

Ideas for Harmonic Analysis in the 21st Century

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Mathematics for Everyone



Private Vorbemerkungen

I was educated in *Classical and “Abstract” Harmonic Analysis*.

Topics of the early conferences:

- Fourier Algebra $\mathbf{A}(G)$
- nilpotent groups, special groups
- multiplier questions
- ideals in Banach algebras, Banach modules (!Cigler)
- amenability

BUT I do not remember any discussion about the “goals of harmonic analysis”.

As a visitor giving a survey talk on “analysis in general” Jean Dieudonne declared **abstract harmonic analysis “off-stream”**.



Private Vorbemerkungen

Some comments made by Hans Reiter gave guidance about the understanding of the “essential problems” ([citations]):

- We are studying **convolution**, hence the Banach algebra $(L^1(G), \|\cdot\|_1)$ in order to understand its ideal structure (because we are interested in spectral analysis);
- We are studying Segal algebras because they are similar
- The natural setting for Fourier Analysis are LCA groups! (A.Weil);
- the existence of null-sets makes existence of the devil evident (hence use approximation arguments);



Private Vorbemerkungen

If one looks at the classical theory of Fourier series, as I had the pleasure to learn it from the late Prof. Leopold Schmetterer (in my 3rd semester) there was no comment concerning convolution or the convolution theorem, and even in the “old testament” (A. Zygmund’s treatise) the index barely provides any hint to convolution.

It was also clear at that time that L^p -spaces (over the torus or the Euclidean space) are the right spaces to analyze operators. Ground-breaking results where the result on pointwise a.e. convergence of $L^2(\mathbb{T})$ -Fourier series.



Private Vorbemerkungen

Presentation of ideas, or formulation of theories can be provided also in quite different formats (think of courses in mathematics, physics or engineering):

- 1 heuristic;
- 2 strictly formal;
- 3 by examples;
- 4 conceptual; (e.g. integral)
- 5 computational recipes (integration methods);
- 6 by demonstrating applications;
- 7 standard research versus *change of paradigms* (Thomas Kuhn!)



Typology of Mathematicians

Research and publication of results is a *human activity*, hence there are different typologies:

- the BIRDS and the FROGS (Notices);
- the theory builders;
- the problem solvers;
- the bridge builders;
- the educators;
- the communicators;
- the perfectionists;
- the systematic types;
- the pioneers;
- the masters of ceremony (closing up);
- the output oriented ones;



Typology of Mathematicians

Research and publication of results is a *human activity*, hence there are also different orientations:

- applied mathematical work;
- application driven research;
- application oriented research
- application motivated research;
- pure/explorative research;

Of course the tools used and the level of precision reached will/may depend on the orientation of the researcher resp. the tradition in the community.



Historical Notes

Clearly engineers and applied people are more interested in “working code” and concrete algorithms with *some justification* while people at the theoretical end may be happy with an *existence and uniqueness result*.

Given the ability to do extensive computations within short time, based on an elaborate theory and using efficient code I want to proclaim *Postmodern Analysis* as the (hopefully/inevitably upcoming) period where all the knowledge is coming together and provides constructive and realizable methods.



Some Heuristic Claims in Fourier Analysis

From the engineering point of view once can describe a TILS (translation invariant linear systems resp. operators) are just convolution operators. This comes from the following *heuristic* consideration, using $T_x f(y) = f(y - x)$, noting that $T_x(\delta_0) = \delta_x$:

$$f = \int_G f(x) \delta_x dx = \int_G T_x(\delta_0) f(x) dx;$$

hence with $\mu := T(\delta_0)$, the **impulse response** we have:

$$Tf = \int_G T(T_x(\delta_0)) f(x) dx = \int_G T_x(\mu) f(x) dx = f * \mu,$$

so at the Fourier transform side we get the **transfer function**

$$\widehat{Tf} = \widehat{\mu} \cdot \widehat{f}.$$



The Scandal

The above reasoning is clearly too vague to be appreciated by mathematicians (and can be only “saved” using distribution theory), and indeed allows for what I. Sandberg the *scandal in system theory*: which consists in based on his (for abstract harmonic analyst not surprising) observation:

There are non-trivial linear operators from $C_b(\mathbb{R}^d)$ into itself, which commute with translations, but are not convolution operators! In fact, one can use on of those translation invariant means σ on $C_b(\mathbb{R}^d)$ (since \mathbb{R}^d is amenable) and map f to the constant times $\sigma(f)$, but clearly it is zero on $C_0(\mathbb{R}^d)$ (they have limit zero at infinity).

Of course this is partially due to the nonseparability of $C_b(\mathbb{R}^d)$ and one *simply has to take* $C_0(\mathbb{R}^d)$ instead to save the situation (see my course notes on the subject).



Some Important Tools in Fourier Analysis

This seems to be like a strong warning to work on save ground, and make sure that all the things that we do “exist in a proper sense” and to avoid problems. It is like saying: you should not make use of an electric device if it is not TÜV approved.

So what are the most important objects and tools that we need in Fourier Analysis resp. in (abstract or applied) Harmonic Analysis:
It is the Fourier transform, convolution, Poisson’s formula etc. ...
Just recall that we need to take *integrals*

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} dt \quad (1)$$

$$f * g(x) = \int_{\mathbb{R}^d} g(x - y) f(y) dy.$$

(2)



Some Important Tools in Fourier Analysis

Therefore it appears completely natural (nowadays at least) to use the *Lebesgue integral* as the perfect integral, resistant to limits (unlike the poor Riemann integral), and providing us with the *Banach space* $L^1(G)$ resp. even the *Hilbert space* $L^2(G)$, with its scalar product and the possibility of proving Plancherel's theorem, from the engineering point of view the all important “energy conservation law” for the Fourier transform.

So we became used to the Fourier transform as a an INTEGRAL transform (which is unitary, properly extended), *turning convolution into pointwise multiplication* (and vice versa).

But does that help when I discuss (e.g.) *chirp signals* of the form $h(x) = e^{ix^2}$ (linear chirps). They satisfy $\mathcal{F}h = h$ and thus obviously they are harmless (bounded!) *Fourier multipliers*.

But does $h * f(x)$ exists almost everywhere for $f \in L^2(\mathbb{R})$?
And also the inverse FT is not really an integral transform!?



Some Important Tools in Fourier Analysis

Even if there is no other immediate use for L^p -space for p different from $p = 1, 2, \infty$ one can argue, that L^p -spaces are very natural scale of spaces, nicely closed under *complex interpolation* (according to Riesz-Thorin).

A sceptical person may ask, what it help a user, to know that a given function is in $L^{3.4}$, but let us avoid this discussion here.

But what are the multipliers between L^p -spaces, or from L^p to L^q (for general $p, q \in [1, \infty]$)?

It turns out they are convolution operators with pseudo-measures resp. quasi-measures (objects which are locally like pseudo-measures, the functionals on the Fourier algebra $\mathbf{A}(G)$). So these limitations just shift the problem areas to other regions. There is also a (tricky) theory of transformable (possibly unbounded) measures by Gil de Lamadrid.



Some Important Tools in Fourier Analysis

Furthermore, let us not forget that we are going to do *FOURIER ANALYSIS* and *FOURIER SYNTHESIS* originally, i.e. we wanted to *filter out the frequencies inside a given signal* using Fourier analysis, and we would like to resynthesize the given signal from those frequencies which are found inside the signal, or it least inside a set of frequencies where we can be sure (for whatever reasons) that it contains all the frequencies in the signal. In fact, this brings us again back to the original meaning of the concept of *sets of spectral synthesis!*

And it is not so difficult when we do everything just over finite group, or more down to earth, if we work with finite signals, viewed as functions on \mathbf{Z}_N or $\mathbf{Z}_M \times \mathbf{Z}_N$ in the case of images.



Abstract versus Conceptual Harmonic Analysis

Branches of Harmonic Analysis

- AHA = Abstract Harmonic Analysis;
- NHA = Numerical (or computational) Harmonic Analysis;
- Applied Harmonic Analysis;
- CONCEPTUAL HARMONIC ANALYSIS
(integration of all this!)

I am going to give some verbal description of my view on Abstract Harmonic Analysis versus a term that I try to propagate more and more, namely “Conceptual Harmonic Analysis”



Books on the market: Harmonic Analysis

Juergen Jost: **Postmodern Analysis** . 2nd ed. [18]

Over the years the concept of Fourier Analysis and Harmonic Analysis have changed very much.

The **“old testament”** by Antoni Zygmund: Trigonometric series. 2nd ed. Vols. I, II Cambridge University Press, (1959, [25])

Edwin Hewitt and Kenneth A. Ross: **Abstract Harmonic Analysis**. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups [17]

Grafakos, Loukas: **Classical and Modern** Fourier Analysis [10] has later been split into two pieces

Hans Reiter [and Jan Stegeman]: **“Classical Harmonic Analysis and Locally Compact Groups”** (2nd ed., 2000, [21]).



Books on the market

More recent books are:

- H. J. Weaver: Applications of Discrete and Continuous Fourier Analysis (1983, [23]).
- R. Shakarchi and Elias M. Stein: Fourier Analysis: An Introduction Princeton University Press, Princeton Lectures in Analysis, (2003, [22]).
- Robert Marks: Handbook of Fourier Analysis and its Applications Oxford University Press, (2009 , [20]).
- M.W. Wong: Discrete Fourier analysis. Pseudo-Differential Operators. Theory and Applications 5. Basel: Birkhäuser (2011, [24]).

More recent books providing all necessary details from measure theory are (among many others);



Books on the market

- John J. Benedetto and Wojciech Czaja Integration and Modern Analysis Birkhäuser, (2009, p.576, [1]);
- and books on frames, bases, or TF-analysis e.g.
- C. Heil: A Basis Theory Primer. Expanded ed., ([16]);
- O. Christensen: An Introduction to Frames and Riesz Bases ([3]);
- K. Gröchenig “Foundations of TF-analysis” ([15]);



Fourier Analysis “in the books”

Let us take a look back into the historical development of
FOURIER ANALYSIS:

First J.B. Fourier’s claim that *every* periodic function can be *expanded* into a Fourier series was challenging the mathematical community to develop proper notions of **functions** and **integrals** (Riemann, Lebesgue) and different types of convergence. The need to properly describe measurability and exceptional sets has certainly greatly influenced the **theory of sets**.

The last century saw the development from **classical Fourier analysis** (over Euclidean spaces \mathbb{R}^d) to what has been sometime called **abstract Harmonic Analysis** (over LCA groups), but also the invention of the theory of **(tempered) distributions** as a tool to extend the domain of definition to include point masses and polynomials.



Fourier Analysis “in the books” II

If we take a look into published books on Fourier Analysis one finds a majority of books following the **historical path**:

- First starting with Fourier series and Hilbert spaces
- Then the Fourier transform, Riemann-Lebesgue theorem
- Fourier inversion, uniqueness
- Plancherel's theorem, convolution theorem
- L^p -spaces, Hausdorff-Young theorem
- Poisson's formula, Sampling, Shannon's theorem
- Fast Fourier Transforms (FFT)
- perhaps some wavelets or other recent stuff
- ideally indications about the extension to distributions



Fourier Analysis “in the books” III

A side aspect or rather an *enabling tool* in performing this task, is the availability of suitable function spaces (in fact Banach spaces of functions or distributions) aside from the classical L^P -spaces, such as

- Wiener amalgam spaces
- Modulation spaces
- Sobolev spaces
- Besov-Triebel-Lizorkin spaces
- and others



What are the Proposed Ingredients?

While most books still follow the historical path, starting from Lebesgue integration theory, going to L^p -spaces and finish by mentioning recent developments like FFT, wavelets or frames I think we should **reconsider the catalogue** of important concepts and how we want to introduce them.

It is just like linear algebra, where the rapid development of numerical linear algebra is having its impact (to some extent) on the content of modern linear algebra books, which try to accommodate the balance between concrete and often simple examples, to abstract viewpoints and then not-so-trivial applications.



The New Catalogue of Terms I

Overall I may suggest a more functional analytic approach to Fourier/Harmonic Analysis. By this I mean to *acknowledge* the fact of life that linear spaces of signals or operators are inherently *infinite dimensional*, so that the idea of working with a fixed basis is not a good idea, but also definitely requires to have infinite series, and thus *completeness* must play an important role. So in principle we need **Banach spaces and their duals**, in many cases it will be convenient to work in a **Hilbert space**. In the case of dual elements we also have to allow considerations of the w^* -convergence of functionals. OVERALL: nothing exotic so far!



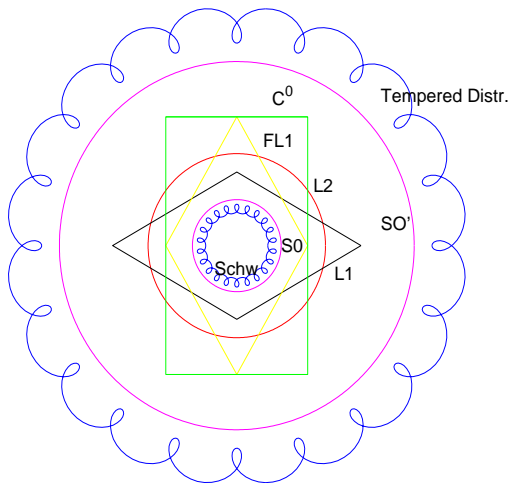
The New Catalogue of Terms II

Concepts and terms which are perhaps less familiar are, some of which already are interesting in a linear algebra setting:

- 1 *Banach algebras* and *Banach modules*;
- 2 approximate units (e.g. Dirac sequences for convolution);
- 3 the four spaces associated with a linear mapping (G. Strang);
- 4 the SVD (singular value decomposition);
- 5 **Banach frames** and Riesz (projection) bases as extensions of generating and lin. indep. sets;
- 6 **Banach Gelfand triples**, retracts in this category;
- 7 unitary Banach Gelfand triple isomorphism;



A first look at the collection of space



Fourier Analysis: Technical requirements I

In order to run this program properly one has to explain the necessary terms first. The Fourier transform appears a priori to be defined as an **integral transform**, namely via the equation

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} dt \quad (3)$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} d\omega. \quad (4)$$

We all know about the short-comings of the (nice and simple) Riemann integral and therefore we advocate the use of Lebesgue integration. BUT it cannot justify the inversion formula!



Fourier Analysis: Technical requirements II

There appear to be many further reasons for assuming that the theory of Lebesgue integration learned properly is a requirement for a deeper understanding of Fourier analysis:

- The Banach space $(L^1(\mathbb{R}^d), \|\cdot\|_1)$ is not only the “proper domain” of the (integral) FT, but also for convolution!



$$f * g(x) := \int_{\mathbb{R}^d} g(x-y)f(y)dy = \int_{\mathbb{R}^d} T_y g(x)f(y)dy \quad (5)$$

- the convolution theorem $\widehat{f * g} = \hat{f} \cdot \hat{g}$.
- checking associativity of convolution requires Fubini;



Fourier Analysis: Technical requirements III

Still there are cases where Lebesgue integration is *not be able to provide good answers*:

- The Fourier inversion problems requires to introduce summability kernels and take limits of functions (in suitable function spaces) which are obtained by the inverse Fourier integral;
- There are discrete measures having well defined Fourier (Stieltjes) transforms, but the corresponding theory of *transformable measures* is equally asymmetric! (developed by Argabright and Gil de Lamadrid)
- Pure frequencies (characters) are not in the domain of classical Fourier transforms (their transforms should be Dirac measures);
- For PDE applications the Schwartz theory of tempered distributions is the right thing anyway!



Fourier Analysis: Technical requirements IV

So what do we need in order to properly define **generalized functions** and in particular **tempered distributions**.

- To define them we *need suitable spaces of test functions*, typically infinitely differentiable functions, with compact support, are sufficiently rapid decay;
- in the case of the Schwartz space of rapidly decreasing functions $\mathcal{S}(\mathbb{R}^d)$ one takes only functions which together with all their (partial) derivatives decay faster than any polynomial;
- in order to properly define the dual space one has to at least indirectly put a convergence on such spaces of test functions in order to select the *continuous linear functionals*;
- the useful test functions spaces (such as $\mathcal{S}(\mathbb{R}^d)$) fulfill extra properties, such as Fourier invariance.



Fourier Analysis for Applied Scientists

If we look into this program and the potential role that Fourier Analysis (and more generally Harmonic Analysis) should play the teaching of Fourier Analysis we may ask ourselves:

- are we teaching all of these concepts to the majority of our master students (or teacher students)?
- is there a chance that applied scientist such as engineers of computer scientists can learn Fourier analysis in this way;
- are there ways of teaching the essence of Fourier analysis in a different way, without making wrong claims, such as:

THE DIRAC FUNCTION IS ZERO EVERYWHERE EXCEPT AT ZERO, BUT IT IS SO IMMENSELY HUGE, SOMEHOW LARGER THAN PLUS INFINITY, SUCH THAT ITS INTEGRAL SATISFIES $\int_{\mathbb{R}} \delta_0(t) dt = 1$.

So should we now forget about Lebesgue's integral??



Assessment of needs: bottom up approach

Let us try a bottom up approach, analyzing the emphasize put on those items that take an important role in typical engineering books (as opposed to the mathematical literature):

- The basic terms are filters, time-invariant systems, etc.
- Characterization of linear time-invariant systems as *convolution operators* by means of the impulse response function;
- Characterization of LTIs using the *transfer function*
- the convolution theorem connecting impulse response and transfer function



Assessment of needs: bottom up approach: II

Asking critically what is needed.

- is Lebesgue integration crucial, or even more important than the concept of “generalized functions”
- is it OK to leave engineers with the all-important “sifting property” of the Dirac δ -distribution/function?
- is there a way to teach distribution theory in a natural context? (i.e. without topological vector spaces)
- What is the role of the FFT? Just the computational little brother of the true Fourier transform, based on Lebesgue integration?



Comparing with the Number Systems

Preparing for the suggestion to make use of so-called *Banach Gelfand Triples* let us recall what the situation is when we **use different number system**. Once we have learned to see the advantages of the different types of numbers, taken from the fields

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

it is easy to work with them, following properly given rules.

- Inversion is only “simple” within \mathbb{Q} !
- Taking square roots (and other things) is only possible within the (positive part of) \mathbb{R} ;
- Once \mathbb{C} is well-defined as pairs of reals, with properly defined addition and multiplication, the field properties of \mathbb{C} follow without reflection at each step *how e.g. inversion is carried out within \mathbb{R} !*



Comparing with the Number Systems, II

So in teaching analysis one has to think about the choice of spending a lot of time on the establishment of the existence and basic properties of the real number system, or rather assume from the beginning that *there exists some totally ordered field, containing the rational numbers as a dense subfield*, and go deeper into analysis:

Better spend time on concepts of differentiability of functions, Taylor expansions of differentiable functions, or other properties which will the students enable to properly solve differential equations or develop realistic models for application areas. Of course we are *not advocating* here to leave the students without an awareness of what has been shown and what is taken as granted, what could be proved in detail, and what is actually carried out step by step.



General Achievements in the Last Century

Recall that we have come a long way from Fourier to Modern Harmonic Analysis:

- in the late 19th century concepts of pointwise convergence had its first high period (De-la-Vallee Poussin, Dirichlet);
- Set theory and integration theory have been founded, then topology and functional analysis have been developed;
- Banach algebras and in particular Gelfand's theory provide proper foundations of Fourier analysis over LCA groups;
- the existence of a Haar measure and Pontrjagin's theorem, as well as Plancherel's theorem have been established;
- L. Schwartz established his theory of tempered distributions, extended by Bruhat to LCA groups;
- Cooley-Tukey invent the FFT around 1965;



Assessment of needs: bottom up approach: III

Given the realistic situation that a typical, even mathematically oriented engineer will

- never be able to follow in all details, but
- nowadays students learn about linear algebra in a more applied spirit, using MATLABTM in their course, and can carry out numerical experiments;
- that a certain understanding of generalized functions or distributions is unavoidable, if the computation of divergent integrals resulting in Dirac's or other computational tricks (which do not even have a precise meaning to those who have digested more theory);



Time-Frequency Analysis and Applications

Most colleagues would accept the statement that Wavelet Analysis has brought some important aspects to real life, e.g. due to its ability of bringing us a “mathematical zoom” and a good way of compressing image information. And clearly Fourier Analysis (in the way it has been performed in the last century) was not so helpful in doing this, but has Fourier Analysis been given a fair chance?

Hasn't Fourier Analysis been carrying the flavour of being import (for theoretical applications) but cumbersome and difficult when it comes to applications. Aren't the troubles that one has with Fourier inversion evident, partially due to the fact that the building blocks (the pure frequency resp. complex exponentials) do *not belong to the Hilbert space* ($L^2(\mathbb{R}^d)$, $\|\cdot\|_2$), hence Fourier Analysis and Fourier Synthesis have only a “formal meaning”?



Applications of Fourier Analysis in Modern Life

Although it is not the usual way to justify the teaching of mathematical topics it is not unreasonable to reconsider those areas where Fourier Analysis and in particular the use of the FFT has a significant impact on our daily life!

Let us just mention a view cases:

- digital signal and image processing
- mobile communication
- medical imaging (e.g. tomography)
- why are WAV files stored using the sampling rate 44100/sec?
- how is MP3 able to compress musical information?

But what is the relationship to the Fourier Transform we are teaching in our analysis classes?



Case Study I: Characterization of TILS

It is one of the first mathematical claims taught to electrical engineers students that every translation invariant linear system T can be viewed as a convolution operator. The standard rule (often with some hand-waving) is the following:

If one treats the case of discrete signals, say over the group \mathbb{Z} of integers, then the *impulse response* is just the output arising if the unit-vector \mathbf{e}_0 is used as input vector. Clearly unit vectors at $n \in \mathbb{Z}$ can be obtained from \mathbf{e}_0 by n units (left or right) thus allowing to obtain a mathematically correct characterization of T

But what kind of problems are put under the rug if we have continuous variables, say we are interested in TILs over the real line \mathbb{R} ? Should/can we simply replace \mathbf{e}_0 by δ_0 ?



Case Study I: Characterization of TILS: II

The usual way , Dirac sequences of box functions... vague and formal arguments..



Case Study I: Characterization of TILS: III

Not too long ago an obviously mathematically trained engineer, Irwin Sandberg, was observing that there is a “scandal”. Between 1998 and 2004 he published a series of papers, e.g. with the title:

Continuous multidimensional systems and the impulse response scandal.

So far the impact of his observations was quite modest, although he demonstrates among others that there are bounded linear mappings from the space $\mathbf{C}_b(\mathbb{R}^d)$ (endowed with the sup-norm) into itself, which are translation invariant but *cannot be represented* in the usual/expected way!

It seems that the community prefers to ignore a pedantic member rather than doing ground work on the basics!



Case Study I: Characterization of TILS: IV

But it is easy to correct the problem by replacing $\mathbf{C}_b(\mathbb{R}^d)$ by $\mathbf{C}_0(\mathbb{R}^d)$, the space of continuous, complex-valued functions vanishing at infinity (mostly because this space is separable!). Recall that by definition (justified by the Riesz-Representation Theorem) the *dual space* to $(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ can be called the space of bounded (regular) Borel measures, denoted by $\mathbf{M}(\mathbb{R}^d)$.

Theorem

There is an isometric isomorphism between the Banach space of bounded linear mappings from $(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ into itself which commutes with all translation operators $T_x, x \in \mathbb{R}^d$, and the space $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$ (with the norm as dual space).

Given $\mu \in \mathbf{M}(\mathbb{R}^d)$ one defines the convolution operator

$$C_\mu f : f \mapsto \mu * f(x) = \mu(T_x \check{f}), \quad \check{f}(x) = f(-x).$$

Case Study I: Characterization of TILS: V

This results requires a couple of auxiliary observations, such as the (norm) density of compactly supported measures within $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$ in order to ensure that C_μ is not only *uniformly continuous* but in fact even in $\mathbf{C}_0(\mathbb{R}^d)$, for every input signal $f \in \mathbf{C}_0(\mathbb{R}^d)$.

On the other hand all one needs is the existence of sufficiently many functions in $\mathbf{C}_0(\mathbb{R}^d)$, i.e. the density of functions with compact support within $(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$. This fact is easily established using the local compactness of the additive group \mathbb{R}^d , and thus the proof of the theorem is valid in the **context of general locally compact Abelian groups G !**

Note that it also does not depend on the Haar measure on G .



Case Study I: Characterization of TILS: VI

There is a number of consequences from this observation, starting from the observation that the simple translation operators T_x describe of course translation invariant operators on $\mathbf{C}_0(G)$, and correspond exactly to the bounded measure $\delta_x : f \mapsto f(x)$.

Using uniform partitions of unity one can show that finite discrete measures are dense (in the w^* -topology) in $\mathbf{C}'_0(\mathbb{R}^d) = \mathbf{M}(\mathbb{R}^d)$, or equivalently any TILS can be approximated in the strong operator topology by finite linear combinations of translation operators: For $\varepsilon > 0$ and any finite subset $F \subset \mathbf{C}_0$ one has for a suitable sequence (a_n) in \mathbb{C} and a finite sequence (x_n) in \mathbb{R}^d such that:

$$\|T(f) - \sum_{k=1}^l a_n T_{x_n} f\|_\infty < \varepsilon \quad \forall f \in F.$$



Case Study I: Characterization of TILS: VII

The most important consequence of the identification of TILS and bounded measures is the fact that one can transfer the Banach algebra properties of the family of all TILS to the bounded measures. *By definition* we define composition of bounded measures (to be called **convolution!**) via the **composition of the corresponding operators**. Obviously convolution is thus an associative and bilinear operation on MbG .

Since the translation operators on a LCA group commute with each other and since they can be used to approximate general TILS one finds out that this convolution is also commutative.

Finally one shows that any $\mu \in \mathbf{C}'_0(G)$ extends to all of $\mathbf{C}_b(G)$ in a unique way, hence $\widehat{\mu} = \mu(\chi_s)$ (with $\chi \in \widehat{G}$) is well defined and one shows the convolution theorem: $\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2}$. Again all this can be done without any integration theory!



Case Study I: Characterization of TILS: VIIa

Here let us introduce some terminology about Banach modules, which may be less familiar to the “general public”:

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ is a *Banach module* over a Banach algebra $(\mathbf{A}, \cdot, \|\cdot\|_{\mathbf{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\mathbf{A} \times \mathbf{B}$ into \mathbf{B} with

$$\|a \bullet b\|_{\mathbf{B}} \leq \|a\|_{\mathbf{A}} \|b\|_{\mathbf{B}} \quad \forall a \in \mathbf{A}, b \in \mathbf{B}$$

which behaves like an ordinary multiplication, i.e. is associative, distributive, etc.:

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \mathbf{A}, b \in \mathbf{B}.$$

Case Study I: Characterization of TILS: VIIb

Following [19] (Katznelson) we define:

Definition

A Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ of locally integrable functions is called a *homogeneous Banach space* on \mathbb{R}^d if it satisfies

- ① $\|T_x f\|_{\mathbf{B}} = \|f\|_{\mathbf{B}} \quad \forall f \in \mathbf{B}, x \in \mathbb{R}^d;$
- ② $\|T_x f - f\|_{\mathbf{B}} \rightarrow 0$ for $x \rightarrow 0, \quad \forall f \in \mathbf{B}.$

Important examples are of course the spaces $\mathbf{B} = L^p(\mathbb{R}^d)$ for $1 \leq p < \infty$ (which obviously are not sitting inside of $L^1(\mathbb{R}^d)$). It can be shown in an elementary way (just using a kind of vector-valued Riemann-integrals) that any such space is a Banach convolution module over $\mathbf{M}(\mathbb{R}^d)$:

$$\|\mu * f\|_{\mathbf{B}} \leq \|\mu\|_{\mathbf{M}} \|f\|_{\mathbf{B}} \quad \forall \mu \in \mathbf{M}(\mathbb{R}^d), f \in \mathbf{B}.$$



Case Study I: Characterization of TILS: VIII

In order not to play down the role of the *Haar measures* on any LCA group let us remind that it is a why which enables the **identification of ordinary functions with bounded measures!** In fact, one can define for $k \in \mathbf{C}_c(\mathbb{R}^d)$ (continuous with compact support) the measure

$$\mu_k : h \mapsto \int_{\mathbb{R}^d} h(x)k(x)dx, \quad h \in \mathbf{C}_0(\mathbb{R}^d),$$

With some nice tricks one then has to show that

$$\|\mu_k\|_{\mathbf{M}} = \int_{\mathbb{R}^d} |k(x)|dx,$$

i.e. the Haar measure (a linear functional) applied to $|k|$!, and from there one can define $\mathbf{L}^1(\mathbb{R}^d)$ as the closure of all those measures in $(\mathbf{M}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}})$ and and closed ideal in $\mathbf{M}(\mathbb{R}^d)$.



Summary so far

Summarizing so far we can claim that it is possible to introduce - even in the full generality of locally compact **Abelian** groups and without the use of the Haar measure basic facts about

- the commutative Banach **convolution algebra** $M(G)$;
- the **Fourier (Stieltjes) transform** satisfying the convolution theorem;
- the extension of the group action on homogeneous Banach spaces to $M(G)$, which is thus taking the role of the *group algebra* over finite Abelian groups.
- the technical tool to achieve this are fine partitions of unity!

This also shows that one should not let oneself be *distracted* from certain technical questions which arise from measure theoretical considerations!



Recalling some Basic Questions

Most often the “naturalness” of the assumption of Lebesgue integrability, i.e. the restriction to $L^1(\mathbb{R}^d)$ is coming from considerations concerning the **existence** (!!??!!) of the Fourier transform or some convolution. But *WHAT DO WE MEAN*, when making one of the following claims?

- The function has a Fourier transform?
- The convolution $f * g$ of two function f, g exists?
- The convolution operator $f \mapsto f * g$ is well defined on a given Banach space of functions $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$;
- what is the validity range for the convolution theorem

$$\widehat{\sigma_1 * \sigma_2} = \widehat{\sigma_1} * \widehat{\sigma_2}$$



Recalling some Basic Questions, II

Let us consider the first question. We have many options

- The Fourier transform is well defined pointwise as a Lebesgue integral. *This is of course just a slightly reformulation of the assumption $f \in L^1(\mathbb{R}^d)$!* (and thus not very interesting);
- But isn't then the Fourier transform of the SINC function: $SINC(t) = \sin(\pi t)/\pi t, t \in \mathbb{R} \setminus \{0\}$ not well defined, *although one could take the Fourier integral in the improper Riemannian sense. Should we prohibit this option, but how far can we go in making exceptions?*
- certainly $SINC$ has an (inverse) Fourier transform (the box-function) in the L^2 -sense.
- in such situations: should we allow a combination of summability methods, but require pointwise convergence?



Recalling some Basic Questions, III

What about the well-definedness of linear operators for bilinear operations such as convolution? When does a function (or distribution) deserve to be called a **convolution kernel** for a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ (such as $L^2(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d)$)?

Again there are various options:

- a) the *strict one*: For every $h \in \mathbf{B}$ the Lebesgue integral $\int_{\mathbb{R}^d} h(x-y)f(y)dy$ exists almost everywhere and defines a new element to be called $f * h \in \mathbf{B}$;
- b) The integral is well-defined for test functions $h \in \mathbf{B}$, but

$$\|f * h\|_{\mathbf{B}} \leq C_f \|h\|_{\mathbf{B}}$$

for all such h , assuming that they are dense in $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$;



Recalling some Basic Questions, IIIb

In many situations there will be now difference, e.g. if both $f, g \in L^2(\mathbb{R}^d)$ then the Cauchy-Schwarz inequality guarantees the existence of convolution integrals and

$$|f * g(x)| \leq \|f\|_2 \|g\|_2 \quad \forall f, g \in L^2$$

But what about functions $g \notin L^2(\mathbb{R}^d)$ which happen to have a bounded Fourier transform, and hence define a bounded Fourier multiplier thanks to Plancherel's Fourier transform. Chirp functions of the form $t \mapsto e^{i\alpha t^2}$ are typical examples, because their FT is another chirp.

In other words, the pointwise concept is imposing restrictions which are not suitable from a functional analytic viewpoint. This was also the cause of troubles in work on tempered L^p -functions by K. McKennon in the 70th ([11]).



The Proper Setting from a TF-viewpoint

Surprisingly many classical questions can be given a more elegant interpretation using function spaces which would be mostly considered as suitable and relevant for TF applications (only), such as the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ (resp. $\mathbf{S}_0(G)$) and its dual. Originally ([?]) it got its symbol because it is the smallest (therefore the index zero!) in a class of so-called Segal algebras as studied by Hans Reiter ([21]).

It is defined as follows: A bounded, continuous and integrable function f belongs $\mathbf{S}_0(\mathbb{R}^d)$ if (and only if) it has a (Riemann-) integrable Short-time Fourier transform $V_{g_0} f$, defined as

$$\|f\|_{\mathbf{S}_0} := \|V_{g_0} f\|_{L^1(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}$$

The assumption is so strong that the same space arises even if one just assumes a priori that f is a tempered distribution.



The Proper Setting from a TF-viewpoint

It is not difficult to show that the space is the same if g_0 is replaced by any other non-zero function from the Schwartz space, or even from $\mathbf{S}_0(\mathbb{R}^d)$ itself. In such a way the space can be defined for general LCA groups (using e.g. smooth and compactly supported functions instead of g_0 , or a some $g \in L^1(G)$ with compactly supported Fourier transform, i.e. a band-limited L^1 -function).

The advantage of the choice $g = g_0$ is the Fourier invariance of g_0 , which in turn implies that $\mathbf{S}_0(\mathbb{R}^d)$ is isometrically invariant under the Fourier transform!

Depending on the viewpoint, one can say that it is one of the few Fourier invariant Banach spaces (aside from $L^2(\mathbb{R}^d)$), sitting inside of $L^2(\mathbb{R}^d)$, or otherwise, that the Fourier transform is not only isometric on $\mathbf{S}_0(\mathbb{R}^d)$, but also isometric in the L^2 -norm and therefore extends to a unitary transformation for $L^2(\mathbb{R}^d)$, viewed as the completion of $\mathbf{S}_0(\mathbb{R}^d)$ with respect to $\|\cdot\|_2$.



Generalized Stochastic Processes

Definition

① Let \mathcal{H} be an arbitrary Hilbert space. A bounded linear mapping $\rho : S_0(G) \rightarrow \mathcal{H}$ is called a **generalized stochastic process** (GSP).

② A GSP ρ is called **(wide sense time-) stationary**, if

$$(\rho(f)|\rho(g)) = (\rho(T_x f)|\rho(T_x g)) \quad \forall x \in G \text{ and } \forall f, g \in S_0(G).$$

③ A GSP ρ is called **(wide sense) frequency stationary**, if

$$(\rho(f)|\rho(g)) = (\rho(M_t f)|\rho(M_t g)) \quad \forall t \in \hat{G} \text{ and } \forall f, g \in S_0(G).$$

④ A time- and frequency-stationary GSP is called **white noise**.

Generalized Stochastic Processes II

Definition

- ① A GSP ρ is called **bounded**, if $\exists c > 0$ such that

$$\|\rho(f)\|_{\mathcal{H}} \leq c\|f\|_{\infty} \quad \forall f \in S_0(G).$$

- ② A GSP ρ is called **variation-bounded (V-bounded)**, if $\exists c > 0$ such that

$$\|\rho(f)\|_{\mathcal{H}} \leq c\|\hat{f}\|_{\infty} \quad \forall f \in S_0(G).$$

- ③ A GSP ρ is called **orthogonally scattered** if

$$\text{supp}(f) \cap \text{supp}(g) = \emptyset \implies \rho(f) \perp \rho(g) \quad \text{for } f, g \in S_0(G).$$

Generalized Stochastic Processes III

Due to the **tensor product property** of S_0 :

$$S_0(G) \hat{\otimes} S_0(G) = S_0(G \times G)$$

(cf. [?] Theorem 7 D) it is justified to hope that the following definition determines an element of $S_0(G \times G)$.

Definition 6: Let ρ be a GSP. The **autocovariance** (or auto-correlation) distribution σ_ρ is defined as:
 $\langle \sigma_\rho, f \otimes g \rangle := (\rho(f) | \rho(\bar{g})) \quad \forall f, g \in S_0(G)$.



Generalized Stochastic Processes IV

Theorem 1: For a GSP ρ the following properties are equivalent:

a) ρ stationary $\iff \sigma_\rho$ diagonally invariant, i.e.

$$L_{(x,x)}\sigma_\rho = \sigma_\rho \quad \forall x \in G;$$

b) ρ bounded $\iff \sigma_\rho$

extends in a unique way to a bimeasure on $G \times G$;

c) ρ orthogonally scattered

$\iff \sigma_\rho$ is supported by the diagonal, i.e.

$\text{supp}(\sigma_\rho) \subseteq \Delta_G := \{(x, x) \mid x \in G\}$;

$\iff \exists$ positive and translation bounded measure τ_ρ with:

$$\langle \sigma_\rho, f \otimes g \rangle = \langle \tau_\rho, fg \rangle \quad \forall f, g \in S_0(G).$$



Further applications areas

There is a large variety of topics, where the same setting appears to be most appropriate, mostly because it is technically easier than the use of **Schwartz-Bruhat** functions, and on the other hand more general ($\mathcal{S}(G) \subset \mathbf{S}_0(G)$), in particular work on non-commutative geometry by Connes and Rieffel (via the work of Franz Luef);



Realizable constructive approaches vs. constructive

We will illustrate this by the following:

Let us look at the spline-type spaces (e.g. cubic splines on \mathbb{R}) in L^p -spaces, for $p \neq 2$. Then these spaces are closed subspaces of $(L^p(\mathbb{R}), \|\cdot\|_p)$, which is uniformly convex, hence there is a best approximation to any given $f \in L^p(\mathbb{R})$ by an element from the corresponding spline-type space. But this is not a constructive approach.

In contrast we have quite explicit methods (so-called constructive approximation) to obtain the (orthogonal) approximation in the case $p = 2$, and the corresponding operator can be shown to be also (uniformly) bounded on the family of L^p -spaces, for the full range of $1 \leq p \leq \infty$.

See [7] and an upcoming Springer BRIEF for discussions of the problems arising with this, for the multi-window case.

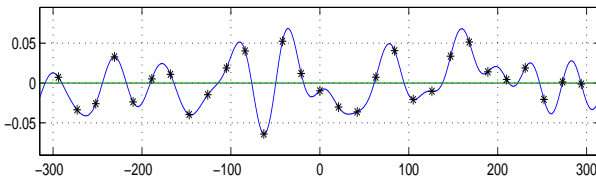


Main aspect: Factorize through finite dimensions

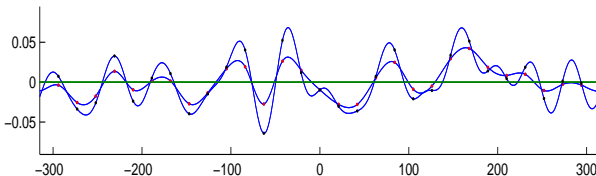
The general idea - at least in the context of Gabor analysis - of this refined approach (merging concepts of functional analysis and numerical analysis) is to approximate a continuous problem by the *corresponding problem* in the finite context, i.e. on a suitably chosen finite Abelian group, with the need to describe also the process of returning from discrete/finite data to the continuous domain. Usually this is done by quasi-interpolation. The most important special case of quasi-interpolation is piecewise linear interpolation, (quasi-interpolation using linear splines), while the operator $Q(d) = \sum d_\lambda T_\lambda \varphi$ is not interpolating anymore (e.g. for $d = (f(\lambda))$) for the case of cubic B-splines.



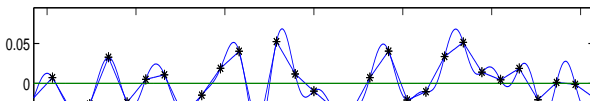
MATLAB illustration of the situation



red: sampling values of cubic quasi-interpolator



piecewise linear interpolation



Connections to related situations

Relevant results (mostly due to Norbert Kaiblinger and hgfei)

- quasi-interpolation in the Fourier algebra (\mathbf{S}_0 -setting);
- in particular: realizing the (continuous) FT using FFT methods
- robustness results (varying the lattice constants, etc.), required to find a suitable finite model (!periodicity assumptions);



Connections to related situations

- Wavelet multipliers (shearlet multipliers, Toeplitz operators etc.);
- Quality of operators depending only on the rough properties of the symbol, at least for “nice” windows;
- Connection between discrete variants of these operators and corresponding continuous frame multipliers;



END of adjusted material



Goals of this lecture

- show that the usual generalizations of linear algebra concepts to the Hilbert space case (namely **linear independence and totality**) are inappropriate in many cases;
- that **frames** and **Riesz bases** (for subspaces) are the right generalization to Hilbert spaces;
- that Hilbert spaces are themselves a too narrow concept and should be replaced **Banach Gelfand Triples**, ideally isomorphic to the canonical ones $(\ell^1, \ell^2, \ell^\infty)$;
- Describing the situation of frames or Riesz bases via commutative diagrams allows to extend this notation to BGTs;
- Demonstrate by examples (Fourier transform, kernel theorem) that this viewpoint brings us very close to the finite-dimensional setting!



OVERVIEW over this lecture 60- MINUTES

- This is a talk about the frames viewing frames as **RETRACTS**, i.e. a construction which can be done in any *category*;
- about the **ubiquity of Banach Gelfand Triples** ;
- provides a setting **very similar to the finite dimensional setting**;
- showing how **easy they are to use**;
- showing some applications in **Fourier Analysis**;
- indicating its relevance for numerical applications;
- and for teaching purposes;
- that it is a good vehicle to transfer algebraic facts (over finite Abelian group to the setting of LCA groups);
- perhaps **change your view on Fourier Analysis**.



General Considerations

One of the specifically interesting aspects of Gabor analysis is the fact, that it can be realized over **general LCA groups**.

Typically we discuss mathematical questions on Gabor Analysis in the context of Euclidean spaces, i.e. for \mathbb{R} or \mathbb{R}^d , while the implementations take place for finite, discrete signals, modeled as function on the cyclic groups, e.g. over \mathbb{Z}_N (cyclic group of order N) or in the case of image processing over $\mathbb{Z}_M \times \mathbb{Z}_N$.

In this setting one can much better see what the linear algebra background is (kernels, linear dependence) and the algebraic properties, while in the general case (much) more functional analysis comes in (a variety of norms, types of convergence, etc.).



ANALYSIS: Calculating with all kind of numbers

We teach in our courses that there is a huge variety of *NUMBERS*, but for our daily life rationals, reals and complex numbers suffice. The most beautiful equation

$$e^{2\pi i} = 1.$$

It uses the exponential function, with a (purely) imaginary exponent to get a nice result, more appealing than (the equivalent)

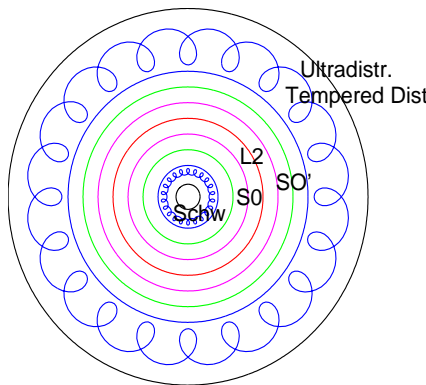
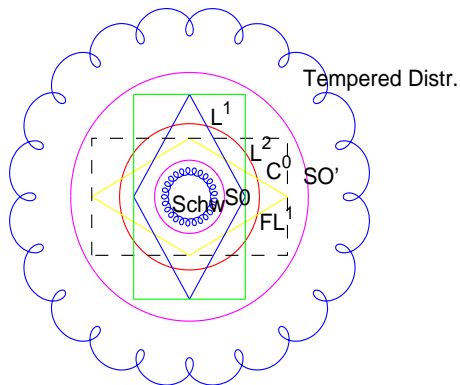
$$\cos(2\pi) + i * \sin(2\pi) = 1 \quad \text{in } \mathbb{C}.$$

But actual computation are done for rational numbers only!! Recall

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$



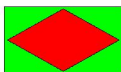
ANALYSIS: Spaces used to describe the Fourier Transform



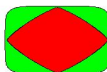
Hausdorff Young Theorem for the Fourier Transform

$$\mathcal{FL}^p(\mathbb{R}^d) \subseteq L^q(\mathbb{R}^d), \quad 1 \leq p \leq 2, \quad \frac{1}{q} + \frac{1}{p} = 1.$$

$p=1$



$p=1.2$



$p=1.4$



$p=1.6$



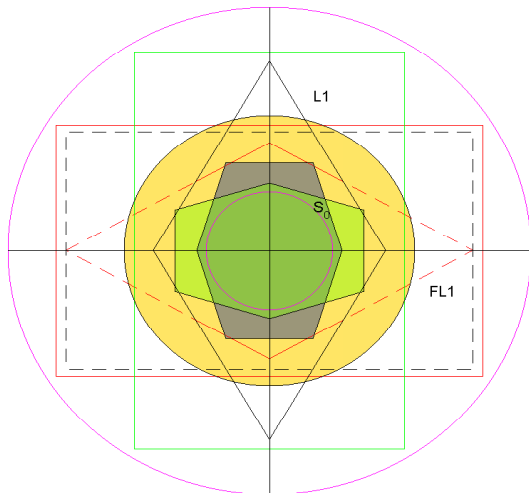
$p=1.8$



$p=2$



Wiener Amalgam spaces, Wiener Algebra, etc.



Frames in Hilbert Spaces: Classical Approach

Definition

A family $(f_i)_{i \in I}$ in a Hilbert space \mathcal{H} is called a *frame* if there exist constants $A, B > 0$ such that for all $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2 \quad (7)$$

It is well known that condition (7) is satisfied if and only if the so-called **frame operator** is invertible, which is given by

Definition

$$S(f) := \sum_{i \in I} \langle f, f_i \rangle f_i, \quad \text{for } f \in \mathcal{H},$$

Frames in Hilbert Spaces: Classical Approach II

The obvious fact $S \circ S^{-1} = Id = S^{-1} \circ S$ implies that the (canonical) *dual frame* $(\tilde{f}_i)_{i \in I}$, defined by $\tilde{f}_i := S^{-1}(f_i)$ has the property that one has for $f \in \mathcal{H}$:

Definition

$$f = \sum_{i \in I} \langle f, \tilde{f}_i \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \tilde{f}_i \quad (8)$$

Moreover, applying S^{-1} to this equation one finds that the family $(\tilde{f}_i)_{i \in I}$ is in fact a frame, whose frame operator is just S^{-1} , and consequently the “second dual frame” is just the original one.



Frames in Hilbert Spaces: Approach III

Since S is *positive definite* in this case we can also get to a more symmetric expression by defining $h_i = S^{-1/2}f_i$. In this case one has

$$f = \sum_{i \in I} \langle f, h_i \rangle h_i \quad \text{for all } f \in \mathcal{H}. \quad (9)$$

The family $(h_i)_{i \in I}$ defined in this way is called the *canonical tight frame* associated to the given family $(g_i)_{i \in I}$. It is in some sense the closest tight frame to the given family $(f_i)_{i \in I}$.



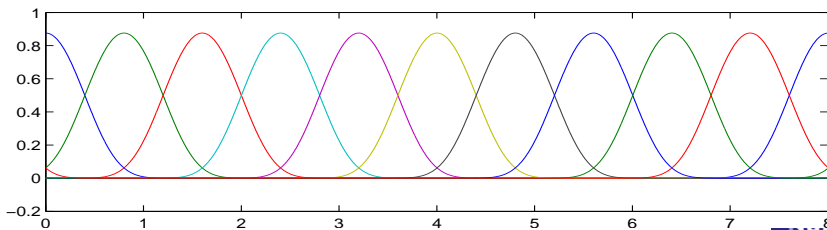
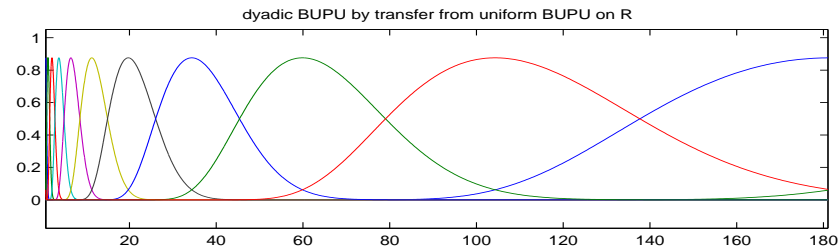
Where did frames come up? Historical views:

I think there is a historical reason for **frames** to pop up in the setting of **separable Hilbert spaces** \mathcal{H} . The first and fundamental paper was by Duffin and Schaeffer ([5]) which gained popularity in the “painless” paper by Daubechies, Grossmann and Y. Meyer ([4]). It gives explicit constructions of tight **Wavelet** as well as **Gabor frames**. For the wavelet case such dual pairs are also known due to the work of Frazier-Jawerth, see [8, 9]. Such characterizations (e.g. via **atomic decompositions**, with control of the coefficients) can in fact seen as prerunners of the concept of **Banach frames** to be discussed below.

These methods are closely related to the Fourier description of function spaces (going back to H. Triebel and J. Peetre) using *dyadic partitions of unity* on the Fourier transform side.



Dyadic Partitions of Unity and Besov spaces



Where did frames come up? Historical views II

The construction of **orthonormal wavelets** (in particular the first constructions by Y. Meyer and Lemarie, and subsequently the famous papers by Ingrid Daubechies), with prescribed degree of smoothness and even compact support makes a big difference to the Gabor case.

In fact, the Balian-Low theorem prohibits the existence of (Riesz- or) orthogonal Gabor bases with well TF-localized atoms, hence one has to be content with **Gabor frames** (for signal expansions) or **Gabor Riesz basic sequences** (for mobile communication such as OFDM).

This also brings up a connection to filter banks, which in the case of Gabor frames has been studied extensively by H. Bölcskei and coauthors (see [2]).



LINEAR ALGEBRA: Gilbert Strang's FOUR SPACES

Let us recall the *standard linear algebra situation*. Given some $m \times n$ -matrix \mathbf{A} we view it as a collection of *column* resp. as a collection of *row vectors*. We have:

$$\mathbf{row-rank}(\mathbf{A}) = \mathbf{column-rank}(\mathbf{A})$$

Each *homogeneous linear system of equations* can be expressed in the form of scalar products¹ we find that

$$\mathbf{Null}(\mathbf{A}) = \mathbf{Rowspace}(\mathbf{A})^\perp$$

and of course (by reasons of symmetry) for $\mathbf{A}' := \mathit{conj}(A^t)$:

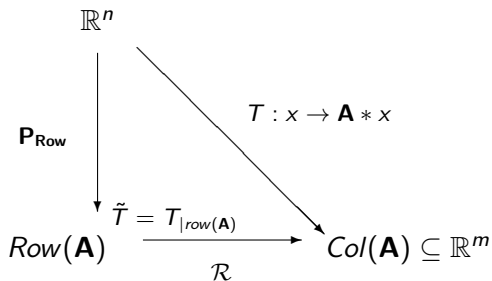
$$\mathbf{Null}(\mathbf{A}') = \mathbf{Colspace}(\mathbf{A})^\perp$$

¹Think of $3x + 4y + 5z = 0$ is just another way to say that the vector $\mathbf{x} = [x, y, z]$ satisfies $\langle \mathbf{x}, [3, 4, 5] \rangle = 0$.

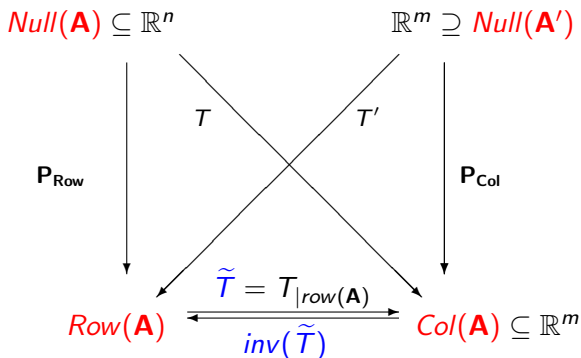


Geometric interpretation of matrix multiplication

Since *clearly* the restriction of the linear mapping $x \mapsto \mathbf{A} * x$



Geometric interpretation of matrix multiplication



$$T = \tilde{T} \circ P_{\text{Row}}, \quad \text{pinv}(T) = \text{inv}(\tilde{T}) \circ P_{\text{Col}}.$$



Four spaces and the SVD

The **SVD** (the so-called Singular Value Decomposition) of a matrix, described in the MATLAB helpul as a way to write A as

$$A = U * S * V'$$

, where the columns of U form an ON-Basis in \mathbb{R}^m and the columns of V form an ON-basis for \mathbb{R}^n , and S is a (rectangular) diagonal matrix containing the non-negative *singular values* (σ_k) of A . We have $\sigma_1 \geq \sigma_2 \dots \sigma_r > 0$, for $r = \text{rank}(A)$, while $\sigma_s = 0$ for $s > r$. In standard description we have for A and $\text{pinv}(A) = A^+$:

$$A * x = \sum_{k=1}^r \sigma_k \langle x, v_k \rangle u_k, \quad A^+ * y = \sum_{k=1}^r \frac{1}{\sigma_k} \langle y, u_k \rangle v_k.$$



Generally known facts in this situation

The **Four Spaces** are well known from **LINEAR ALGEBRA**, e.g. in the **dimension formulas**:

ROW-Rank of \mathbf{A} equals COLUMN-Rank of \mathbf{A} .

The defect (i.e. the dimension of the Null-space of \mathbf{A}) plus the dimension of the range space of \mathbf{A} (i.e. the column space of \mathbf{A}) equals the dimension of the domain space \mathbb{R}^n . Or in terms of linear, homogeneous equations: The dimension of set of all solution to the homogeneous linear equations equals the number of variables minus the dimension of the column space of \mathbf{A} .

The SVD also shows, that the isomorphism between the Row-space and the Column-space can be described by a diagonal matrix, if suitable orthonormal basis for these spaces are used.



Consequences of the SVD

We can describe the quality of the isomorphism \tilde{T} by looking at its condition number, which is σ_1/σ_r , the so-called **Kato-condition number** of T .

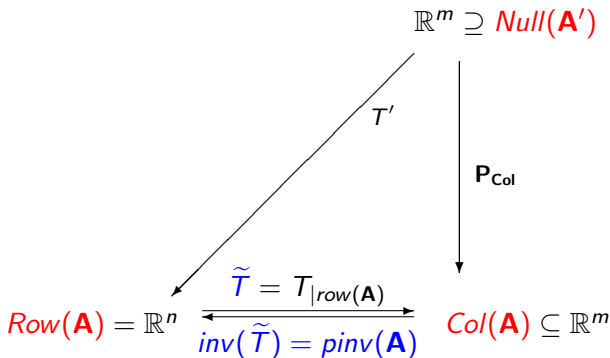
It is not surprising that for **normal matrices** with $A' * A = A * A'$ one can even have diagonalization, i.e. one can choose $U = V$, because

$$\text{Null}(A) =_{\text{always}} \text{Null}(A' * A) = \text{Null}(A * A') = \text{Null}(A').$$

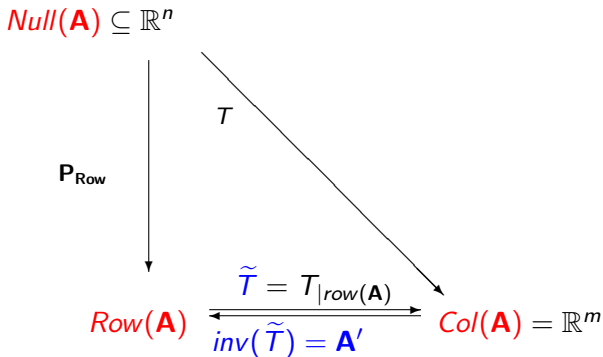
The most interesting cases appear if a matrix has **maximal rank**, i.e. if $\text{rank}(\mathbf{A}) = \min(m, n)$, or equivalently if one of the two Null-spaces is trivial. Then we have either linear independent columns of \mathbf{A} (injectivity of $T \gg$ RIESZ BASIS for subspaces) or the columns of \mathbf{A} span all of \mathbb{R}^m (i.e. $\text{Null}(A') = \{0\}$): FRAME SETTING!



Geometric interpretation: linear independent set > R.B.

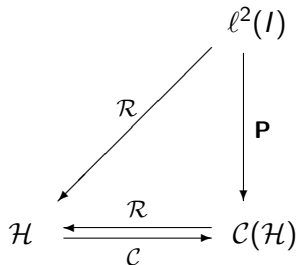


Geometric interpretation: generating set > FRAME



The frame diagram for Hilbert spaces:

If we consider \mathbf{A} as a collection of column vectors, then the role of \mathbf{A}' is that of a coefficient mapping: $f \mapsto (\langle f, f_i \rangle)$.



This diagram is **fully equivalent** to the frame inequalities (6).



Riesz basic sequences in Hilbert spaces:

The diagram for a Riesz basis (for a subspace), nowadays called a **Riesz basic sequence** looks quite the same.

In fact, from an abstract sequence there is no! difference, just like there is no difference (from an abstract viewpoint) between a matrix \mathbf{A} and the transpose matrix \mathbf{A}' .

However, it makes a lot of sense to think that in one case the collection of vectors (making up a Riesz BS) spans the (closed) subspace of \mathcal{H} by just taking all the infinite linear combinations (series) with ℓ^2 -coefficients.

In this way the **synthesis mapping** $\mathbf{c} \mapsto \sum_i c_i g_i$ from $\ell^2(I)$ into the closed linear span is *surjective*, while in the frame case the **analysis mapping** $f \mapsto (\langle f, g_i \rangle)$ from \mathcal{H} into $\ell^2(I)$ is injective (with bounded inverse).



A hierarchy of conditions 1

While the following conditions are equivalent in the case of a finite dimensional vector space (we discuss the frame-like situation) one has to put more assumptions in the case of separable Hilbert spaces and even more in the case of Banach spaces.

Note that one has in the case of an infinite-dimensional Hilbert space: A set of vectors $(f_i)_{i \in I}$ is **total** in \mathcal{H} if and only if the analysis mapping $f \mapsto (\langle f, g_i \rangle)$ is **injective**. In contrast to the frame condition nothing is said about a series expansion, and in fact for better approximation of $f \in \mathcal{H}$ a completely different finite linear combination of g_i 's can be used, without any control on the ℓ^2 -norm of the corresponding coefficients.

THEREFORE one has to make the assumption that the range of the coefficient mapping has to be a *closed subspace* of $\ell^2(I)$ in the discussion of *frames in Hilbert spaces*.



A hierarchy of conditions 2

In the case of Banach spaces one even has to go one step further. Taking the **norm equivalence** between some Banach space norm and a corresponding sequence space norm in a suitable Banach space of sequences over the index set I (replacing $\ell^2(I)$ for the Hilbert space) is **not enough!**

In fact, making such a definition would come back to the assumption that the coefficient mapping $\mathcal{C} : f \mapsto (\langle f, g_i \rangle)$ allows to identify with some **closed subspace** of that Banach space of sequences. Although in principle this might be a useful concept it would not cover typical operations, such as taking Gabor coefficients and applying localization or thresholding, as **the modified sequence** is then typically **not** in the range of the sampled STFT, but resynthesis should work!



A hierarchy of conditions 3

What one really needs in order to have the diagram is the identification of the Banach space under consideration (modulation space, or Besov-Triebel-Lozirkhin space in the case of wavelet frames) with a close and **complemented subspace** of a larger space of sequences (taking the abstract position of $\ell^2(I)$).

To assume the existence of a left inverse to the coefficient mapping allows to establish this fact in a natural way. Assume that \mathcal{R} is the left inverse to \mathcal{C} . Then $\mathcal{C} \circ \mathcal{R}$ is providing the projection operator (the orthogonal projection in the case of $\ell^2(I)$, if the canonical dual frame is used for synthesis) onto the range of \mathcal{C} . The converse is an easy exercise: starting from a projection followed by the inverse on the range one obtains a right inverse operator \mathcal{R} .



A hierarchy of conditions 4

The above situation (assuming the validity of a diagram and the existence of the reconstruction mapping) is part of the definition of Banach frames as given by K. Gröchenig in [12].

Having the classical situation in mind, and the *spirit of frames in the Hilbert spaces case* one should however add two more conditions:

In order to avoid trivial examples of Banach frames one should assume that the associated Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ of sequences should be assumed to be **solid**, i.e. satisfy that $|a_i| \leq |b_i|$ for all $i \in I$ and $b \in \mathbf{B}$ implies $a \in \mathbf{B}$ and $\|a\|_{\mathbf{B}} \leq \|b\|_{\mathbf{B}}$.

Then one could identify the reconstruction mapping \mathcal{R} with the collection of images of unit vectors $h_i := \mathcal{R}(\vec{e}_i)$, where \vec{e}_i is the unit vector at $i \in I$. Moreover, **unconditional convergence of a series** of the form $\sum_i c_i h_i$ would be automatic.



A hierarchy of conditions 6

Instead of going into this detail (including potentially the suggestion to talk about [unconditional Banach frames](#)) I would like to emphasize another aspect of the theory of Banach frames. According to *my personal opinion* it is not very interesting to discuss individual Banach frames, or the existence of *some Banach frames* with respect to *some abstract Banach space of sequences*, even if the above additional criteria apply.

The *interesting cases* concern situations, where the coefficient and synthesis mapping concern a whole [family of related Banach spaces](#), the setting of Banach Gelfand triples being the minimal (and most natural) instance of such a situation.

A comparison: As the family, consisting of father, mother and the **child** is the foundation of our social system, Banach Gelfand Triples are the prototype of *families*, sometimes *scales of Banach spaces*, the “child” being of course our beloved **Hilbert space**.



Banach Gelfand Triples and Rigged Hilbert space

The next term to be introduced are **Banach Gelfand Triples**.

There exists already and established terminology concerning triples of spaces, such as the **Schwartz triple** consisting of the spaces $(\mathcal{S}, L^2, \mathcal{S}')(\mathbb{R}^d)$, or triples of weighted Hilbert spaces, such as $(L^2_w, L^2, L^2_{1/w})$, where $w(t) = (1 + |t|^2)^{s/2}$ for some $s > 0$, which is - via the Fourier transform isomorphic to another (“Hilbertian”) Gelfand Triple of the form $(\mathcal{H}_s, L^2, \mathcal{H}'_s)$, with a Sobolev space and its dual space being used e.g. in order to describe the behaviour of elliptic partial differential operators.

The point to be made is that suitable Banach spaces, in fact imitating the **prototypical** Banach Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$ allows to obtain a surprisingly large number of results resembling the finite dimensional situation.



A Classical Example related to Fourier Series

There is a well known and classical example related to the more general setting I want to describe, which - as so many things - go back to N. Wiener. He introduced (within $L^2(\mathbb{T})$) the space $(\mathbf{A}(\mathbb{T}), \|\cdot\|_{\mathbf{A}})$ of **absolutely convergent Fourier series**. Of course this space sits inside of $(L^2(\mathbb{T}), \|\cdot\|_2)$ as a dense subspace, with the norm $\|f\|_{\mathbf{A}} := \sum_{n \in \mathbb{Z}} |\hat{f}(n)|$.

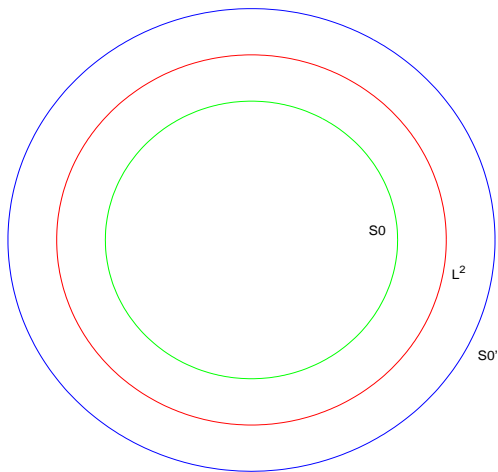
Later on the discussion about Fourier series and generalized functions led (as I believe naturally) to the concept of **pseudo-measures**, which are either the elements of the dual of $(\mathbf{A}(\mathbb{T}), \|\cdot\|_{\mathbf{A}})$, or the (generalized) inverse Fourier transforms of bounded sequences, i.e. $\mathcal{F}^{-1}(\ell^\infty(\mathbb{Z}))$.

In other words, this extended view on the Fourier analysis operator $\mathcal{C} : f \mapsto (\hat{f}(n))_{n \in \mathbb{Z}}$ on the BGT $(\mathbf{A}, L^2, \mathbf{PM})$ into $(\ell^1, \ell^2, \ell^\infty)$ is the **prototype** of what we will call a **BGT-isomorphism**.



The visualization of a Banach Gelfand Triple

The S_0 Gelfand triple



Rethinking shortly the Fourier Transform

Since the Fourier transform is one of the central transforms, both for abstract harmonic analysis, engineering applications and pseudo-differential operators let us take a look at it first. People (and books) approach it in different ways and flavours:

- It is defined as integral transform (Lebesgue!?)
- It is computed using the FFT (what is the connection)
- Should engineers learn about tempered distributions?
- How can we reconcile mathematical rigor and still stay in touch with applied people (physics, engineering).



The finite Fourier transform (and FFT)

For practical applications the discrete (finite) Fourier transform is of utmost importance, because of its **algebraic properties** [joint diagonalization of circulant matrices, hence fast multiplication of polynomials, etc.] and its **computational efficiency** (FFT algorithms of signals of length N run in $N \log(N)$ time, for $N = 2^k$, due to recursive arguments).

It maps a vector of length n onto the values of the polynomial generated by this set of coefficients, over the unit roots of order n on the unit circle (hence it is a Vandermonde matrix). It is a **unitary matrix** (up to the factor $1/\sqrt{n}$) and maps **pure frequencies onto unit vectors** (engineers talk of *energy preservation*).



The Fourier Integral and Inversion

If we define the Fourier transform for functions on \mathbb{R}^d using an integral transform, then it is useful to assume that $f \in L^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} dt \quad (10)$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} d\omega, \quad (11)$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in L^1(\mathbb{R}^d)$. One often speaks of **Fourier analysis** being the first step, and the Fourier inversion as a method to build f from the pure frequencies (we talk of **Fourier synthesis**).



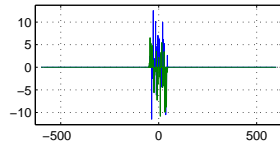
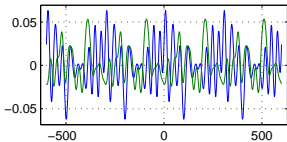
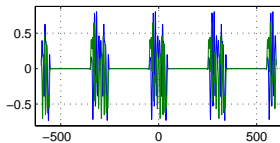
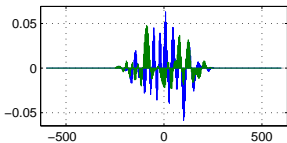
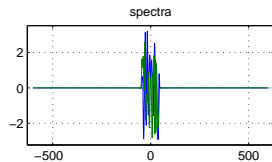
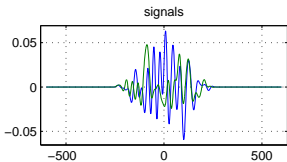
The classical situation with Fourier

Unfortunately the Fourier transform does not behave well with respect to L^1 , and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

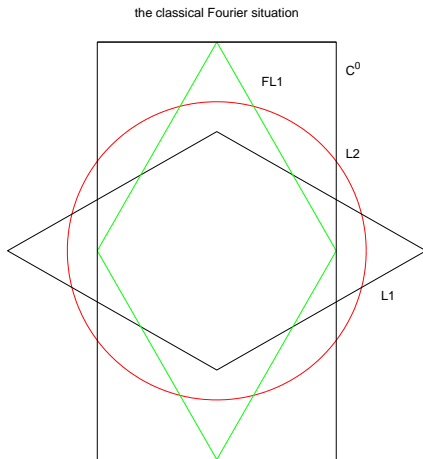
- 1 For $f \in L^1(\mathbb{R}^d)$ we have $\hat{f} \in C_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in L^1(\mathbb{R}^d)$);
- 2 The Fourier transform f on $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ is isometric in the L^2 -sense, but the Fourier integral cannot be written anymore;
- 3 Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
- 4 L^p -spaces have traditionally a high reputation among function spaces, but tell us little about \hat{f} .



Effects of Sampling and Periodization: Poisson's formula



A schematic description of the situation



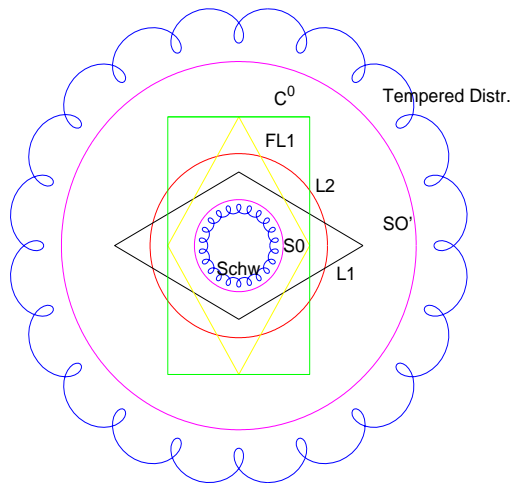
The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce **generalized functions**. In order to do so one has to introduce **test functions**, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The “good ones” are admitted and called **generalized functions**, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as $5/7$ is a complex number!).

If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. But there are easier **alternatives**.



A schematic description of the situation



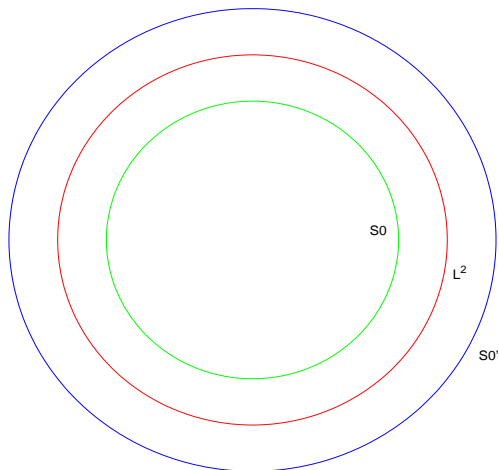
The Banach space $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ or $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$), which will serve our purpose. Its dual space $(\mathcal{S}'_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}'_0})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant “objects” (in fact so-called local pseudo-measures or quasimeasures, originally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is correspondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to **Banach Gelfand Triples**, mostly related to $(\mathcal{S}_0, L^2, \mathcal{S}'_0)(\mathbb{R}^d)$.



repeated: SOGELFTR

The S_0 Gelfand triple

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t).$$

Behavior under Fourier transform

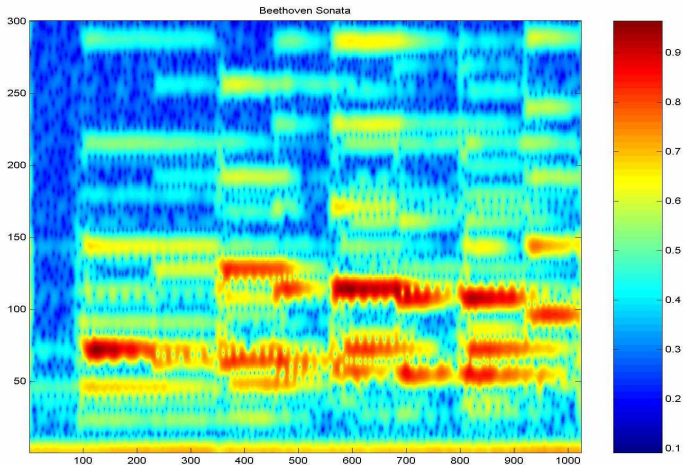
$$(\widehat{T_x f}) = M_{-x} \hat{f} \quad (\widehat{M_\omega f}) = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \quad \lambda = (t, \omega);$$



A Typical Musical STFT



A Banach Space of Test Functions (Fei 1979)

A function in $f \in L^2(\mathbb{R}^d)$ is in the subspace $\mathbf{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the “window”) in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{\mathbf{S}_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \mathcal{S}(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $\mathbf{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.



Basic properties of $M^1 = \mathcal{S}_0(\mathbb{R}^d)$

Lemma

Let $f \in \mathcal{S}_0(\mathbb{R}^d)$, then the following holds:

- (1) $\pi(u, \eta)f \in \mathcal{S}_0(\mathbb{R}^d)$ for $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and
 $\|\pi(u, \eta)f\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.
- (2) $\hat{f} \in \mathcal{S}_0(\mathbb{R}^d)$, and $\|\hat{f}\|_{\mathcal{S}_0} = \|f\|_{\mathcal{S}_0}$.

In fact, $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p -spaces (and their Fourier images).

There are many other independent characterizations of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1 -coefficients, or as $\mathcal{W}(\mathcal{FL}^1, \ell^1) = M_{1,1}^0(\mathbb{R}^d)$.



Basic properties of $M^\infty(\mathbb{R}^d) = \mathcal{S}'_0(\mathbb{R}^d)$

It is probably no surprise to learn that the dual space of $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$, i.e. $\mathcal{S}'_0(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts

$$\pi(\lambda), \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

As an amalgam space one has

$\mathcal{S}'_0(\mathbb{R}^d) = \mathbf{W}(\mathcal{FL}^1, \ell^1)' = \mathbf{W}(\mathcal{FL}^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of **translation bounded quasi-measures**, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on \mathbb{R}^d with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in \mathcal{S}_0(\mathbb{R}^d)$).

Consequently norm convergence in $\mathcal{S}'_0(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain **atomic characterizations** of $(\mathcal{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathcal{S}_0})$ imply that w^* -convergence is in fact equivalent to **locally uniform convergence** of the STFT. – Hifi recordings!



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space B , which is dense in some Hilbert space \mathcal{H} , which in turn is contained in B' is called a **Banach Gelfand triple**.

Definition

If $(B_1, \mathcal{H}_1, B'_1)$ and $(B_2, \mathcal{H}_2, B'_2)$ are Gelfand triples then a linear operator T is called a **[unitary] Gelfand triple isomorphism** if

- ① A is an isomorphism between B_1 and B_2 .
- ② A is [a unitary operator resp.] an isomorphism between \mathcal{H}_1 and \mathcal{H}_2 .
- ③ A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between B'_1 and B'_2 .

Banach Gelfand Triples, etc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space \mathcal{H} can be used to establish such a unitary isomorphism, by choosing as \mathbf{B} the space of elements within \mathcal{H} which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.

For the case of the Fourier system as CONB for $\mathcal{H} = \mathbf{L}^2([0, 1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\mathbf{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\mathbf{PM}(\mathbb{U}) = \mathbf{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\mathbf{A}, \mathbf{L}^2, \mathbf{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.



The BGT $(\mathcal{S}_0, L^2, \mathcal{S}'_0)$ and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.

For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the “ $ax+b$ ”-group) one finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler’s formula) to cos and sin functions in order to build a **Wilson ONB** for $L^2(\mathbb{R}^d)$. In this way another BGT-isomorphism between $(\mathcal{S}_0, L^2, \mathcal{S}'_0)$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.



The Fourier transform as BGT automorphism

The **Fourier transform** \mathcal{F} on \mathbb{R}^d has the following properties:

- ① \mathcal{F} is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
- ② \mathcal{F} is a unitary map between $L^2(\mathbb{R}^d)$ and $L^2(\widehat{\mathbb{R}}^d)$,
- ③ \mathcal{F} is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}'_0(\mathbb{R}^d)$ onto $\mathbf{S}'_0(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \quad (12)$$

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}'_0(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, L^2, \mathbf{S}'_0)(\mathbb{R}^d)$.



The w^* – topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, L^2, \mathbf{S}'_0)(\mathbb{R}^d)$ correspond to norm convergence in $(L^1, L^2, L^\infty)(\mathbb{R}^{2d})$.

The **FOURIER transform**, viewed as a BGT-automorphism is **uniquely determined** by the fact that it maps **pure frequencies** onto the corresponding **point measures** δ_ω .

This is a typical case, where we can see, that the w^* -continuity plays a role, and where the fact that $\delta_x \in \mathbf{S}'_0(\mathbb{R}^d)$ as well as $\chi_s \in \mathbf{S}'_0(\mathbb{R}^d)$ are important.

In the STFT-domain the w^* -convergence has a particular meaning: a sequence σ_n is w^* -convergent to σ_0 if $V_g(\sigma_n)(\lambda) \rightarrow V_g(\sigma_0)(\lambda)$ uniformly over compact subsets of the TF-plane (for one or any non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$).



Classical Results in Banach Space Theory

Wiener's inversion theorem:

Theorem

Assume that $h \in \mathbf{A}(\mathbb{U})$ is free of zeros, i.e. that $h(t) \neq 0$ for all $t \in \mathbb{U}$. Then the function $g(t) := 1/h(t)$ belongs to $\mathbf{A}(\mathbb{U})$ as well.

The proof of this theorem is one of the nice applications of a spectral calculus with methods from Banach algebra theory. This result can be reinterpreted in our context as a results which states:

Assume that the pointwise multiplication operator $f \mapsto h \cdot f$ is invertible as an operator on $(L^2(\mathbb{U}), \|\cdot\|_2)$, and also a BGT-morphism on $(\mathbf{A}, L^2, \mathbf{PM})$ (equivalent to the assumption $h \in \mathbf{A}(\mathbb{U})!$), then it is also continuously invertible as BGT-morphism.



SOGTr-results in Banach Triple terminology

In the setting of $(\mathbf{S}_0, L^2, \mathbf{S}'_0)$ a quite similar results is due to Gröchenig and coauthors:

Theorem

Assume that for some $g \in \mathbf{S}_0$ the Gabor frame operator $S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$ is invertible at the Hilbert space level, then S defines automatically an automorphism of the BGT $(\mathbf{S}_0, L^2, \mathbf{S}'_0)$. Equivalently, when $g \in \mathbf{S}_0$ generates a Gabor frame (g_λ) , then the dual frame (of the form (\tilde{g}_λ)) is also generated by the element $\tilde{g} = S^{-1}(g) \in \mathbf{S}_0$.

The first version of this result has been based on matrix-valued versions of Wiener's inversion theorem, while the final result (Gröchenig and Leinert, see [14]) makes use of the concept of *symmetry* in Banach algebras and Hulanicki's Lemma.



Classical Results in Banach Space Theory 2

Theorem (Theorem by S. Banach)

*Assume that a linear mapping between two Banach spaces is continuous, and invertible as a mapping between sets, then it is automatically an isomorphism of Banach spaces, i.e. the inverse mapping is automatically linear and **continuous**.*

So we have invertibility only in a more comprehensive category, and want to conclude invertibility in the given smaller (or richer) category of objects.



Other relations to the finite-dimensional case

The paper [13]: Gabor frames without inequalities Int. Math. Res. Not. IMRN, No.23, (2007) contains another collection of statements, showing the strong analogy between a finite-dimensional setting and the setting of Banach Gelfand triples: The main result (Theorem 3.1) of that paper shows, that the Gabor frame condition (which at first sight looks just like a two-sided norm condition) is in fact equivalent to **injectivity of the analysis mapping** (however at the “outer level”, i.e. from $\mathbf{S}_0'(\mathbb{R}^d)$ into $\ell^\infty(\mathbb{Z}^d)$), while it is also equivalent to **surjectivity of the synthesis mapping**, but this time from $\ell^1(\mathbb{Z}^d)$ onto $\mathbf{S}_0(\mathbb{R}^d)$.



Kernel Theorem for general operators in $\mathcal{L}(\mathbf{S}_0, \mathbf{S}'_0)$

Theorem

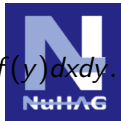
If K is a bounded operator from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}'_0(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathbf{S}'_0(\mathbb{R}^{2d})$ such that $\langle Kf, g \rangle = \langle k, g \otimes f \rangle$ for $f, g \in \mathbf{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.

Formally sometimes one writes by “abuse of language”

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \mathbf{S}'_0(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, y)f(y)dy g(x)dx = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$



Kernel Theorem II: Hilbert Schmidt Operators

This result is the “outer shell” of the Gelfand triple isomorphism. The “middle = Hilbert” shell which corresponds to the well-known result that Hilbert Schmidt operators on $L^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $L^2(\mathbb{R}^{2d})$ -kernels. The complete picture can be best expressed by a unitary Gelfand triple isomorphism. First the innermost shell:

Theorem

*The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S \rangle_{\mathcal{HS}} = \text{trace}(T * S')$ as scalar product on \mathcal{HS} and the usual Hilbert space structure on $L^2(\mathbb{R}^{2d})$ on the kernels. An operator T has a kernel in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ if and only if the T maps $\mathbf{S}'_0(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$, boundedly, but continuously **also from w^* -topology into the norm topology of $\mathbf{S}_0(\mathbb{R}^d)$.***



Kernel Theorem III

Remark: Note that for such **regularizing** kernels in $K \in \mathbf{S}_0(\mathbb{R}^{2d})$ the usual identification. Recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the n -th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x, y) = T(\delta_y)(x) = \delta_x(T(\delta_y)).$$

Note that $\delta_y \in \mathbf{S}'_0(\mathbb{R}^d)$ implies that $K(\delta_y) \in \mathbf{S}_0(\mathbb{R}^d)$ by the regularizing properties of K , hence pointwise evaluation is OK. With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathbf{S}_0, L^2, \mathbf{S}'_0)(\mathbb{R}^{2d})$ into the **Gelfand triple of operator spaces**

$$(\mathcal{L}(\mathbf{S}'_0, \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}'_0)).$$



AN IMPORTANT TECHNICAL warning!!

How should we **realize** these various BGT-mappings?

Recall: How can we **check numerically** that $e^{2\pi i} = 1$??

Note: we can only do our computations (e.g. multiplication, division etc.) properly in the rational domain \mathbb{Q} , we get to \mathbb{R} by approximation, and then to the complex numbers applying “the correct rules” (for pairs of real numbers).

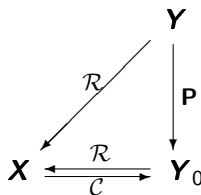
In the BGT context it means: All the (partial) Fourier transforms, integrals etc. only have to be meaningful at the \mathbf{S}_0 -level (using simply Riemannian integrals!), typically isometric in the L^2 -sense, and extend by duality considerations to \mathbf{S}'_0 when necessary, using w^* -continuity!

The Fourier transform is a good example (think of Fourier inversion and summability methods), similar arguments apply to the transition from the integral kernel of a linear mapping to its Kohn-Nirenberg symbol., for example.

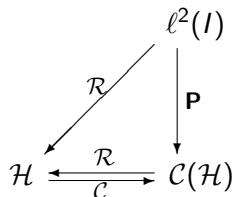


Frames and Riesz Bases: the Diagram

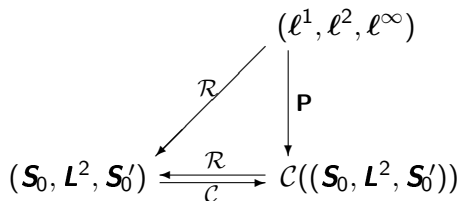
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in \mathbf{Y} onto the range \mathbf{Y}_0 of \mathcal{C} , thus we have the following commutative diagram.



The frame diagram for Hilbert spaces:



The frame diagram for Hilbert spaces (S_0, L^2, S'_0) :



Verbal Description of the Situation

Assume that $g \in \mathbf{S}_0(\mathbb{R}^d)$ is given and some lattice Λ . Then (g, Λ) generates a Gabor frame for $\mathcal{H} = \mathbf{L}^2(\mathbb{R}^d)$ if and only if the coefficient mapping \mathcal{C} from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ as a left inverse \mathcal{R} (i.e. $\mathcal{R} \circ \mathcal{C} = Id_{\mathcal{H}}$), which is also a GTR-homomorphism back from $(\ell^1, \ell^2, \ell^\infty)$ to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)$. In practice it means, that the dual Gabor atom \tilde{g} is also in $\mathbf{S}_0(\mathbb{R}^d)$, and also the canonical tight atom $S^{-1/2}$, and therefore the whole procedure of taking coefficients, perhaps multiplying them with some sequence (to obtain a Gabor multiplier) and resynthesis is well defined and a BGT-morphism for any such pair.



Gabor frame matrix representations

Much in the same way as basis in \mathbb{C}^n are used in order to describe linear mappings as matrices we can also use Gabor frame expansions in order to describe (and analyze resp. better understand) certain linear operators T (slowly variant channels, operators in Sjostrand's class, connected with another family of modulation spaces) by their **frame matrix expansion**. Working (for convenience) with a Gabor frame with atom $g \in \mathbf{S}_0(\mathbb{R}^d)$ (e.g. Gaussian atom, with $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$), and form for $\lambda, \mu \in \Lambda$ the infinite matrix

$$a_{\lambda, \mu} := [T(\pi(\lambda)g)](\pi(\mu)g).$$

This makes sense even if T maps only $\mathbf{S}_0(\mathbb{R}^d)$ into $\mathbf{S}'_0(\mathbb{R}^d)$!



Gabor frame matrix representations II

For any good Gabor family (tight or not) the mapping $T \mapsto \mathbf{A} = (a_{\lambda, \mu})$ is itself defining a frame representation, hence a retract diagram, from the operator BGT $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ into the $(\ell^1, \ell^2, \ell^\infty)$ over \mathbb{Z}^{2d} !

In other words, we can recognize whether an operator is regularizing, i.e. maps $\mathbf{S}'_0(\mathbb{R}^d)$ into $\mathbf{S}_0(\mathbb{R}^d)$ (with w^* -continuity) if and only if the matrix has coefficients in $\ell^1(\mathbb{Z}^{2d})$.

Note however, that invertibility of T is NOT equivalent to invertibility of \mathbf{A} ! (one has to take the pseudo-inverse).



The Spreading Representation

The kernel theorem corresponds of course to the fact that every linear mapping T from \mathbb{C}^n to \mathbb{C}^n can be represented by a uniquely determined matrix \mathbf{A} , whose columns \mathbf{a}_k are the images $T(\vec{e}_k)$. When we identify \mathbb{C}^N with $\ell^2(\mathbf{Z}_N)$ (as it is suitable when interpreting the FFT as a unitary mapping on \mathbb{C}^N) there is another way to represent every linear mapping: we have exactly N cyclic shift operators and (via the FFT) the same number of frequency shifts, so we have exactly N^2 TF-shifts on $\ell^2(\mathbf{Z}_N)$. They even form an orthonormal system with respect to the Frobenius norm, coming from the scalar product

$$\langle \mathbf{A}, \mathbf{B} \rangle_{Frob} := \sum_{k,j} a_{k,j} \bar{b}_{k,j} = \text{trace}(A * B')$$

This relationship is called the **spreading representation** of the linear mapping T resp. of the matrix \mathbf{A} . It can be thought as a kind of operator version of the Fourier transform.



The unitary spreading BGT-isomorphism

Theorem

*There is a natural (unitary) Banach Gelfand triple isomorphism, called the **spreading mapping**, which assigns to operators T from $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ the function or distribution $\eta(T) \in (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^{2d})$. It is uniquely determined by the fact that $T = \pi(\lambda) = M_\omega T_t$ corresponds to $\delta_{t,\omega}$.*

Via the symplectic Fourier transform, which is of course another unitary BGT-automorphism of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^{2d})$ we arrive at the **Kohn-Nirenberg calculus** for pseudo-differential operators. In other words, the mapping $T \mapsto \sigma_T = \mathcal{F}_{\text{symp}} \eta(T)$ is another unitary BGT isomorphism (onto $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^{2d})$, again).



Consequences of the Spreading Representation

The analogy between the ordinary Fourier transform for functions (and distributions) with the spreading representation of operators (from nice to most general within our context) has interesting consequences.

We know that Λ -periodic distributions are exactly the ones having a Fourier transform supported on the orthogonal lattice Λ^\perp , and periodizing an L^1 -function corresponds to sampling its FT.

For operators this means: an operator T commutes with all operators $\pi(\Lambda)$, for some $\Lambda \triangleleft \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, if and only if $\text{supp}(\eta(T)) \subset \Lambda^\circ$, the **adjoint lattice**. The Gabor frame operator is the Λ -periodization of $P_g : f \mapsto \langle f, g \rangle g$, hence $\eta(S)$ is obtained by multiplying $\eta(P_g) = V_g(g)$ pointwise by $\bigsqcup_{\Lambda^\circ} = \sum_{\lambda^\circ \in \Lambda^\circ} \delta_{\lambda^\circ}$.



Consequences of the Spreading Representation 2

This observation is essentially explaining the Janssen representation of the Gabor frame operator (see [6]).

Another analogy is the understanding that there is a class of so-called **underspread operators**, which are well suited to model slowly varying communication channels (e.g. between the basis station and your mobile phone, while you are sitting in the - fast moving - train).

These operators have a known and very limited support of their spreading distributions (maximal time- and Doppler shift on the basis of physical considerations), which can be used to “sample” the operator (pilot tones, channel identification) and subsequently decode (invert) it (approximately).



Summability of sequences and quality of operators

One can however also fix the Gabor system, with both analysis and synthesis window in $\mathfrak{S}_0(\mathbb{R}^d)$ (typically one will take g and \tilde{g} respectively, or even more symmetrically a tight Gabor window). Then one can take the multiplier sequence in different sequence spaces, e.g. in $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.

Lemma

Then the mapping from multiplier sequences to Gabor multipliers is a Banach Gelfand triple homomorphism into Banach Gelfand triple of operator ideals, consisting of the Schatten class $S_1 =$ trace class operators, $\mathcal{H} = \mathcal{HS}$, the Hilbert Schmidt operators, and the class of all bounded operators (with the norm and strong operator topology).

Automatic continuity ($>$ Balian-Low)

In contrast to the pure Hilbert space case (the box-function is an ideal orthonormal system on the real line, but does *NOT allow* for any deformation, without losing the property of being even a Riesz basis):

Theorem (Fei/Kaiblinger, TAMS)

Assume that a pair (g, Λ) , with $g \in \mathbf{S}_0(\mathbb{R}^d)$ defines a Gabor frame or a Gabor Riesz basis respectively [note that by Wexler/Raz and Ron/Shen these two situations are equivalent modulo taking adjoint subgroups!], then the same is true for slightly perturbed atoms or lattices, and the corresponding dual atoms (biorthogonal generators) depend continuously in the $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$ -sense on both parameters.

THANK YOU!

Thank you for your attention!

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Furthermore there are various talks given in the last few years on related topics (e.g. Gelfand triples), that can be found by searching by title or by name in

http://www.univie.ac.at/nuhag-php/nuhag_talks/



Selection of bibliographic items, see www.nuhag.eu



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Invertibility, Surjectivity and Injectivity

In another, very recent paper, Charly Groechenig has discovered that there is another analogy to the finite dimensional case: There one has: A square matrix is invertible if and only if it is surjective or injective (the other property then follows automatically). We have a similar situation here (systematically describe in Charly's paper):

K.Groechenig: Gabor frames without inequalities,
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Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called **kernel theorem**. It uses $\mathbf{B} = \mathcal{L}(\mathbf{S}'_0, \mathbf{S}_0)$.

Theorem

*There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ and $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^{2d})$. This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Moreover, the **spreading mapping** is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators $\pi(\lambda)$ and the corresponding point masses δ_λ .*

The w^* -topology: a natural alternative

It is not difficult to show, that the norms of $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}'_0)(\mathbb{R}^d)$ correspond to norm convergence in $(\mathbf{L}^1, \mathbf{L}^2, \mathbf{L}^\infty)(\mathbb{R}^{2d})$.

Therefore it is interesting to check what the w^* -convergence looks like:

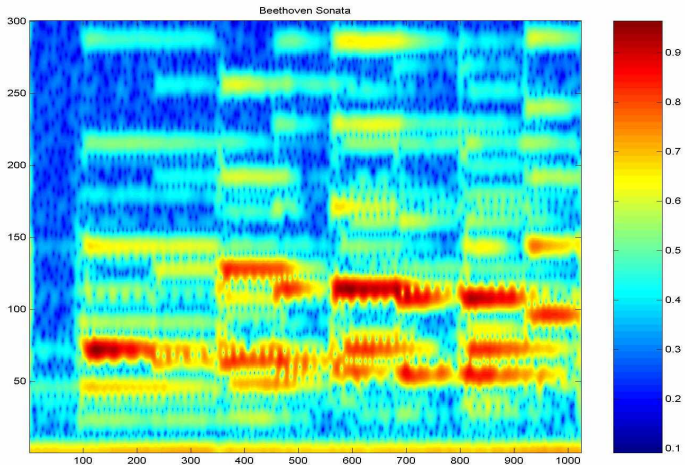
Lemma

For any $g \in \mathbf{S}_0(\mathbb{R}^d)$ a sequence σ_n is w^ -convergent to σ_0 if and only the spectrograms $V_g(\sigma_n)$ converge uniformly over compact sets to the spectrogram $V_g(\sigma_0)$.*

The **FOURIER transform**, viewed as a BGT-automorphism is **uniquely determined** by the fact that it maps **pure frequencies** onto the corresponding **point measures** δ_ω .



A Typical Musical STFT



The w^* -topology: dense subfamilies

From the practical point of view this means, that one has to **look at the spectrograms** of the sequence σ_n and verify whether they look closer and closer the spectrogram of the limit distribution $V_g(\sigma_0)$ over compact sets.

The approximation of elements from $\mathbf{S}_0'(\mathbb{R}^d)$ takes place by a bounded sequence.

Since any Banach-Gelfand triple homomorphism preserves this property (by definition) one can reduce many problems to w^* -dense subsets of $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$.

Let us look at some concrete examples: **Test-functions, finite discrete measures $\mu = \sum_i c_i \delta_{t_i}$, trigonometric polynomials $q(t) = \sum_i a_i e^{2\pi i \omega_i t}$, or discrete AND periodic measures** (this class is invariant under the generalized Fourier transform and can be realized computationally using the FFT).



The w^* – topology: approximation strategies

- How to approximate general distributions by test functions: Regularization procedures via product convolution operators, $h_\alpha(g_\beta * \sigma) \rightarrow \sigma$ or TF-localization operators: multiply the STFT with a 2D-summability kernel before resynthesis (e.g. partial sums for Hermite expansion);
- how to approximate an L^1 -Fourier transform by test functions: and classical summability
- how to approximate a test function by a finite discrete sequence using quasi-interpolation (N. Kaiblinger):

$$Q_\Psi f(x) = \sum_i f(x_i)\psi_i(x).$$

