism:
Theorem 5. *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with* $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $S_0(\mathbb{R}^{2d})$ *if and only if the corresponding operator* K *maps* $S_0'(\mathbb{R}^d)$ *into* $S_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* $w*-$*topology into the norm topology of* $S_0(\mathbb{R}^d)$*.*

Remark: Note that for "regularizing" kernels in $S_0(\mathbb{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Since $\delta_y \in S_0'(\mathbb{R}^d)$ and thus $K(\delta_y) \in S_0(\mathbb{R}^d)$ the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfandtriple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ into $\big(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))\big)$.

The Kohn Nirenberg Symbol and Spreading Function

The Kohn-Nirenberg symbol of an operator (respectively its *symplectic* Fourier transform can be obtained from the kernel using some automorphism and a partial Fourier transform, which again provide unitary Gelfand isomorphisms. The symplectic Fourier transform is another unitary Gelfand Triple automorphism of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$. It can be characterized in the following way:

Theorem 6. *The correspondence between an operator from the Banach Gelfand triple* $\big(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))\big)$ *and the corresponding spreading distribution in* $S_0(\mathbb{R}^{2d})$ *is the uniquely defined Gelfand triple isomorphism between* $\big(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))\big)$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ *which maps the time-frequency shift operators* $M_y \circ T_x$ *onto the Dirac measure* $\delta_{(x,y)}$.

Wiener amalgam convolution and pointwise multiplier results imply that

$$S_0(\mathbb{R}^d) \cdot (S_0'(\mathbb{R}^d) * S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d) \quad \text{and} \quad S_0(\mathbb{R}^d) * (S_0'(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d)$$

because, e.g. $S_0(\mathbb{R}^d) * S_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}L^1, \ell^1) * \boldsymbol{W}(\mathcal{F}L^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}L^1, \ell^\infty)$.

Lemma 4. *Let* $(\mathbf{\Psi}_{\mathbf{k},\mathbf{n}})$ *be an orthonormal Wilson basis for* $L^2(\mathbb{R}^d)$. *Then the coefficient mapping* $D : f \mapsto \langle f, \mathbf{\Psi}_{k,n}\rangle$ *induces a unitary Gelfand triple isomorphism between*

$$(S_0, L^2, S_0')(\mathbb{R}^d) \quad \text{and} \quad (\ell^1, \ell^2, \ell^\infty)(\mathbb{Z}^d \times \mathbb{N}^d).$$

"PAUSE! "

From this identification one can get some understanding of the kernel theorem: one can express the w^*-to-norm continuous operators from $S_0'(\mathbb{R}^d)$ into $S_0(\mathbb{R}^d)$ with the w^*-to-norm continuous operators from ℓ^∞ into ℓ^1 which turns out to be exactly the space of (bi-infinite) matrices in $\ell^1(\mathbb{Z}^d \times \mathbb{Z}^d)$. "PAUSE! "Another way of compactly describing a typical result from Gabor analysis is the following:

Theorem 7. *Assume that* $g \in S_0(\mathbf{R}^d)$ *and that* $(g_\lambda)_{\lambda\in\Lambda}$ *is a Gabor frame (for* $\boldsymbol{L}^2(\mathbb{R}^d)$*). Then the dual Gabor atom* $\widetilde{g}$ *automatically belongs to* $S_0(\mathbf{R}^d)$ [1], *and the coefficient mapping* $\mathcal{C} : f \mapsto R_\Lambda(V_g f) = V_g f|_\Lambda$ *maps* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$ *into* $(\ell^1, \ell^2, \ell^\infty)$. *The reconstruction mapping* $\mathcal{R}$ *uses the dual Gabor atom* $\widetilde{g}$ *via* $\mathcal{R}(\mathbf{c}) = \sum_{\lambda\in\mathbf{\Lambda}} \mathbf{c}_\lambda \widetilde{\mathbf{g}}_\lambda$.

[1] due to Gröchenig and Leinert

As an example we mention the KN symbol of a rank-one operator $f \otimes \overline{g}$, which describes the mapping $h \mapsto \langle h, g \rangle f$, is equal to

$$\sigma(f \otimes \overline{g})(x,\omega) = f(x)\overline{\hat{g}(\omega)}e^{-2\pi i x\cdot\omega}, \quad (x,\omega) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \tag{5}$$

the Rihaczek distribution of f against g. For $f, g \in S_0(\mathbb{R}^d)$ we have that the KN-symbol $\sigma(f \otimes \overline{g})$ is in $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$ which in turn implies (using the last equation) that $(x,\omega) \mapsto e^{2\pi i x\cdot\omega}$ is a pointwise multiplier on $S_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Kohn-Nirenberg and Spreading Symbols of Operators

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

Kohn-Nirenberg correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator with *symbol* σ

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}d\omega$$

is called the *pseudodifferential operator* with Kohn-Nirenberg symbol σ.

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \sigma(x,\omega) e^{-2\pi i (y-x)\cdot\omega} d\omega \Big) f(y) dy$$
$$= \int_{\mathbb{R}^d} k(x,y) f(y) dy.$$

2. Formulas for the (integral) kernel k: $k = \mathcal{T}_a \mathcal{F}_2 \sigma$

$$k(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x)$$
$$= \widehat{\sigma}(\eta, y-x) e^{2\pi i \eta\cdot x} d\eta.$$

3. The *spreading representation* of the same operator arises from the identity

$$K_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$

$\widehat{\sigma}$ is called the spreading function of the operator K_σ.

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *Rihaczek distribution* is defined by

$$R(f,g)(x,\omega) = e^{-2\pi i x\cdot\omega}\overline{\widehat{f}(\omega)}g(x).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, R(f,g)\rangle = \langle K_\sigma f, g\rangle$$

is well-defined and describes a uniquely defined operator from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Weyl correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator

$$L_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi,u)e^{-\pi i \xi\cdot u} f(x)\,du\,d\xi$$

is called the *pseudodifferential operator* with *symbol* σ. The map $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and σ the Weyl symbol of the operator L_σ.

$$
\begin{aligned}
L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \widehat{\sigma} e^{-\pi i u\cdot\xi} T_{-u} M_\xi f(x)\,du\,d\xi \\
&= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \widehat{\sigma}(\xi, y-x) e^{-2\pi i \xi \frac{x+y}{2}} \Big) f(y)\,dy.
\end{aligned}
$$

2. Formulas for the kernel k from the KN-symbol: $k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma$

$$
\begin{aligned}
k(x,y) &= \mathcal{F}_1^{-1}\widehat{\sigma}\Big(\frac{x+y}{2}, y-x\Big) \\
&= \mathcal{F}_2\sigma\Big(\frac{x+y}{2}, y-x\Big) \\
&= \mathcal{F}_2^{-1}\sigma\Big(\frac{x+y}{2}, y-x\Big) \\
&= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma.
\end{aligned}
$$

3. $\langle L_\sigma f, g\rangle = \langle k, g \otimes \overline{f}\rangle .(Weyloperatorvs.kernel)$

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *cross Wigner distribution* of f and g is defined by

$$W(f,g)(x,y) = \int_{\mathbb{R}^d} f(x+t/2)\overline{g}(x-t/2)e^{-2\pi i\omega\cdot t}dt = \mathcal{F}_2\mathcal{T}_s(f\otimes\overline{g})(x,\omega).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, W(f,g)\rangle = \langle L_\sigma f, g\rangle$$

is well-defined and describes a uniquely defined operator L_σ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

$$(\mathcal{U}\sigma)(\xi,u) = \mathcal{F}^{-1}(e^{\pi i u\cdot\xi}\widehat{\sigma}(\xi,u)).$$
$$K_{\mathcal{U}\sigma} = L_\sigma$$

describes the connection between the Weyl symbol and the operator kernel.

In all these considerations the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be correctly replaced by $S_0(\mathbf{R}^d)$ and the tempered distributions by $S_0'(\mathbf{R}^d)$.

Useful operator representations (thanks to G. Pfander)

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s)f(s)\,ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega)\widehat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x)f(x-t)\,dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu)f(x-t)e^{2\pi i x\cdot\nu}\,dt\,d\nu$
	$=$
	$\int\int \eta_H(t,\nu)M_\nu T_t f(x),\,dt\,d\nu\,,$

Time-Frequency Concentration via Modulation Spaces

Recall that the Gauss function is given by $g_0(t) = e^{-\pi t^2}$, $z = (x, \xi) \in \mathbb{R}^d$. The Short-Time Fourier Transform (STFT) with Gaussian window is therefore

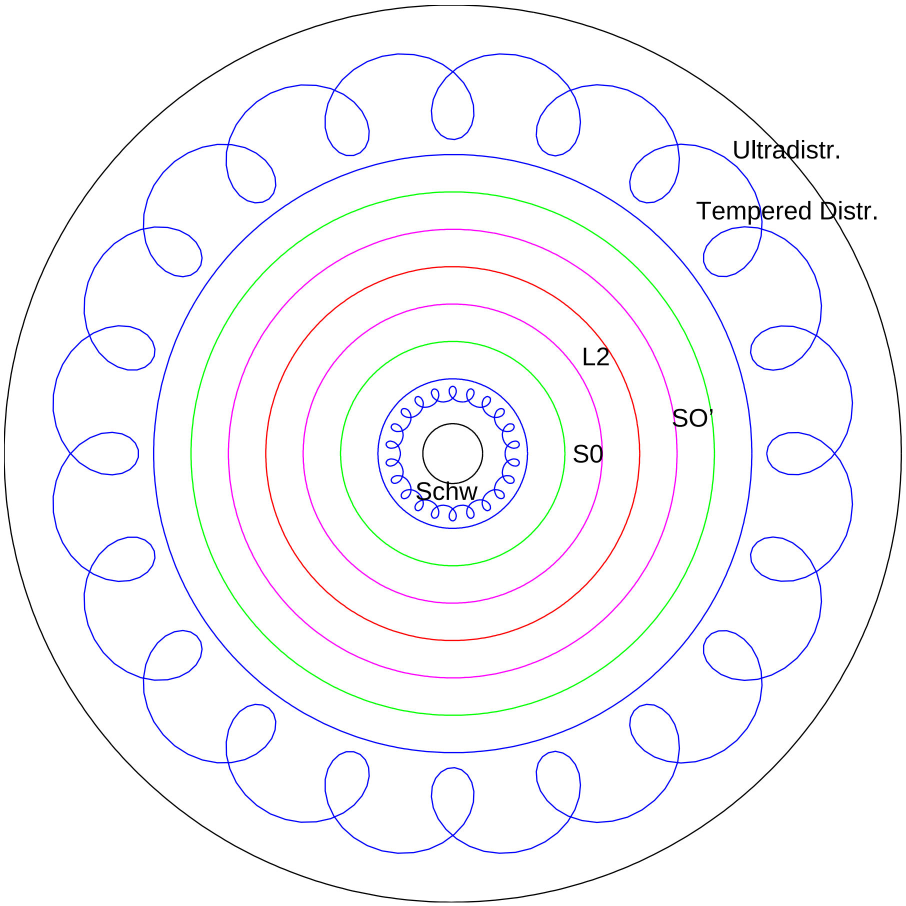
$$\begin{aligned} V_{g_0} f(z) &= \int_{\mathbb{R}^d} f(t)\, g_0(t - x)\, e^{2\pi i t \cdot \xi}\, dt \\ &= \langle f, M_\xi T_x g_0 \rangle \end{aligned}$$

Modulation space M_v^1 ("good pulses") consists of all functions such that

$$\|f\|_{M_v^1} := \int_{\mathbb{R}^d} |V_{g_0} f(z)|\, v(z)\, dz < \infty$$

Typical examples for such weight functions v on phase space are either weights depending on frequency only, such as $v(x, \xi) = (1 + |x|)^s$ (leading to the "classical" modulation spaces), or more interesting (because they lead to Fourier invariant spaces) weights which are radial: $v_s := (1 + |x|^2 + |\xi|^2)^{s/2}$. The intersection of all spaces $M_{v_s}^1$ is just the Schwartz space $\mathcal{S}(\mathbb{R}^d)$.

A Collection of Fourier Invariant Spaces



Frames and Riesz Bases

A family is a Riesz basis in a Hilbert space if it behave like a finite linear independent sequence in a Hilbert space: The set of all linear combinations (infinite linear combinations with square summable coefficients) is a closed subspace (whose orthogonal complement is the null-space of the adjoint mapping). It always has a biorthogonal Riesz basis (obtained via the inverse Gram matrix).

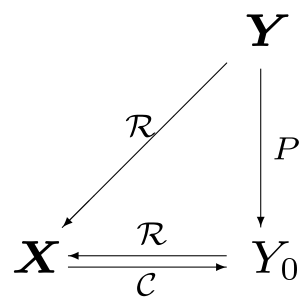
A family is a frame in a Hilbert spaces, if it behaves like a generating set (in the sense that the set of all linear combinations with square summable coefficient equals the whole Hilbert space. There is always a dual frame (obtained by inverse frame operator).

There is also a natural "duality between these problems, either in the abstract Hilbert space setting, or better understandable in the context of matrices. A rectangular $m \times n$ matrix A of *maximal rank* is either linear independent $m \geq n$ or a generating set $n \geq m$, and therefore its transpose (conjugate) is of the "other type". In this contact the frame matrix is just the Gramian to A'.

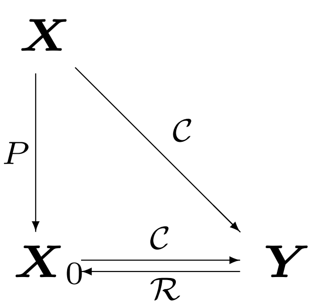
Frames and Riesz Bases: Commutative Diagrams

Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range Y_0 of $\mathcal{C}$ in $\boldsymbol{Y}$:



Riesz Basis case: E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$:



Unconditional Banach Frames

A suggestion for making the bring the well established notion of Banach frames closer to the setting we are used from the Hilbert space and ℓ^2-setting:

Definition 8. *A mapping* $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ *defines an unconditional (or solid) Banach frame for* $\mathbf{B}$ *w.r.t. the sequence space* $\mathbf{Y}$ *if*

1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, *with* $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \| \|_{\mathbf{Y}})$ *is a solid Banach space of sequences over* I, *with* $\mathbf{c} \mapsto c_i$ *being continuous from* $\mathbf{Y}$ *to* $\mathbb{C}$ *and solid, i.e. satisfying* $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$ *(hence, w.l.o.g.,* $\mathbf{e}_i \in \mathbf{Y}$*),*
3. *finite sequences are dense in* $\mathbf{Y}$ *(at least* W^**).*

Corollary 5. *By setting* $h_i := \mathcal{R}e_i$ *we have* $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ *unconditional in* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$*, hence* $f = \sum_{i \in I} T(f)_i h_i$ *as unconditional series.*

"PAUSE! "We may talk about Gelfand frames (or Banach frames for Gelfand triples) resp. Gelfand Riesz bases (as opposed to a Riesz projection basis for a given pair of Banach spaces).

Spline type Spaces: Reconstruction from Local Averages

Theorem 8 (Reconstruction in spline type spaces from local averages, Aldroubi/Fei). *Let the weight m, the lattice Λ, and the generator ϕ in Wiener's algebra be given. Then there exists a density $\gamma = \gamma(\phi) > 0$ and $a_0 > 0$ such that any $f \in V_m^p(\phi)$ can be recovered from the data $\{\langle f, \psi_{x_j}\rangle : j \in J\}$ on any γ-dense set $X = \{x_j : \; j \in J\}$ and for any $0 < a < a_0$, by the following iterative algorithm :*

$$f_1 = PA_X f, \quad f_{n+1} = PA_X(f - f_n) + f_n \ ,$$

where P is the operator onto $V_m^p(\phi)$. In this case, the iterate f_n converges to f in the W_m^p-norm, hence both in the L_m^p-norm, and uniformly over compact sets. If furthermore $m(x) \geq 1$ for all $x \in \mathbf{R}^d$, then $L_m^p \subset L^p(\mathbf{R}^d)$ and one has uniform convergence.
The convergence is geometric, that is,

$$\|f - f_n\|_{L_m^p} \leq \|f - f_n\|_{W_m^p} \leq C_1 \alpha^n \|f - f_1\|_{W_m^p}$$

for some $\alpha = \alpha(\gamma, a, \phi) < 1$ and $C_1 < \infty$.

Riesz Projection bases for Spline-type spaces

Think of a transation invarant (say wavelet) closed subspace V with a Riesz (or even orthonormal) basis of the form $(T_\lambda \varphi)_{\lambda\in\Lambda}$. If φ is of some mild quality, namely $\varphi \in \boldsymbol{W}(Ł^2, \ell^1)$ then we have $\varphi * \varphi^* \in \boldsymbol{W}(\mathcal{F}L^1, \ell^1) = S_0(\mathbf{R}^d)$, hence the sampled autocorrelation function is in ℓ^1. "PAUSE! "
The orthonormal projection from the Hilbert space $f \mapsto P_V$ is obtained by the mapping

$$f \mapsto \sum_{\lambda\in\Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi.$$

"PAUSE! "But this mapping is not only well defined on $\boldsymbol{L}^2$, but also on a $\boldsymbol{L}^p$, for the full range of $1 \leq p \leq \infty$ and - again due to the properties of Wiener amalgams brings us for $f \in (\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ coefficients which are in $(\ell^1, \ell^2, \ell^\infty)$, which in turn implies that the function $\sum_{\lambda\in\Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi$ is a well defined element of $\boldsymbol{W}(\boldsymbol{C}^0, (\ell^1, \ell^2, \ell^\infty))$.

Fundamental facts in Gabor Analysis

Here is a list of "fundamental questions" in Gabor analysis:

1. Given a "sufficiently nice" atom, can we show, that it provides us with good Gabor frames (regular or irregular)?
2. Given the fact that the Gabor frame operator for a regular Gabor family $g_\lambda = \pi(\lambda)g$ (with λ from a lattice Λ in phase space) commutes with TF-shifts from Λ, how can we derive Janssen's representation from this fact (and what can be said about the convergence of Janssen's representation)?
3. An important ingredient for obtaining the Janssen is the fact that the periodization of an operator along a TF-lattice Λ corresponds exactly to sampling in the spreading domain. How can that be described in mathematically correct terms?
4. Recalling that the dual frame for a regular Gabor frame is a Gabor frame with $\tilde{g} = S^{-1}g$ as generator. How can one ensure good properties of $\tilde{g}$, given a good quality of g.
5. In which way do we have robustness of the choice of dual Gabor atoms, in the sense that "similar lattices" induce similar dual Gabor atoms?

Sufficient Dense Lattices Λ provide Gabor Frames

It is just a simple reformulation that a (regular or irregular) Gabor family $(g_i) = g_{\lambda_i}$ is a (Gabor) frame if and only if the Gabor frame operator

$$S : f \mapsto \sum_{i \in I} \langle f, g_i \rangle g_i$$

is a bounded and positive definite (hence invertible) operator on the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$. The usual way to understand S is to view it as a composition of the coefficient operator $T : f \mapsto (\langle f, g_i \rangle)_{i \in I}$ and its adjoint T^*, given by $(c_i) \mapsto \sum_i c_i g_i$. Obviously T can be reinterpreted as the mapping $f \mapsto (V_g f(\lambda_i))_{i \in I}$.
Since it is well known that for any two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ one has $V_g f \in \boldsymbol{C}^b(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^{2d})$ one would not expect that there is a problem with sampling. At least for the regular case, i.e. if $\{\lambda_i, i \in I\} = \Lambda$ is a lattice. However, examples by F-Janssen show that one may have a situation, where the family $(g_\lambda)_{\lambda \in \Lambda}$ is a Bessel family (equivalent to the boundedness of T) for any rational lattice Λ, whereas it is not a Bessel family for a "dense set" of irrational lattices.

Sufficient Dense Lattices Λ provide Gabor Frames, ctd.

It is now maybe not so surprising the hear, that windows in $S_0(\mathbf{R}^d)$ are the good ones: Not only can one claim that the coefficient mapping is bounded for *any lattice* Λ, but even more generally for arbitrary reatively separated point sets $(\lambda_i)_{i\in I} \subset \mathbf{R}^d \times \widehat{\mathbf{R}}^d$
This may not come as a surprise as soon as one has understood that $V_g f$ is not only in $\boldsymbol{C}^b(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^{2d})$, but that

$$g \in S_0(\mathbf{R}^d) \Rightarrow V_g f \in \boldsymbol{W}(\mathbb{C}^0, \ell^2)(\mathbf{R}^d \times \widehat{\mathbf{R}}^d) \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d)$$

This in term can be justified by the (non-trivial) fact that $g \in S_0(\mathbf{R}^d)$ if and only if $V_g g$ (not just $V_{g_0} g \in \boldsymbol{L}^1(\mathbb{R}^{2d})$) if and only if $V_g g \in \boldsymbol{W}(\mathbb{C}^0, \ell^1)(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$, combined with a *twisted* convolution relation between short time Fourier transforms:

$$V_g f = V_g f *_{tw} V_g g.$$

The fact that $V_g f \in \boldsymbol{W}(\mathbb{C}^0, \ell^2)$ (even in $\boldsymbol{W}(\mathcal{F}L^1, \ell^2)$!) in connection with the reproducing formula is also the basis for coorbit theory which shows that *any sufficiently dense* family $(\lambda_i)_{i\in I} \subset \mathbf{R}^d \times \widehat{\mathbf{R}}^d$ generates a Gabor frame (g_i).

Automatic Quality of Irregular Gabor Families

It is one of the standard results of classical coorbit theory (Fei/Gro) that - again given sufficient density of the (possibly irregular) family $(\lambda_i)_{i\in I}$ - the Gabor frames (g_i) arising are not just frames for the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, but also allow to characterize various coorbit spaces. In the present situation coorbit spaces characterized by the behaviour of STFTs of their elements are just the modulation spaces.

One concrete and compact way to express the essential feature in the context of Banach Gelfand triples is the following result, which can be seen as examples of Banach frame features of coorbit theory:

Theorem 9. *For the (irregular or regular) Gabor described above, with Gabor atom $g \in S_0(\mathbf{R}^d)$ and (λ_i) sufficiently dense one has: the mapping $T : f \mapsto (V_g f(\lambda_i))_{i\in I}$ describes a retract from the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$ into the Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$.*

In particular one can recognize the membership of f in one of the space $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$ by checking the membership of the family $(V_g f(\lambda_i))_{i\in I}$ in one of the spaces from $(\ell^1, \ell^2, \ell^\infty)$, and resynthesis from the sampling values can be achieved in a stable way at each level.

Robustness and Jitter Stability of such Gabor Frames

Once one has found that a family $(g_i) = (\pi(\lambda_i)g)_{i \in I}$ is a Gabor frame, what the *quality of that Gabor frame is*. How robust is it against minor perturbation of the underlying family (λ_i)?
It is natural to ask whether it is true that not only Gabor families which are obtained from an exact lattice are frames, but also those obtained by a (sufficiently small) uniform jitter error.
"PAUSE! "That such a theorem cannot hold for general L^2-frames is easily seen by considering the complete orthonormal system on $\mathbb{R}$ obtained from $g = \mathbf{1}_{[0,1)}$ and the critical lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$. "PAUSE! "
In contrast, one has for $g \in S_0(\mathbf{R}^d)$: Assume that $(\pi(\lambda_i)g$ is a Gabor frame, then there exists $\delta > 0$ such that for any family (λ_i') with $|\lambda_i' - \lambda_i| < \delta$ for all $i \in I$ one has: $\pi(\lambda_i')g$ is a Gabor frame as well. "PAUSE! "
Even more is true: not only are those frames off all those families of points "sufficiently close" are all of similar quality, they are also preservering the property of characterizing the Gefand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$ by having the *canonical coefficients* in $(\ell^1, \ell^2, \ell^\infty)$.

Gabor Atom, the Tight, and the Dual Atom

For the *regular case*, i.e. for the case that the discrete point set generating the regular Gabor family $(g_\lambda)_{\lambda\in\Lambda}$ from one or several Gabor atoms the Gabor frame operator has the crucial addition property of commuting with the TF-shifts used in building it. Although the mapping $\lambda \mapsto \pi(\lambda)$ is not a unitary group representation, but still a projective representation (with some phase factors arising in the composition law), one arrives via resummation at the crucial fact that "PAUSE! "

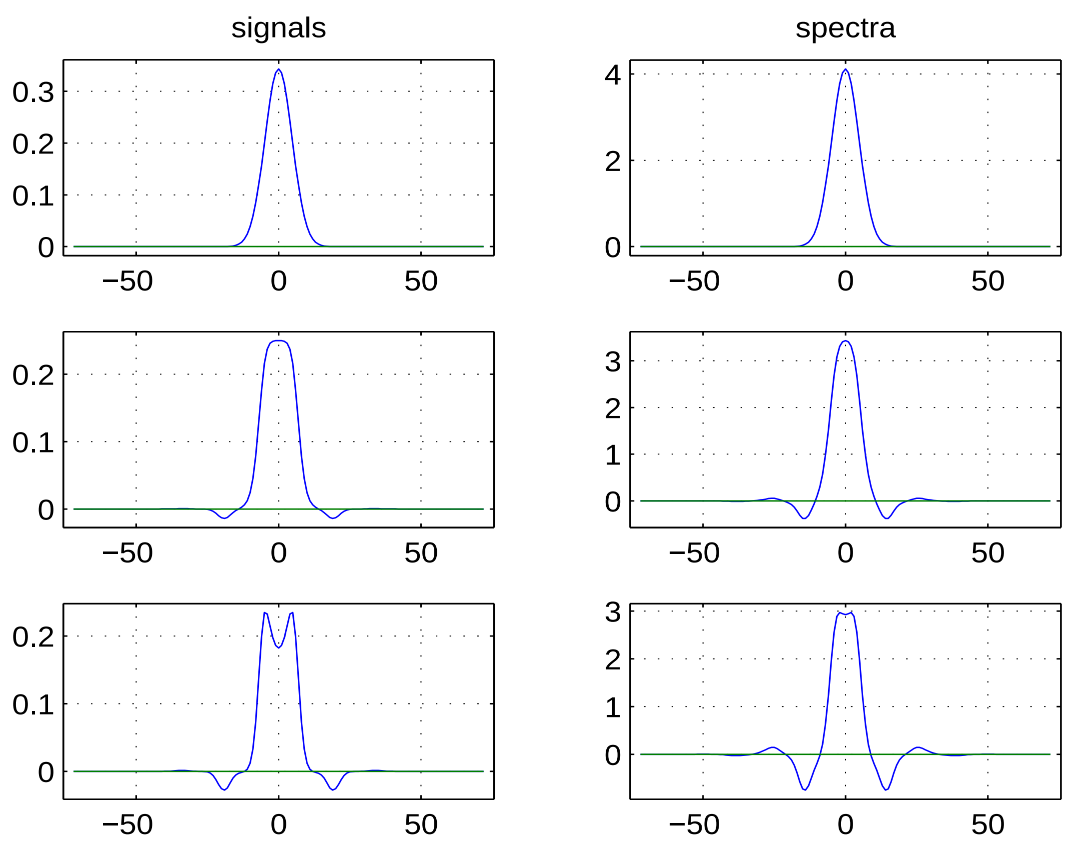
$$S \circ \pi(\lambda) = \pi(\lambda) \circ S \quad \forall \lambda \in \Lambda$$

"PAUSE! "

Obviously this commutation rule implies the same property for the inverse frame operator $S^{-1} \circ \pi(\lambda) = \pi(\lambda) \circ S^{-1} \quad \forall \lambda \in \Lambda$ and the usual simple description of recovery as $f = S(S^{-1}f) = S^{-1}(Sf)$ realizes to the representations using the dual Gabor atom $\widetilde{g}$:
"PAUSE! "

$$f = \sum_{\lambda\in\Lambda} V_{\widetilde{g}} f(\lambda) \cdot \pi(\lambda) g = \sum_{\lambda\in\Lambda} V_g f(\lambda) \cdot \pi(\lambda) \widetilde{g}.$$

Gabor Atom, the Tight, and the Dual Atom



Janssen's Representation of the Gabor Frame Operator

Definition 9. *For a given TF-lattice Λ the adjoint TF-lattice Λ° is defined as follows:*

$$\Lambda^\circ := \{\lambda^\circ \mid \pi(\lambda) \circ \pi(\lambda^\circ) = \pi(\lambda^\circ) \circ \pi(\lambda), \quad \forall \lambda \in \Lambda\}$$

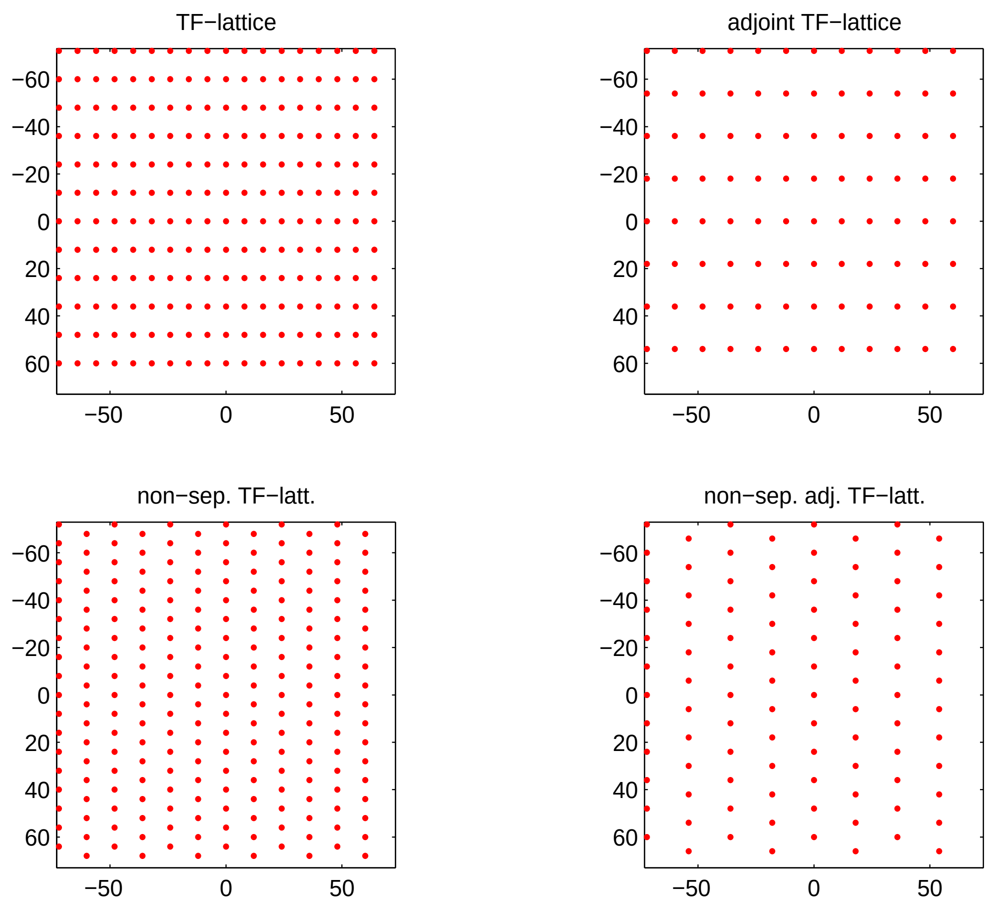
The so-called Janssen representation of the Gabor frame operator

Note that in the case of a separable group $\Lambda = \Lambda_1 \times \Lambda_2$, with Λ_1 and Λ_2 being lattices in the time- and frequency domain respectively it is easy to derive Walnut's representation from the Janssen's representation, describing S as a superposition of the form

$$Sf = \sum_{\lambda_1 \in \Lambda_1} G_{\lambda_1} T_{\lambda_1} f,$$

where the pointwise multipliers G_{λ_1} are Λ_1-periodic functions.
Concerning the convergence of this series it is not hard to show that $g \in \boldsymbol{W}(\boldsymbol{C}^0, \ell^1)(\mathbf{R}^d)$ or just $g \in \boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbf{R}^d)$ implies that $\sum_{\lambda_1 \in \Lambda_1} \|G_{\lambda_1}\|_\infty < \infty$, and consequently that S is bounded on any $\boldsymbol{L}^p, 1 \leq p \leq \infty$.

Adjoint Lattices, Janssen's Representation



Local properties of STFTs with $S_0(\mathbf{R}^d)$- windows

Corollary 6. *Let* $g \in S_0(\mathbf{R}^d)$. *Then* $|V_g f|^2 \in S_0(\mathbb{R}^{2d}) \subset \boldsymbol{W}(\boldsymbol{C}^0, \ell^1)$ *for* $f \in L^2(\mathbb{R}^d)$.
"PAUSE! "

This will be an important fact in the background of the following study: Consider the rank-one operators $P_\lambda : f \mapsto \langle f, g_\lambda \rangle g_\lambda$, for $\lambda \in \Lambda$. For g normalized in $\boldsymbol{L}^2$ these are the projections on the 1D-space generated by g_λ, and for $g \in S_0(\mathbf{R}^d)$ they are "good quality operators in $\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d))$. "PAUSE! "

We them as elements of $\mathcal{HS}$ and want to find out, whether they are a Riesz basis in $\mathcal{HS}$ by checking for the invertibility of their Gram matrix.

$$\langle P_\lambda, P_{\lambda'} \rangle_{\mathcal{HS}} = |\langle g_\lambda, g_{\lambda'} \rangle|^2_{\boldsymbol{L}^2} = |V_g g(\lambda - \lambda')|^2$$

this in turn is a circulant matrix, and its invertibility is equivalent to the fact that the $\Lambda^\perp$ periodic version of $\mathcal{F}_\Lambda(|V_g g|^2)$ is free of zeros (note that we can apply Wiener's inversion theorem because $\mathcal{F}_\Lambda(|V_g g|^2) \in S_0(\mathbf{R}^d)$, hence its periodization has an absolutely convergent Fourier series (as well as its inverse with respect to convolution).

Gabor Multipliers: Overview of Questions

$$G_m(f) = \sum_{\lambda \in \Lambda} m_\lambda V_g f(\lambda) g_\lambda = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda(f)$$

where we assume for simplicity that $g \in S_0(\mathbf{R}^d)$ generates a tight Gabor frame, or equivalently, we assume that $m \equiv 1$ gives us the identity operator. "PAUSE! "

- Gabor multiplier results can be obtained from the mapping properties of the $\mathcal{C} : f \mapsto V_g f|_\Lambda$ and the synthesis mapping $\mathcal{R} : c \mapsto \sum c_\lambda g_\lambda$.
- in addition one may ask in which sense the quality of the Gabor multipliers depends on the ingredients;
- what can be said about the linear mapping from sequences (m_λ) to operators G_m (injectivity, etc.);
- best approximation by Gabor multipliers;
- questions of stability (condition numbers);

Gabor Multipliers: Basic Facts

Theorem 10. *The mapping GM from the "upper symbol" (m_λ) to the Gabor multiplier G_m (for arbitrary $g \in \boldsymbol{L}^2(\mathbb{R}^d)$) is a Gelfand triple isomorphism from the Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ to the Gelfand triple of operator spaces on $\boldsymbol{L}^2(\mathbb{R}^d)$ consisting of $(\mathcal{S}_1, \mathcal{HS}, \mathcal{B}(\boldsymbol{L}^2))$.*

For $g \in S_0(\mathbf{R}^d)$ we have something stronger: "PAUSE! "

Theorem 11. *Assume $g \in S_0(\mathbf{R}^d)$. Then the mapping GM from the "upper symbol" (m_λ) to the Gabor multiplier G_m is a bounded linear Gelfand triple mapping from the Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ to the Gelfand triple of operator spaces $\big(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))\big)$.*

"PAUSE! "

Gabor Multipliers: Basic Facts II

Theorem 12. *Assume that* $g \in S_0(\mathbf{R}^d)$ *and that* (P_λ) *is a Riesz basis in* $\mathcal{HS}$, *then the mapping* GM *defines a Gelfand Riesz basis for* $(\ell^1, \ell^2, \ell^\infty)$ *into* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0{}')$.
In particuar, the orthogonal projection $T \mapsto P(T)$, *mapping a given Hilbert Schmidt operator to its coefficients of the best approximation by Gabor multipliers with respect to the given Gabor frame generated from* (g, Λ) *is extending to a Gelfand triple mapping from* $\big(\mathcal{L}(S_0'(\mathbb{R}^d), S_0(\mathbb{R}^d)), \mathcal{HS}, \mathcal{L}(S_0(\mathbb{R}^d), S_0'(\mathbb{R}^d))\big)$ *to* $(\ell^1, \ell^2, \ell^\infty)$.

Janssen's Representation of the (regular) Gabor Frame Operator II

$$S = \sum_{\lambda \in \Lambda} g_\lambda \otimes g_\lambda^*$$

can be written as

$$S = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ)$$

Sometimes one finds in the literature the so-called condition (A) (due to Tolimieri-Orr, which simply means that the coefficient sequence $(V_g g(\lambda^\circ))$ belongs to $\ell^1(\Lambda^\circ)$

Oviously this is true for the case that $g \in S_0(\mathbf{R}^d)$, because then $V_g g \in S_0(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ and consequently sampling to any discrete subgroup gives absolutely summable coefficients.

Stability of Gabor Frames with respect to Dilation

Recent results (Trans. Amer. Math. Math. Soc.), obtained together with N. Kaiblinger.

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

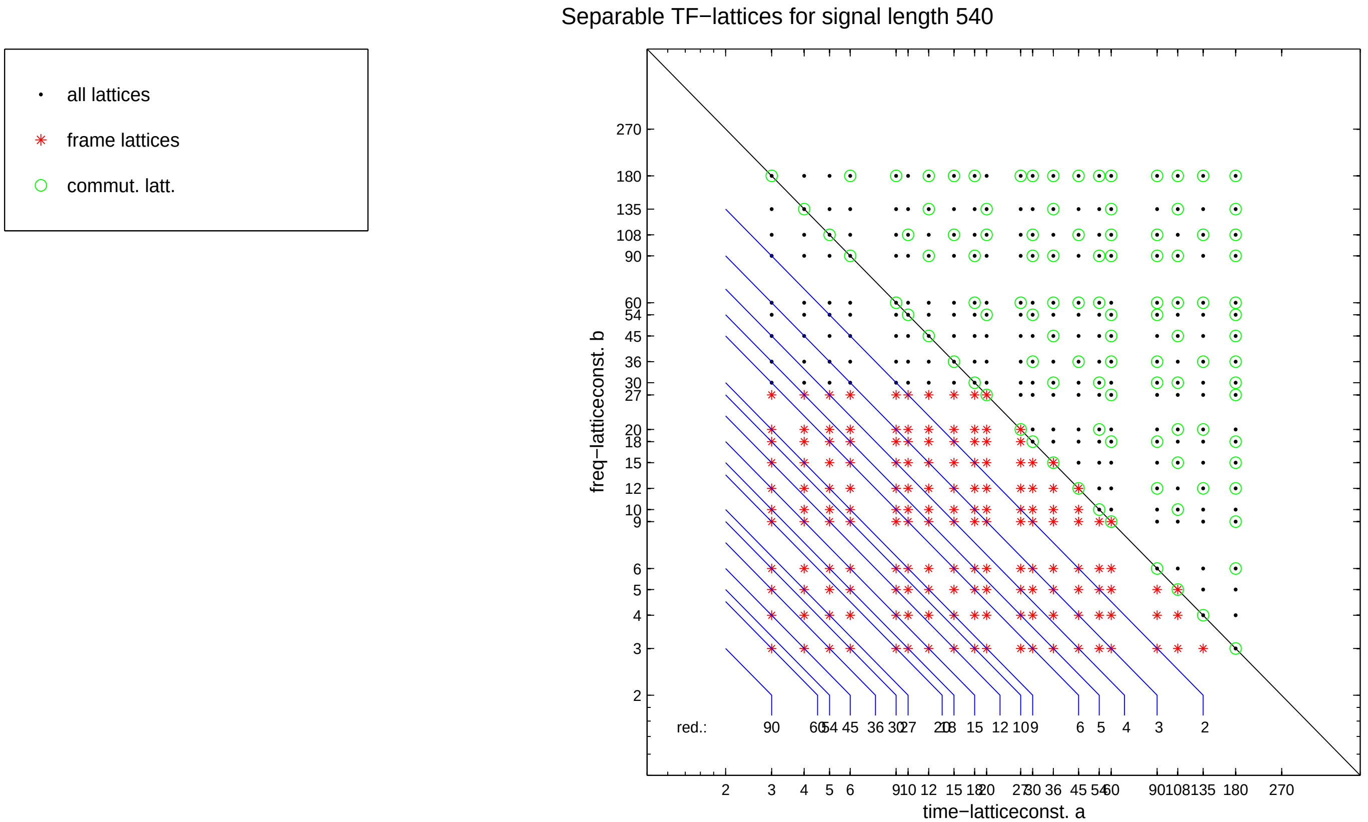
$$F_g = \big\{(g, L) \in X \times GL(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k \in \mathbb{Z}^{2d}}\big\}. \quad (6)$$

The set F_L^2 need not be open (even for good ONBs!). But we have: "PAUSE! "

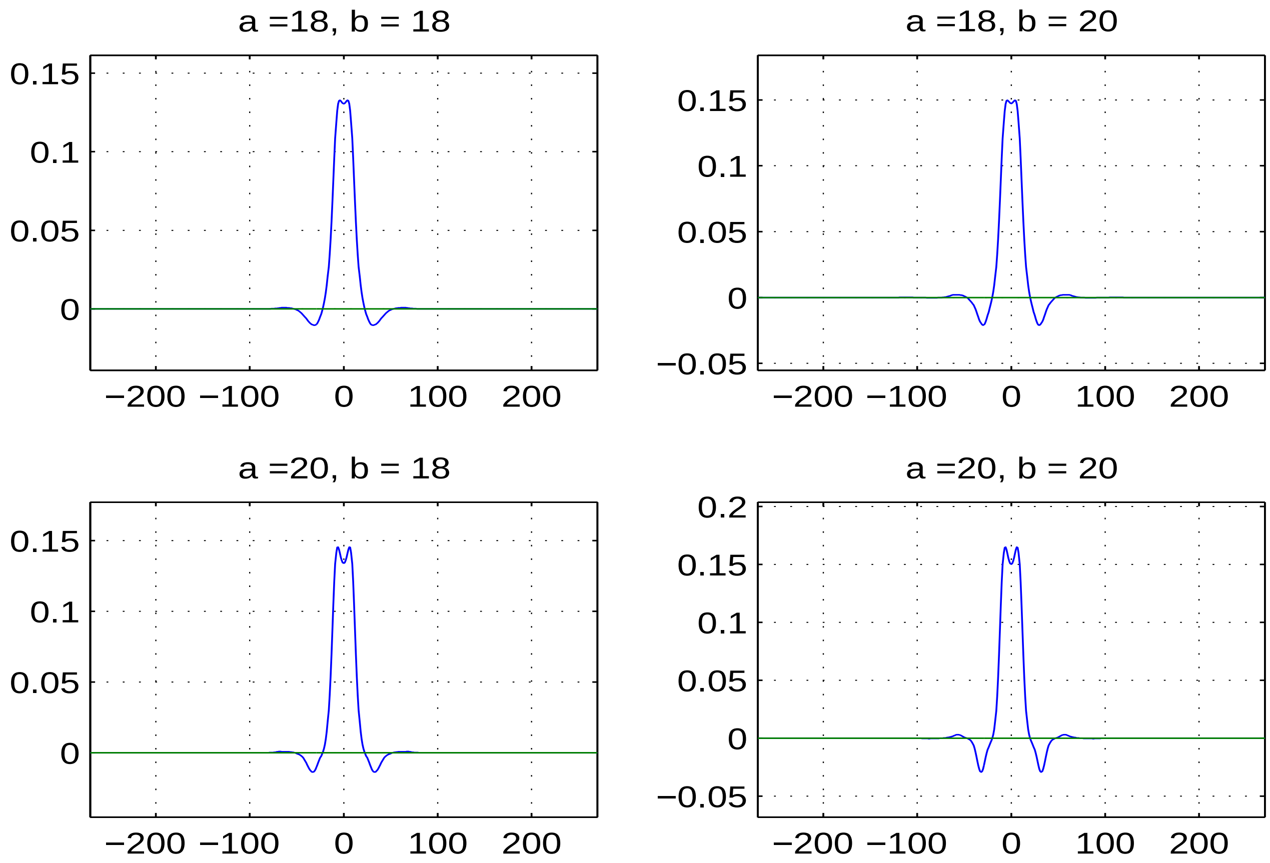
Theorem 13. (i) *The set $F_{S0}(\mathbf{R}^d)$ is open in $S_0(\mathbf{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *$(g, L) \mapsto \widetilde{g}$ is continuous mapping from $F_{S0}(\mathbf{R}^d)$ into $S_0(\mathbf{R}^d)$.*
"PAUSE! " There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Corollary 7. (i) *The set $F_{\mathcal{S}}$ is open in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *The mapping $(g, L) \mapsto \widetilde{g}$ is continuous from $F_{\mathcal{S}}$ into $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.*

A Discrete Version: Each Point "is" a Lattice, $n = 540$



On the continuous dependence of dual atoms on the TF-lattice



Why is it so relevant to know it for the $S_0(\mathbf{R}^d)$ norm?

Isn't the description above self-referential? Wouldn't it be reasonable to look out for the same results for "more standard" function spaces? (assuming that we are only interested in the $\boldsymbol{L}^2$*-setting* "PAUSE! "NO! "PAUSE! "

- Simply because the continuous dependence is not valid in the ordinary $\boldsymbol{L}^2$-setting!
- Even if it was true for some other norm it would not imply, that the overall system, i.e. the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} V_h(f) g_\lambda$$

would *not* be close to the Identity operator in the *operator norm on* $\boldsymbol{L}^2(\mathbb{R}^d)$, for all functions h which arise as dual windows for a pair (g', Λ'), with g' close to g and Λ' close to Λ (in the sense of having very similar generator!).

Other ongoing activities

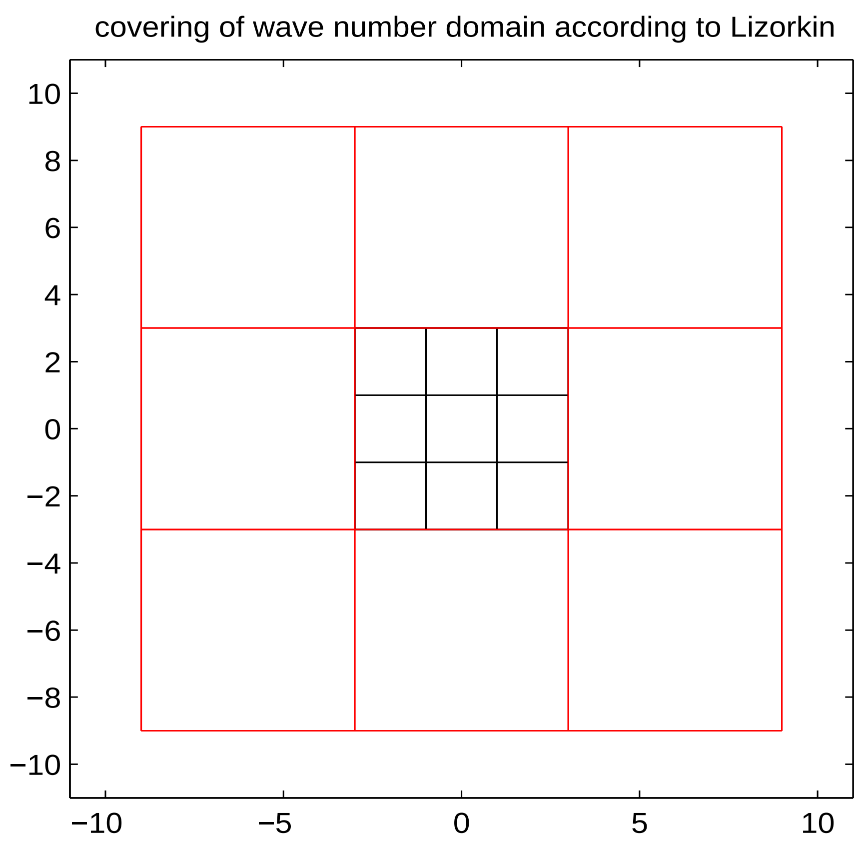
Together with N. Kaiblinger a paper on quasi-interpolation is on the way. It is shown that piecewise linear interpolation resp. quasi-interpolation (using for example cubic splines), i.e. operators of the form

$$Q_h f = \sum_{k \in \mathbb{Z}^d} f(hk) T_{hk} D_h \psi$$

are norm convergent to $f \in S_0(\mathbf{R}^d)$ in the $\boldsymbol{S}_0$-norm.
This is an important step for his work on the approximation of "continuous Gabor problems" by finite ones (handled computationally using MATLAB, for example).

In his recent PhD thesis (Nov. 05) F. Luef has established very interesting connections between the existing body of Gabor analysis and early work by A. Connes and M. Rieffel. The results will be published in several papers. Among others he could show that the use of $S_0(\mathbf{R}^d)$ or $\boldsymbol{S}_0(G)$, for G a LCA group, allows to verify many of the results obtaine by Rieffel using the (complicated) Schwartz-Bruhat space, in an easier way.

Lizorkin Decomposition for Besov Spaces



General decomposition spaces (continuous context)

Definition 10. *Let (X, Σ, μ) be a measure space. A family $(U_x)_{x \in X}$ of measurable subsets of X is called an admissible continuous covering system of X if one has:*

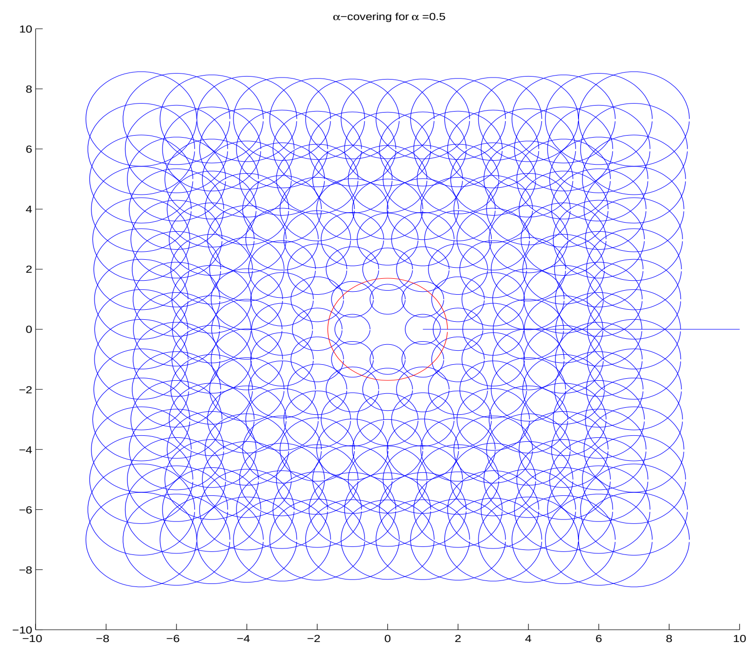
1. *$x \in U_x$ and $0 < \mu(U_x) < \infty$ for all $x \in X$;*
2. *$\exists C > 0$ s.t. $\mu(W_x) \leq C\mu(U_x)$, where $W_x := U_x^{(3)}$, and for general $s \geq 0$, $U_x^{(s)} = \bigcup U_{x_s}$ with the union being taken over all $x_s \in X$ such that $\exists (x_k)_{k=0}^s$: $x_0 = x$, $U_{x_{k-1}} \bigcap U_{x_k} \neq \emptyset$ for $k = 1, 2, \ldots, s$ (hence $U_x^{(0)} = U_x$). $U_x^{(k)}$ will be called the continuous cluster around x of order s.*

Theorem 14. *Let $(U_x)_{x \in X}$ be an admissible continuous covering system of a measure space (X, Σ, μ). Then there exist families $(x_i)_{i \in I}$ in X such that $Q = \left(U_{x_i}^{(s)}\right)_{i \in I}$ defines an admissible discrete covering of X such that*

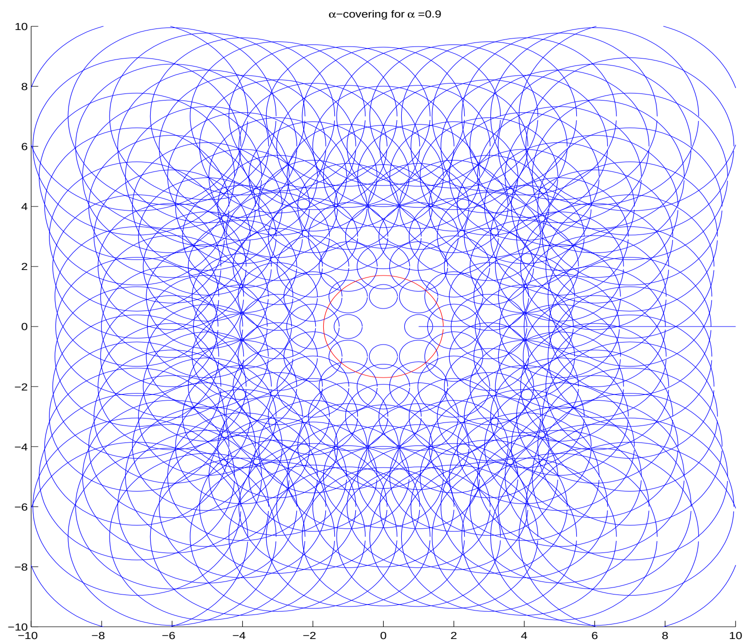
$$\exists C > 0 \; : \quad \forall i : \quad \#\{j \,|\, U_{x_i}^{(s)} \cap U_{x_j}^{(s)} \neq \emptyset\} \leq C < \infty.$$

It is possible to define bounded admissible partitions of unity and equivalence of coverings. Of course different families (x_i) as described above define equivalent coverings.

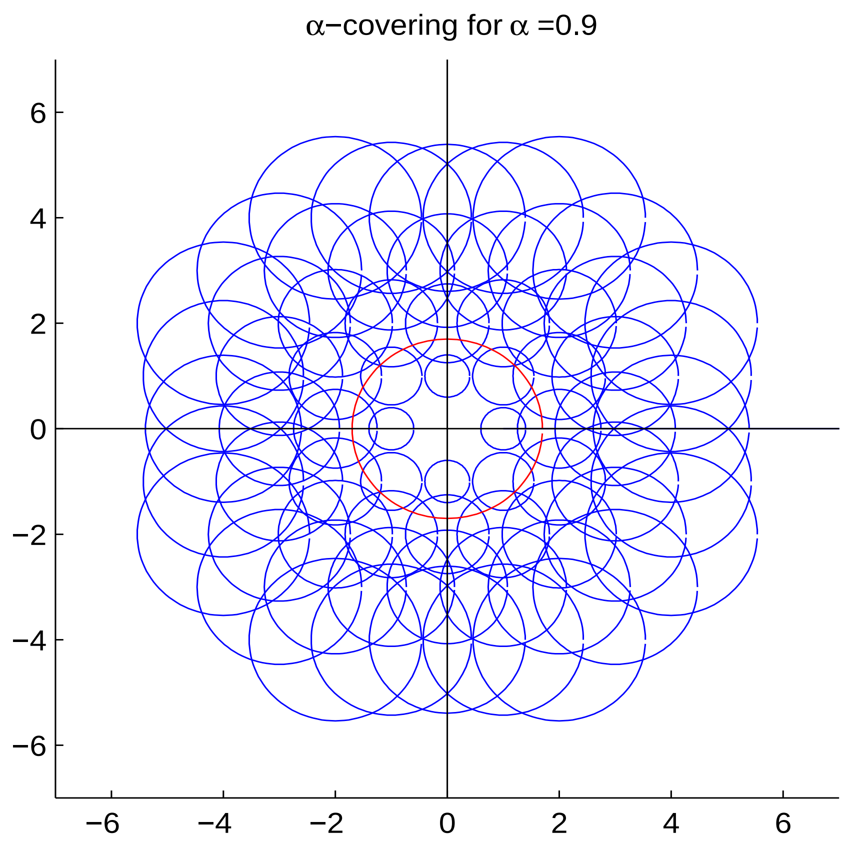
α-modulation spaces $\alpha = 0.5$



α-modulation spaces $\alpha = 0.9$



α-modulation spaces , $\alpha = 0.9$



α-modulation spaces

The theory of α-modulation spaces (originally developed in the PhD of Peter Gröbner, Vienna 1992) are an attempt to intermediate between the "ordinary modulation spaces" making use of uniform coverings of the frequency domains, and the Besov spaces as characterized by the so-called Triebel Lizorkin representation, which in turn is equivalent to a covering obtained by a selection from the continuous admissible covering which is obtained by the system of balls (the limiting case $\alpha = 1$), e.g. $\alpha = 0.5$:

$$\left(B_{R(x)}(x)\right)_{x\in\mathbb{R}^d}, \quad \text{with} \quad R(x) := \sqrt{(1+|x|^2}$$

The theory of quilted Gabor frames (developed by Monika Dörfler) aims at transfering similar ideas to the time-frequency context, in order to gain more flexibility by using adaptive tilings of phase space.