45 Years of Fourier Analysis in Vienna: Changes during an Academic Life Time

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Abstract

This talks gives a *personal account of my experiences in the field of Fourier Analysis*, starting in 1970 (course by L. Schmetterer) as a third semester student at this university, and ending with an outlook about the future of the field in the years to come, and my plans for further contributing to the subject.

Trained under H. Reiter in *Abstract Harmonic Analysis* I decided to turn towards applications after my habilitation in 1979, which finally led me to time-frequency analysis, computational harmonic analysis, and the use of MATLAB for the development of efficient algorithms based on principles of Harmonic Analysis.

Nowadays I see myself as an *application oriented harmonic analyst*, i.e. as a mathematician interested in mathematical structures which allow to provide the correct description of applied problems. More recently I have developed some general ideas concerning the way how Fourier Analysis could and should be thought, both to Engineering and Mathematics Students.

Personalities in Fourier Analysis

Joseph B. Fourier (1768 1830), franz. Mathematiker
Peter G. Lejeune Dirichlet (1805 to 1859), deutsch.
Bernhard Riemann (1826 to 1866), deutsch.
Henri L. Lebesgue (1875 to 1941), franz.
Jacques Hadamard (8.Dez.1865; to 1963), franz.
Alfred Haar (1885 to 1933), ungar.
Johann Radon (!16.Dez.1887 in Tetschen; to 1956)
Norbert Wiener (1894 to 1964), amer.
Antoni Zygmund (1900 to 1992), US-amer.
Israel M. Gelfand (1913 to 2009), russ.
Andre Weil (1906 to 1998), franz.
Alberto Calderon (1920 to 1998), arg.
Hans Reiter (Mathematiker) (1921 to 1992), öster.
Lennart Carleson (1928...), schwed. Mathematiker.

Abbreviated History of Fourier Analysis

1. 19-th century: Dirichlet (pointwise convergence), Riemann integral;
2. early 20-th century: *Lebesgue integral*, Hilbert spaces, Banach spaces, Plancherel theorem;
3. mid 20th: Haar integral, LCA groups, Gelfand-theory;
4. mid 20th: L. Schwartz theory of *tempered distributions*, applications to PDE (Hörmander), Sobolev spaces;
5. 20th: $\boldsymbol{L}^p$-spaces, BV-norm: Riesz representation theory; Wiener: $\boldsymbol{A}(\mathbb{U})$, Besov: $\boldsymbol{B}^s_{p,q}$: Functional Analysis;
6. Cooley-Tukey: 1965: the FFT algorithm;
7. late 20th century: time-frequency analysis, wavelet theory (late 80th).

Books on Fourier Analysis

At the time of my early studies (beginning 1970) a number of books had been of relevance the spirit of BOURBAKI and:

1. Hans Reiter's: Classical Harmonic Analysis and Locally Compact Groups (1968, Oxford, [10, 12]);
2. Walter Rudin: Fourier Analysis on Groups (1962,[13]);
3. E.Hewitt and K.A.Ross: Abstract Harmonic Analysis (1963, 1970; [6, 7]) [the forbidden book!];
4. A.Zygmund: Trigonometric series.Vols. I, II 1959, [16];
5. I.Gelfand, G.Shilov, D.Raikov: Commutative Normed Rings;
6. Nina Bari: Trigonometrische Reihen (1958);
7. Y.Katznelson: An Introduction to Harmonic Analysis (1968, [8]);
8. Andre Weil, L'integration dans les Groupes Topologiques et ses Applications ([15]).

My teachers at that time:

L.Schmetterer was teaching us:

1. Analysis course: Banach spaces, Lebesgue integral, $BV(\mathbb{R})$;
2. Fourier series course (1970): $\boldsymbol{L}^p(\mathbb{U})$, Fourier series, conjugate Fourier series;
3. basic principles of Functional Analysis (spring 1971);
4. Seminar: Fourier Analysis on Groups (W.Rudin books);

J.Cigler was teaching "distribution theory" around that time, and when **H.Reiter** arrived (Sept.1971) his book was already out of print, and there had been new lecture notes on *Segal Algbras* ([11]); he had a course and a seminar in that year on topics of *harmonic analysis on groups*.

Schmetterer's Classical Fourier Analysis

Schmetterer was providing us (after the Fourier Analysis class) with a quite modern summary of functional analysis, but his course on Fourier series was very *classical in style*.
Instead of the *complex exponential* function he was still using the representation of a function by it's Fourier series as a *series*, using the terms $\cos(kx)$ and $\sin(kx), k \geq 0$, using appropriate normalization.
What was *really puzzling* was the fact that he kept *emphasizing* that the function $f(x)$ is somehow only "formally represented" by its Fourier series (while we just had learned that a seris is the limit of its partial sums!).
There was no convolution (equally it is not found in Zygmund's monumental treatise), but a lot of explicit computations, and applications of Hoelder's inequality.

Convolution and the Fourier Transform

Hans Reiter kept emphasizing, that the main purpose of Fourier Analysis over LCA groups (the setting suggested by A.Weil, [15]) was to study the Banach algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, with convolution as *multiplication*

$$f * g(x) = \int_G g(x-y)f(y)dy, \quad f, g \in \boldsymbol{L}^1(G).$$

Thanks to the convolution theorem: $\mathcal{F}(f * g) = \mathcal{F}(f) \cdot \mathcal{F}(g)$, it appeared that the Fourier transform is the next important tool to study $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, which is defined via

$$\mathcal{F}(f)(s) = \hat{f}(s) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i s \bullet t} dt,$$

where $s \bullet t := \sum_{k=1}^d s_k t_k$ is the usual scalar product.

Convolution and the Fourier Transform II

More precisely, the main goal appeared to be the study of *closed ideals* in $\boldsymbol{L}^1(G)$, or equivalently, closed translation invariant subspaces, which where studied by functional analytic methods, using the duality $\boldsymbol{L}^1(G)' = \boldsymbol{L}^\infty(G)$.
It was somehow related to the question of *spectral synthesis*, which can be described in the following way:
Given a closed ideal $I \lhd \boldsymbol{L}^1(G)$ we define $\mathrm{cosp}(I)$ as

$$\{s \,|\, \hat{f}(s) = 0, \ \forall f \in I\}.$$

The question arising now is: If $\mathrm{cosp}(I) = \mathrm{cosp}(J)$, or in other words. Given $E = \mathrm{cosp}(I)$, does this imply

$$I := \{f \in \boldsymbol{L}^1(\mathbb{R}^d), \hat{f}(s) = 0, \quad \forall s \in E\}?$$

Famous Results at that time:

Some of the result which were communicated as important and "extremly deep" at that time, also appreciated somehow by the students, have been:

- the non-spectral synthesis of L.Schwartz (1948) for $\mathbb{R}^3$ and the generalization by P.Malliavian (1959) ([9, 14]);
- Lennart Carleson's result about the almost everywhere convergence of Fourier series in $\boldsymbol{L}^2(\mathbb{U})$ in 1966 ([1]);
- Charles Fefferman: The Fourier multiplier problem for the ball (on $\boldsymbol{L}^p(\mathbb{R}^d)$), published in Ann.Math. 1971 ([3]);
- Per Enflos ([2]) result, that not every Banach space has a basis (1973)

Segal Algebras

One of the aspects that was brought in by H. Reiter was the study of alternative *Banach convolution algebras* (BCA), always with the view-point that by their similarity to $\boldsymbol{L}^1(G)$ one could learn more about this Banach algebra. There have been mainly two types of BCAs which are similar in a different way:

1. So called **Segal algebras** $(S, \|\cdot\|_S)$, which are dense in are complete with another norm, which is more "sensible" than the ordinary $\boldsymbol{L}^1$-norm, or with

$$\|f\|_1 \leq C\|f\|_1, \ \forall f \in S.$$

   More precisely, they are *Banach ideals* in $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, satisfying

$$\|g * f\|_S \leq \|g\|_1 \|f\|_S, \quad \forall g \in \boldsymbol{L}^1(G), f \in S.$$

Beurling Algebras

On the other hand we have the so-called **Beurling algebras** which are weighted $\boldsymbol{L}^1(G)$-algebras, with respect to *submultiplicative weights* w, with $w(x) > 0$ and

$$w(x+y) \leq w(x)w(y), \quad x, y \in \mathbb{R}^d,$$

e.g $w_s(x) = (1+|x|)^s, s \geq 0$. Such a Beurling algebra is then defined (together with the natural norm) via:

$$\boldsymbol{L}^1_w(G) := \{f \mid fw \in \boldsymbol{L}^1(G)\} \text{ with } \|f\|_{1,w} := \|fw\|_1.$$

They are *not* ideals in $\boldsymbol{L}^1(\mathbb{R}^d)$, but they contain also bounded approximate identities (so-called bounded Dirac sequences).

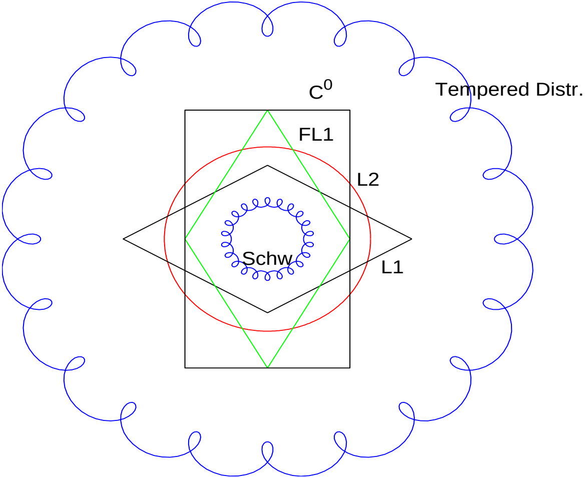
Open Problems at that time?

Although I was not really prepared for doing research I must say that I did not get the impression that there had been many open questions in Fourier analysis at that time, rather some "left over" question that had to be settled, but no "big goal" or "theory building to be developed".
Also *Jean Dieudonné*, when visiting our institute was explaining that "Abstract Harmonic Analysis" is "off-stream" mathematics, not quite encouraging for a young researcher (although I did not accept his statement, but I shared his view that some of the extremly finetuned questions were just "difficult" but not relevant, e.g. Rider-sets, Helson-sets, whatever you name it...).
There were a few problems about (Fourier) multipliers, but H.Reiter himself did not continue to work on Segal algebras!

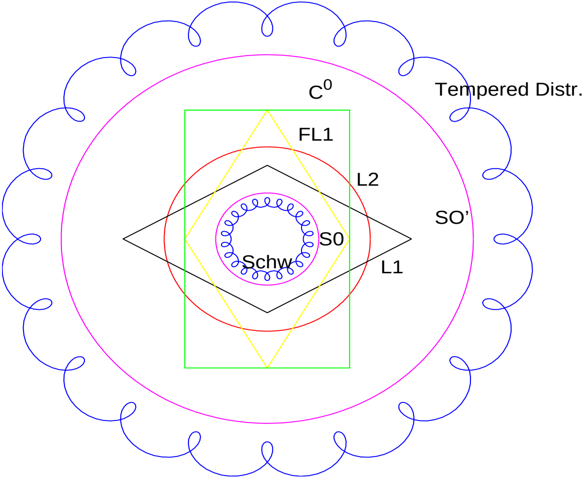
Banach spaces of Functions and Distributions

I found it natural to work out general principles for more general Banach spaces of functions (or their duals: Banach spaces of distributions), including weighted $\boldsymbol{L}^p$-spaces which are BCAs, or intersections of Segal algebras with Beurling algebras, Banach spaces obtained by decomposition methods (so-called *Wiener amalgam spaces* have been developed in the early 80th, just after my habilitation), they where in turn the basis for the development of so-called *modulation spaces*, which can be defined over LCA groups (in analogy to Besov spaces).

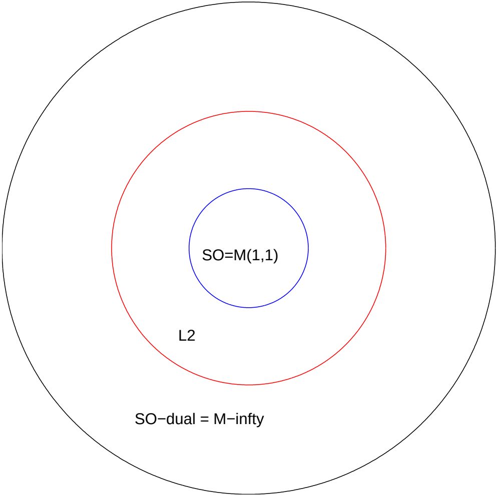
The Fourier Transform and Function Spaces



The new view on the Fourier Transform



The Banach Gelfand Triple (S_0, L^2, S_0')



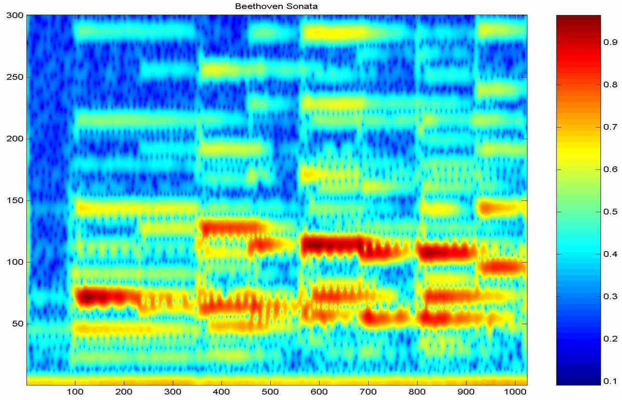
The New Setting developed via Time-Frequency Analysis

Aside from the various technical terms coming up I hope to convey implicitly a few other messages:

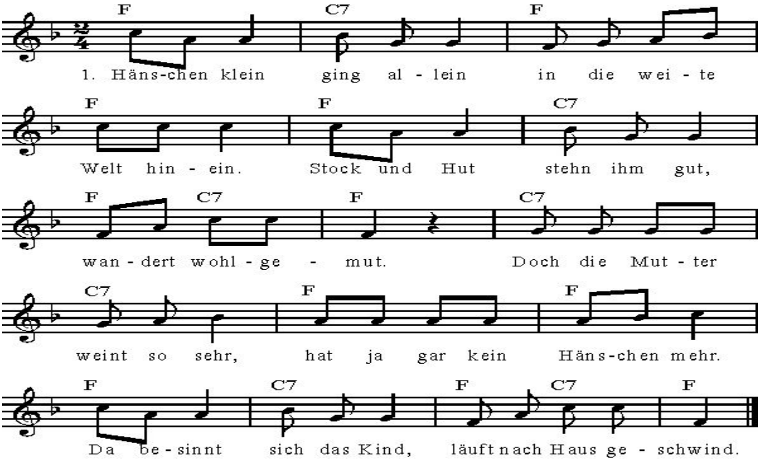
- staying with Banach spaces and their duals one can do amazing things (without touching the full theory of topological vector spaces, Lebesgue integration, or usual distribution theory);
- alongside with the norm topology just the very natural w^*-topology, just in the form of pointwise convergence of functionals, for the dual space has to be kept in mind (allowing thus among other to handle non-reflexive Banach spaces);

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) pictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



compared to musical score ...



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle, \;\; \lambda = (t,\omega);$$

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, *then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\widehat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\widehat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{U}) = \boldsymbol{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

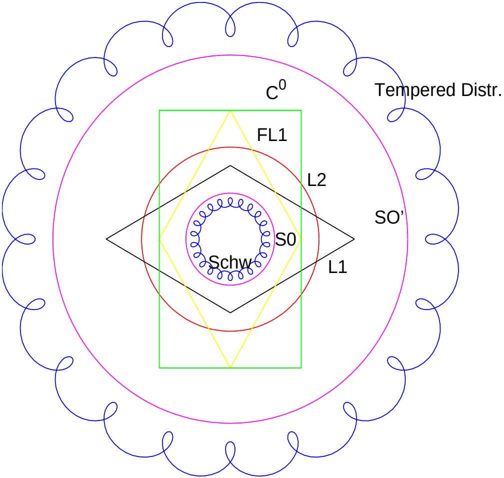
1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

A schematic description of the situation



The Kernel Theorem

Theorem

If K is a bounded operator from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathbf{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \mathbf{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

This result is the "outer shell" of the Gelfand triple isomorphism.

Kernel Theorem II

The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels. The complete picture can be expressed by a unitary Gelfand triple isomorphism.

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

Representations of translation invariant Systems

For applications so-called TILS (*translation invariant linear systems*) are of great importance. Engineers use very vague arguments to convince their students that any such system "is a convolution operator" by some *impulse response*, described (equivalently) as a Fourier multiplier (by the *transfer function*, the Fourier transform of the impulse response).

Theorem

Every bounded linear operator T from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$ which commutes with translations, i.e. with $T \circ T_z = T_z \circ T$, is a convolution operator by some $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$. In other words, one has

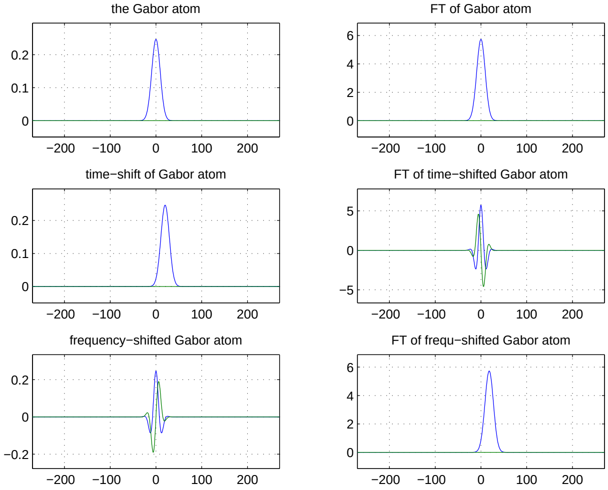
$$Tf(x) = \sigma(T_x f^{\checkmark}), x \in \mathbb{R}^d, f \in \boldsymbol{S}_0(\mathbb{R}^d),$$

where $f^{\checkmark}(x) = f(-x)$.
Moreover, there is norm equivalence between the two spaces, with the operator norm for T and the norm in $\boldsymbol{S}_0'(\mathbb{R}^d)$ for σ.

Dennis Gabor's suggestion of 1946

There is one very interesting example (the prototypical problem going back to D. Gabor, 1946): Consider the family of all time-frequency shifted copies of a standard Gauss function $g_0(t) = e^{-\pi|t|^2}$ (which is invariant under the Fourier transform), and shifted along $\mathbb{Z}$ ($T_n f(z) = f(z - n)$) and shifted also in time along $\mathbb{Z}$ (the modulation operator is given by $M_k h(z) = \chi_k(z) \cdot h(z)$, where $\chi_k(z) = e^{2\pi ikz}$).
Although D. Gabor gave some heuristic arguments suggesting to expand every signal from $\boldsymbol{L}^2(\mathbb{R})$ in a unique way into a (double) series of such "Gabor atoms", a deeper mathematical analysis shows that we have the following problems (the basic analysis has been undertaken e.g. by A.J.E.M. Janssen in the early 80s):

TF-shifted Gaussians: Gabor families



Problems with the original suggestion

Even if one allows to replace the time shifts from along $\mathbb{Z}$ by time-shifts along $a\mathbb{Z}$ and accordingly frequency shifts along $b\mathbb{Z}$ one faces the following problems:

1. for $a \cdot b = 1$ (in particular $a = 1 = b$) one finds a *total* subset, which is not a frame nor Riesz-basis for $\boldsymbol{L}^2(\mathbb{R})$, which is redundant in the sense: after removing one element it is still total in $\boldsymbol{L}^2(\mathbb{R})$, while it is not total anymore after removal of more than one such element;
2. for $a \cdot b > 1$ one does not have anymore totalness, but a Riesz basic sequence for its closed linear span ($\subsetneq \boldsymbol{L}^2(\mathbb{R})$);
3. for $a \cdot b < 1$ one finds that the corresponding Gabor family is a *Gabor frame*: it is a redundant family allowing to expand $f \in \boldsymbol{L}^2(\mathbb{R})$ using ℓ^2-coefficients (but one can remove infinitely many elements and still have this property!);

Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?);
- It is computed using the FFT (what is the connection);
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).

The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of upmost importance, because of its algebraic properties [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its computational efficiency
(FFT algorithms of signals of length N run in $Nlog(N)$ time, for $N = 2^k$, due to recursive arguments).
It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a unitary matrix (up to the factor $1/\sqrt{n}$) and maps pure frequencies onto unit vectors (engineers talk of *energy preservation).*

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\widehat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t}\, dt \tag{2}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \widehat{f}(\omega) \cdot e^{2\pi i t \cdot \omega}\, d\omega, \tag{3}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$. One often speaks of Fourier analysis followed by Fourier inversion as a method to build f from the pure frequencies (Fourier synthesis).

The classical situation with Fourier

Unfortunately the Fourier transform does not behave well with respect to $\boldsymbol{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ we have $\widehat{f} \in \boldsymbol{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\widehat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^d)$ is isometric in the $\boldsymbol{L}^2$-sense, but the Fourier integral cannot be written anymore;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\boldsymbol{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\widehat{f}$.

Basic properties of $S_0(\mathbb{R}^d)$ resp. $S_0(G)$

THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $S_o(G)$; $[with(\alpha^* f)(x) = f(\alpha(x))], x \in G$.
- $\mathcal{F}S_o(G) = S_o(\hat{G})$; (Invariance under the Fourier Transform)
- $T_H S_o(G) = S_o(G/H)$; (Integration along subgroups)
- $R_H S_o(G) = S_o(H)$; (Restriction to subgroups)
- $S_o(G_1)\hat{\otimes}S_o(G_2) = S_o(G_1 \times G_2)$. (tensor product stability);

Basic properties of dual space $S_0(\mathbb{R}^d)'$

THEOREM: (Consequences for the dual space)

- $S_o'(G)$ is a Banach space with a translation invariant norm;
- $S_o'(G) \subseteq \mathcal{S}'(G)$, i.e. $S_o'(G)$ consists of tempered distributions;
- $P(G) \subseteq S_o'(G) \subseteq Q(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq S_o'(G)$; (contains translation bounded measures);
- $\mathcal{M}_T(G) \subseteq S_o'(G)$ (contains "transformable measures" by Gil-de-Lamadrid).

Basic properties of $S_0(\mathbb{R}^d)'$ continued

THEOREM:

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in S_o(\hat{G}), \sigma \in S_o'(G)$, satisfies
$$\mathcal{F}(S_o'(G)) = S_o'(\hat{G}).$$
- $\sigma \in S_o'(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in S_o'(G/H)$ such that
$$\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle .$$
- $S_o'(H)$ can be identified with a subspace of $S_o'(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in S_o'(G)$ one has $\sigma \in i_H(S_o'(H))$ iff $\text{supp}(\sigma) \subseteq H$.

The Usefulness of $\mathbf{S}_0(\mathbb{R}^d)$

Theorem

Poisson's formula *For $f \in \mathbf{S}_0(\mathbb{R}^d)$ and any discrete subgroup H of $\mathbb{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{s \in H^\perp} \hat{f}(s). \qquad (4)$$

By duality one can express this situation as the fact that the Comb-distribution $\mu_{Z^d} = \sum_{k \in Z^d} \delta_k$, as an element of $\mathbf{S}_0(\mathbb{R}^d)'$ is invariant under the (generalized) Fourier transform. This in turn gives a correct mathematical argument for the fact that the sampling over Z, which corresponds to the mapping $f \mapsto f \cdot \mu_{Z^d} = \sum_{k \in Z^d} f(k)\delta_k$ corresponds to convolution with μ_{Z^d} on the Fourier transform side gives periodization along $(Z^d)^\perp = Z^d$ of the Fourier transform $\hat{f}$.

What has changed over the years?

`There is a well-known joke about mathematics: A guy has been frozen for 100 years, and when he comes back to life ...but he still recognizes the mathematics that he has learned at school.`

Obviously life has changed dramatically in many respects, especially in the last 25 years (1989 the internet just started, and I was getting my first computer, a 386-machine, which was the "strongest computer in the institute" until mid 1991!). But in which way does it effect mathematics? How can new electronic tools help us in our work as mathematicians?

I recall, that H. Reiter was complaining that I was wasting money because I tried to collect papers by copying them from the library, instead of having the few relevant volumes on my desk as he used to have it. Currently the NuHAG database has 17154 entries, with many PDF files directly accessible.

What are the Benefits of the new Opportunities?

- **Mathematical software** (in my case MATLAB, OCTAVE, GEOGEBRA) allow to present mathematical contents *visually*, bringing back the (often convincing and easy to memorize) geometric contents of linear algebra or signal processing;
- **Electronic databases** give access to a huge pile of literature (e.g. on wavelets), but *information and experience* are a combination of availability of data plus methods to extract valuable derived data from them! In fact, mathematics (!big data) is playing more and more role here;
- **Simulations** and experiments are done much easier, but have to be done correctly (e.g. NuHAG TBs);
- **Internet platforms** such as YOUTUBE or MOOC platforms like COURSERA allow to share information world-wide, to communicate best practice, etc..

What are the Things to be observed in the Future?

Here are a few observations and recommendations, especially to the younger colleagues:

- While an ever increasing number of papers is written, more and more mathematicians are active as researchers, one sometimes has the feeling the *orientation is lost*; bare feasibility in the technical sense may give a published paper, but will it contribute to the overall body of knowledge?
- We are used to continue research in well established directions. But sometimes the answers come from the outside world, from cross-over, by establishing connections between distant fields (e.g. the theory of polynomials has allowed to solve the *Kadison-Singer conjecture*, which implies the *Feichtinger Conjecture*, and overall looking into applications is often very fruitful (in the long run);

What are the Things to be observed in the Future?

- I like the view-point that all to often we are following a producers view-point (ref. to J.Buhmann, ETH). Pharmaceutical industry tells us how much better the new drug is compared to the old one. But the problem of sick people is to find out what their health problem is, and which doctor could help them! (cf. [4] or [5]);
- We have algorithms, function spaces, literature on operators and operator algebras with more and more parameters, but is there anything like a consumer report, helping us to find out, which one of these results are helpful for applications, which ones require just a lot of time to digest their content? Do we have ideas of preparing something like *consumer's reports*?
- The same problem with algorithms and existing code in whatever programming language (although the situation is more mature in computer science).

Further Resources from the NuHAG web-page

Our next conference takes place in June 2016 in Strobl:

www.nuhag.eu/strobl16

A number of talks describing the technical side of the idea of CONCEPTUAL HARMONIC ANALYSIS can be found from the NuHAG Talk-Server (search for "CONCEPTUAL" in the title).

`Feedback on this presentation as well as the NuHAG web-page etc. is very welcome.`

I hope to see NuHAG providing a group providing a good enviroment to its members, attracting good mathematicians from all over the world, and produce interesting new mathematics and results which help applied scientists in their work.

I will certainly contribute my share, of course a bit more from the background from now on.

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