Function spaces: Concepts, Goals and Applications

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COMMENT for those who did not attend

THIS IS THE MATERIAL RELATED TO A 55 MINUTE TALK DELIVERED AT THE ABBANEUM, THE OLD LECTURE HALL OF MATHEMATICS, AT FRIEDRICH SCHILLER UNIVERSITY, JENA, GERMANY, ON FEB. 12TH, 2016, ON THE OCCASION OF HANS TRIEBEL'S 60TH BIRTHDAY

A GOOD PART OF THE MATERIAL IN THIS NOTE HAS NOT BEEN EFFECTIVELY DISCUSSED DURING THE PRESENTATION (AND A FEW COMMENTS GIVEN AT THE TALK ARE NOT DOCUMENTED HERE!)

This presentation can be downloaded from

http://www.univie.ac.at/nuhag-php/program/talks_details.php?id=3077

HGFei, 12.02.2016, 21:16.

By Now FUNCTION SPACES is an Established Field

Function Spaces have played a crucial role in the genesis of *functional analysis* from its very beginning, but equally for the development of the modern analysis of partial differential (and pseudo-) differential operators, distribution theory, interpolation theory, or approximation theory.
During the last 50 years, partially through the heroic work of Hans Triebel, whose 80th birthday is the reason for this meeting, the theory has almost developed into an independent *branch of analysis*, with a lot of expertise and knowledge being accumulated, published in journals, at conferences and in particular in books.
It is the purpose of this talk to recapitulate a little bit some of the concepts which have lead to the rich zoo of function spaces which we find currently in the literature, but also shed some light on the problem: "find the right space for a operator"!

My Personal Approach

Very much like the "Theory of Function Spaces" is a branch of *functional analysis*, with a focus on particular emphasis of Banach spaces (of generalized functions), as e.g. opposed to experts in C^*-algebras or operators ideals and so on, for *me personally Harmonic Analysis* is the branch of functional analysis which has to do with group actions, with the decomposition of spaces into invariant building blocks, so it has close relationships to the theory of group representations.

The core of my work is in the intersection of the two fields:

(Function spaces) $\cap$ (Harmonic Analysis)

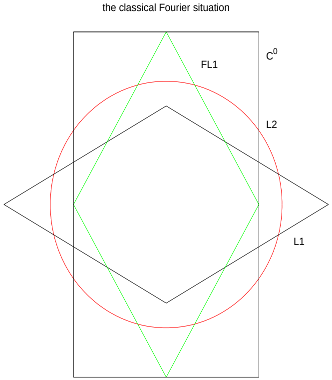
Consequently I was mostly interested in Banach spaces of functions over groups which are invariant with respect to various group actions, mostly translation, but also multiplication with pure frequences or dilations.

Comparison with Car Industry

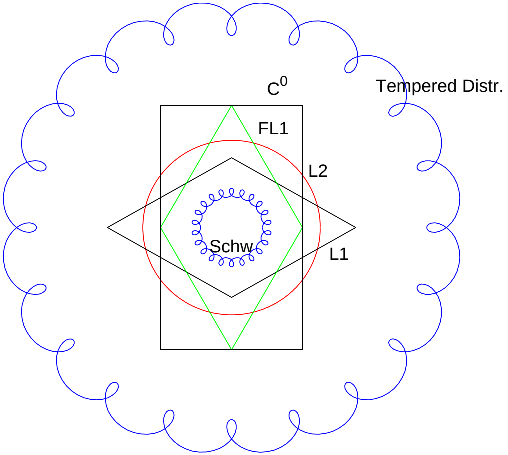
The following comparison is essentially used in [12], a paper about "Choosing Function Spaces". There I compare the situation in the theory of function spaces with *car industry*, where we see the following (somehow natural) steps of development:

1. A few people are able to build cars;
2. There is a broad competition of care manufacturers;
3. Mister Ford comes up and tells us industrial production;
4. Modern Times: consolidation, reduction to few mainstream models plus a variety of cars for "fans";
5. close to reality: individual, just in time fabrication

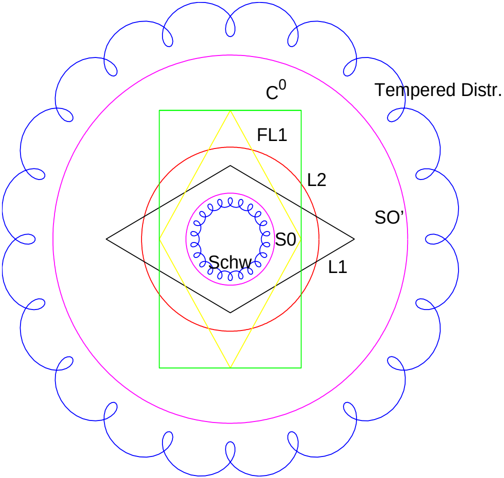
Classical: $L^1, L^2, \mathcal{F}L^1, C_0$: Riemann-Lebesgue, 1907?



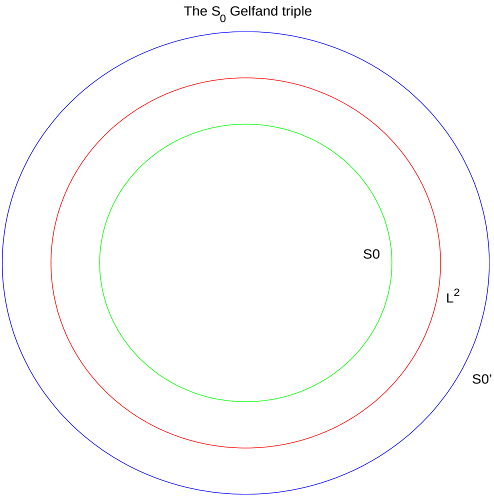
Introducing Schwartz spaces $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$, 1957



The full scenario, including $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$, 1979



Reduction to the essentials: Banach Gelfand Triple: 1998



What are Function Spaces Good For??

Citation to a comment by *Yves Meyes*, made in 1987 (during a meeting with HGFei, just after the first wavelet ONB had been announced, see [36]):

Function spaces are only good for the description of operators. (so the implicit statement: the independent study of function spaces is perhaps questionable).

Even if I was opposed to this claim at this time I am willing to acknowledge that the "production of function spaces just for their own sake" may lead nowhere.

Background of Y. Meyer's statement was certainly the history of Fourier analysis, e.g. at that time the recent appreciation of the real Hardy-space $\left(\boldsymbol{H}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{H}^1}\right)$ or $\left(\boldsymbol{BMO}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{BMO}}\right)$.

Overview over the Talk

There is a number of questions which we would like to address

- What are function spaces?
- A little bit of history of function spaces!
- Which construction principles are relevant?
- Characterization of function spaces via series expansions!
- What about the "usefulness" for specific function spaces; (the "information content" of function spaces);
- Connections between operators and function spaces.

Obviously we will not be able to give exhaustive answers to any of these questions, but perhaps we succeed in presenting some interesting viewpoints and challenging perspectives.

What are function spaces?

The *naive* answer is of course: SPACES OF FUNCTIONS such as

1. $\left(\boldsymbol{C}_0(\mathbb{R}^d), \, \|\cdot\|_\infty\right)$ or
2. $\left(\boldsymbol{L}^1(\mathbb{R}^d), \, \|\cdot\|_1\right), \left(\boldsymbol{L}^2(\mathbb{R}^d), \, \|\cdot\|_2\right)$
   which are "equivalence classes of measurable functions"!
3. $(\boldsymbol{BV}(\mathbb{R}), \|\cdot\|_{\boldsymbol{BV}})$, the space of functions of bounded variation
4. $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$, the Fourier algebra;
5. Sobolev spaces $\left(\mathcal{H}_s(\mathbb{R}^d), \, \|\cdot\|_{\mathcal{H}_s}\right)$,
   which is the inverse Fourier image of $\boldsymbol{L}^2_{w_s}(\mathbb{R}^d)$;
6. Lipschitz spaces;
7. Morrey-Campanato spaces, etc. ([1]);
8. Banach Function Spaces (Lux./Zaanen), [35, 52].

More generally: Space of Generalized Functions?

However with every FUNCTION SPACE one also should look at the dual space, which means that we should also consider spaces such as

1. $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$;
2. Sobolev spaces of negative order;
3. Banach spaces of (tempered distribution) are considered as "Function Spaces" as well;

In fact, distributions, viewed as "generalized functions" constitute Banach spaces of interest for many areas of analysis (sometimes modulo elementary classes, e.g. modulo polynomials: $\dot{\boldsymbol{B}}^s_{p,q}$), homogeneous Besov spaces. *Standard situations* like this have been used in [8] or [2].

or Space of Generalized Functions modulo polynomials?

I am mentioning this aspect, following an interesting discussion with Prof. Triebel (yesterday) about his most recent work about *tempered homogeneous function spaces* ([48, 49])
which are a very good example of setting, where the problem (in this case Navier-Stokes equations) dictate the setting, and where an expert can find the right setting to formulate and solve the problem, while more naive approaches may fail of have limited success.
Similar problems (related to homogeneous versus non-homogeneous function space) arise in the PhD thesis and recent work of Felix Voigtlaender (RWTH Aachen, Nov. 2015, PhD, [51]).

A little bit of history of function spaces!

The first approach to smoothness resulting in the definition of Sobolev spaces and Besov spaces (Besov, Taibleson, Stein) came from the idea of generalized smoothness, expressed by (higher order) difference expression and the corresponding *moduli of continuity*, e.g. describing smoothness by the decay of the modulus of continuity (via the membership in certain weighted $\boldsymbol{L}^q$-spaces on $(0, 1]$). Alternatively there is the line described in the book of S.Nikolksii characterizing smoothness (equivalently) by the degree of approximation using band-limited functions (S. M. Nikol'skij [37]). Fractional order Sobolev spaces can be expressed in terms of weighted Fourier transforms. Many references!

The second and third age

The second age is characterized by the Paley-Littlewood characterizations of Besov or Triebel-Lizorkin spaces using dyadic decompositions on the Fourier transform side, as used in the work of J.Peetre ([38]) and H.Triebel ([42, 43, 41, 39, 44]) , the masters of interpolation theory. Their contribution was to show that these families of function spaces are stable under interpolation.
The third age is - from our point of view - the characterization of function spaces in the context of coorbit spaces ([13, 14], using irreducible integrable group representations of locally compact groups).
Let us also remind that the concept of retracts plays an important role in the context of interpolation theory (see the book of Bergh-Löfström), and can be used to characterize *Banach frames* and *Riesz projection bases* ([24]).

Setting: The Role of Banach Gelfand Triples

Although it is quite natural to view FUNCTION SPACES on $\mathbb{R}^d$ typically as Banach spaces which are sitting between $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ and $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, this approach requires to work with *topological vector spaces*, dual spaces of such spaces, families of seminorms. This setting requires a good background in analysis. In addition, it is not so easy to transfer to the setting of **LCA (locally compact Abelian) groups,** where the space of *Schwartz-Bruhat* functions is much more complicated.

It has turned out that within the family of modulation spaces one can find appropriate spaces one can find relatively simple Banach spaces which can be used as a replacement for the pair of Schwartz spaces. Depending on the viewpoint one can use the *Shubin classes* $\boldsymbol{Q}_s(\mathbb{R}^d)$, or the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual, $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ ([6]).

The $\boldsymbol{L}^p$-spaces are representatives for a huge class of *Banach function spaces*, namely *solid, rearrangement invariant* spaces, which obviously allow all the time-frequency shift operators to act *isometrically* on them:

$$\pi(\lambda) = M_\omega \circ T_t, \quad \text{for} \lambda = (t, \omega),$$

defined for $t, \omega, x \in \mathbb{R}^d$ by

$$M_\omega(f)(x) = e^{2\pi i \omega \cdot x} \cdot f(x), \quad T_t f(x) = f(x - t).$$

Within this class we have also Lorentz and Orlicz spaces, but the fact that the Fourier transform intertwines time- with frequency shift operators ensures that for example also the Fourier transformed spaces $\mathcal{F}(\boldsymbol{L}^p)$ are isometrically TF-shift invariant.

Minimal and Maximal Space in this Family

It is one of the important facts allowing us to establish a **Banach Gelfand Triple** which is quite universally useful for problems in Fourier Analysis, for the mathematical justifation of engineering principles (impulse response, sampling, Shannon's theorem), or for Abstract Harmonic Analysis, *that the family of isometrically TF-invariant function spaces has a* **smallest member**, namely the *Segal algebra* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, and a **biggest member**, namely the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.
In other words: If $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a Banach space of (tempered) distributions, which has the property $\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \forall f \in B$, then we can ensure that one has continuous embeddings

$$\boldsymbol{S}_0(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{S}_0'(\mathbb{R}^d).$$

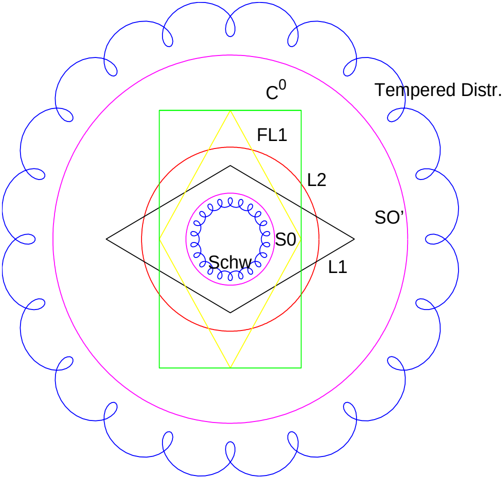
Banach Gelfand Triples: Further Properties

The setting of functions spaces between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ has various additional properties:

- With each spaces $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ also $\mathcal{F}(\boldsymbol{B})$ is in the family;
- With each pair of spaces both the pointwise and the convolution operators between them (the "multiplier spaces") are in the family;
- With each space, also the Wiener Amalgam space $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$ is in the family;
- Modulation spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$, for $s = 0$ are in this family, as inverse FT of $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)$.

Since $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ can be defined (with similar properties) for general LCA we also have a suitable framework for AHA discussions.

The full scenario once more: $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$



Reminding about a few construction principles?

Again referring to [12] (on "Choosing Function Spaces...") I remind you that I was trying to identify the main construction principles, and was able to give a list of **22 such principles**. Among them of course *duality considerations* and various *interpolation methods*.

Please help me to complete the list (there are experts in this room who will certainly be able to come up with 10 other principles).

Let me just mention three less popular ones:

- Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$, (local and global components);
- Coorbit spaces (defined by the behavior of some transform, CWT (cont. wavelet transform), STFT,...);
- Decomposition spaces (like α-modulation spaces), obtained by decomposing the frequency domain;

Characterization of function spaces via series expansions!

While it is very natural to look not only at bases in finite dimensional vector space, but also make use of "*generating systems*", the corresponding principle in the context of (abstract and concrete) Hilbert spaces (such as $\boldsymbol{L}^2(\mathbb{R}^d)$) and Banach spaces (say Besov-Triebel-Lizorkin spaces, or modulation spaces) has been investigated in more detail in the last few decades.
The corresponding (non-orthogonal, but stable) expansion systems are now known as **frames** resp. as **Banach frames**. Frame theory is usually related to the so-called set of *frame inequalities*, but it should be viewed as the situation where the coefficient mapping $f \mapsto (\langle f, g_i \rangle)_{i \in I}$ allows to identify the Hilbert space $\mathcal{H}$ with a closed (hence complemented) subspace of $\ell^2(I)$. **Banach frames** are the natural generalization of this concept, allowing more general Banach spaces of sequences (instead of $\ell^2(I)$).

Banach frames for Besov-Triebel-Lizorkin spaces

Before the "*wavelet wave took off*", i.e. before the discovery (or was it an invention?) of orthonormal wavelet bases, which are now known to be Riesz projection bases for the whole family of classical function spaces (BTL-spaces) the literature had *atomic decompositions* based on the Paley-Littlewood characterization of these spaces.

The series of paper ([25, 26, 27]) described a pair of analyzing and synthesizing functions which together allowed to create a series expansion of all the tempered distributions, with the coefficients in a suitable space if and only if the function itself was in one of the classical spaces. Moreover, the test functions are dense in those spaces if and only if the finite sequences are dense in the corresponding (mixed norm weighted) solid Banach space of sequences.

Families of Function Spaces

One thing which I have certainly learned from Hans Triebel and his extensive publication record is the fact that

Function Spaces come in Families!

Typically such families are closed under (conditional) duality, i.e. with every space also the dual space (of the closure of the test functions) belongs to the same family. Moreover, important families are closed under (mostly complex) interpolation.
A good example are of course the families of $\boldsymbol{L}^p$-spaces, or weighted $\boldsymbol{L}^p$-spaces, the family of Sobolev spaces (derived from $\boldsymbol{L}^2(\mathbb{R}^d)$), with the parameter $s \in \mathbb{R}$, etc..
The same is true for Wiener amalgam spaces, *modulation spaces*, coorbit spaces in general, etc.

Development: from a Historical Viewpoint

Formulated in the language of *everyday life* (which is unfortunately not so widespread used within mathematics!) we can say the following about Function Spaces:

1. At the beginning of the theory (ca. 100 years ago) only few spaces appeared, such as $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for $1 \leq p \leq \infty$, and together with the development of *functional analysis* the role of Banach spaces and in particular Hilbert spaces became clear. The proof of basic properties was not obvious;
2. Meanwhile one can draw from a huge arsenal of general properties which are even taught in (real and functional) analysis courses to graduate students;
3. Interpolation theory allows us to fabricate large families of new spaces from given ones;

Development: New Questions Coming Up

While we started in the last century with an *economy of demand* (where every product that *can be produced* found its *user*), we are currently in a situation of *abundance*.

1. We have a huge variety of "products" (i.e. a multitude of parametrized families of function spaces), so that very few individuals have a complete overview over this huge "zoo"!
2. **It is more important to find information about the usefulness of particular spaces for concrete applications!** So we need a **consumer reports**!
3. Any possible ranking (hopefully not be a single number) has to have clearly defined parameters (similar to the assessment of cars, washing mashines or computers...);

A NEW ASPECT: Performance versus Cost Function

There is another factor known from daily life: It is not only the **performance** of a product which is relevant, but also the **cost factor**. It might be nice to own a fancy car, or buy a fancy tool, but it might be too costly or difficult to obtain. Of course, fans/specialists will go for such "special objects" (sometimes one has the impression: "for any price"!!), but a normal user will be more interested in the *relationship between cost and performance!* A tool which has a wide range of applications, a fair price and which is easy to use will certainly be preferred by the majority of consumers compared to an expensive special tool, or a tool which needs a long period of exercise to learn it. Life is simply too short to concentrate on learning tools, except there is a very high reward for being able to use it skillfully.

Cost Function: Complexity, Time to Learn

Evidently the description of a (or reading about) function space does not *cost money*, but there are other parameters in the scientific reality which contribute to *usability* and *convenience* of use of different types of function spaces.
We think that the following parameters could be relevant for the use concrete function spaces:

1. the difficulty of understanding the concept/defs.;
2. the range of usefulness;
3. the relevance of the application areas;

To have an idea how such concepts apply let us think of concrete examples, such as the Lebesgue spaces $\boldsymbol{L}^p$, $1 \leq p \leq \infty$, or the concept of distributions (with applications in PDE), etc.

How to Satisfy Costumers: Modularity, concepts ...

As a community we have to think not only about the

- invention of new spaces, but also
- a systematic presentation of the different characterizations of these spaces;
- a comparison of their relative position (embedding theorems);
- discussion of possible applications

The goal is not a maximal list of statements about function spaces, but rather a minimal list of crucial statements which are easily combined to practically useful statements.

SUMMARY and Recommendations

Since by now the "Theory of Function Spaces" is a well established area within Mathematical Analysis it is less of a question "how to describe function spaces" or "how to construct new spaces" it is becoming more and more relevant to put new work in this field in the context of the current setting. There are still many open problems and tasks to be performed, but thinking of these tasks as part of an overall program as outlined above may help to gain perspective and context of what is done.

PS: A number of talks on related subjects, in particular on Banach Gelfand triples (in the title) are found at the NuHAG Talk-Server:
`http://www.univie.ac.at/nuhag-php/nuhag_talks/`

SUMMARY: Work to be Done

It is just a quick selection of ideas (not an exhaustive list) of what kind of tasks can be performed:

- Description of (?really) new functions spaces;
- Comparison of existing spaces (optimal embedding or non-inclusion results);
- (real world) applications of function spaces (!big demand, if it comes to rather specific spaces);
- boundedness (and/or invertibility) of given (families) of operators on suitable function spaces;
- atomic characterizations, Banach frames, etc.

What about the usefulness for specific function spaces?

The *standard approach* to function spaces is to check, whether they are well defined, what their basic properties are, how to characterize them in different ways, and what their dual is. When we have *parametrized spaces* (such as the families $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ or $(\boldsymbol{F}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{F}^s_{p,q}}))$ it makes sense to verify, that different parameters correspond indeed to different spaces, i.e. equality of spaces occurs if and only if the parameters are equal (and between families if and only if $p = 2 = q$). However, one should not forget to study possible applications, to find out which characterization is most suitable for a simple proof of an interesting result. But if the spaces are too complicated or the operator too specific, maybe completely artificial, one *should not talk about applications!*.

Banach Gelfand triple (iso)morphisms

Once we have Banach Gelfand triples, consisting of a triple of Banach spaces,

$$(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\mathcal{H}, \|\cdot\|_{\mathcal{H}}) \hookrightarrow (\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$$

it appears to be clear, what a *morphism* is (so that we have a *category* of objects, like compatible pairs in interpolation theory). The important observation was (a while ago) that one has to add the w^*-w^*-continuity of a bounded linear operator from $\boldsymbol{B}_1'$ to $\boldsymbol{B}_2'$, to get a "nice theory" (and a kind of reflexivity, since the dual space of $(\boldsymbol{B}', w^*)$ is just $\boldsymbol{B}$ itself).
We use the terminology of a *unitary BGTR-isomorphism* for operators T which are unitary operators at the Hilbert space level, and are BGT-isomorphisms in general (i.e. restrict nicely to the small spaces, typically test-functions, and extend, uniquely in the sense of w^*-extensions, to $\boldsymbol{B}'$).

Applications: within and outside of Analysis

Fortunately there is a long list of such applications. Due to my education (as abstract harmonic analyst) these applications have nothing to do with PDE, but on the other hand a lot with real world applications in the engineering domain, and of course "within mathematics", and (via *coherent states*, i.e. time-frequency shifted Gaussians as "continuous frame"), also with theoretical physics (quantum mechanics), where one finds a large variety of "interesting integral representations" and claims.

Usefulness of Concrete Function Spaces

I will concentrate on three families of spaces:

1. **Wiener amalgam spaces: [7]**
   Shannon't theory for $L^p_w(G)$, iterative algorithms for irregular sampling, Coorbit theory (Wiener amalgams over LC groups)([13]);
2. **spline-type space** (cubic splines, shift invariant spaces), robustness results for minimal norm interpolation ([23]);
3. **Modulation spaces** have become a standard tool for time-frequency analysis: pseudo-differential operators, spreading functions for the discussion of slowly varying channels ($>$ patents!) ([10, 11]);
4. **The Banach Gelfand Triple** (S_0, L^2, S_0'):
   $>$ wide range of applictions, e.g. Gabor Analysis ([4]);

Usefulness of $S_0(\mathbb{R}^d)$ for Classical Fourier Analysis

The most useful single space appears to be the Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, which coincides with the modulation space $\boldsymbol{M}^1(\mathbb{R}^d)$, consisting of all $\boldsymbol{L}^2$-functions with *integrable STFT*.

As F. Weisz has shown all the *classical summability kernels* are in this class, and all functions in this class make up good summability kernels! (see e.g. [22]).
The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ appears to be the right space for the validity of *Poisson's formula* (which may fail, see [33]).

Using $\boldsymbol{S}_0(\mathbb{R}^d)$ one can also show how to *approximate the $\hat{f}$ starting from samples of f*, via the FFT and quasi-interpolation ([34, 19]).
It can be used to define and discuss in a distributional setting (over LCA groups) *generalized stochastic processes* ([32, 17]).

Usefulness of (S_0, L^2, S_0') for Fourier Analysis

The Banach Gelfand Triple is immensely useful for the description of all kinds of results related to classical, but also abstract or numerical harmonic analysis (obviously one needs distribution theory to describe simultaneously discrete and continuous, periodic and non-periodic "signals") (see [3]: unified signal analysis).
The w^*-convergence allows to explain properly (and mathematically correct) how the Fourier transform arises (as a w^*-limit) from the classical theory of Fourier series. If we periodize an L^1-function more and more course, the Fourier transform (in the S_0'-sense) gets supported on a refined lattice ...
This setting should be digestible to engineers, and I am working on a presentation of the material which is both mathematically sound and helpful for the engineers, using such formulas:

$$\int_{\mathbb{R}} e^{2\pi i s t} ds = \delta_0 .$$

Usefulness of $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ for Gabor Analysis

Obviously $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ (even for general LCA groups, where it can replace the Schwartz-Bruhat space) are very useful within TF-analysis, more specifically in Gabor analysis, which can be seen as the theory of reconstruction of a spectrogram (or STFT) from its *regular or irregular samples*, i.e. from the sampling values along a lattice $\Lambda = \boldsymbol{A}(\mathbb{Z}^{2d})$.
Equivalently we are asking, wether the *Gabor family* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a frame, resp. a BGT-frame for the BGT triple.
The answer is quite general and nice: If the frame operator is invertible as operator on $\boldsymbol{L}^2(\mathbb{R}^d)$ it is also invertible as $BGT-morphism$, and hence the *dual atom* $\tilde{g} := S^{-1}(g)$ also belongs to $\boldsymbol{S}_0$ if $g \in \boldsymbol{S}_0$ (see [29], a variant of Wiener's lemma).

Usefulness of (S_0, L^2, S_0') for Gabor Analysis ctd.

Modern Gabor analysis, which includes considerations of stability and robustness of the Banach frame expansions obtained via time-frequency analysis is using $\boldsymbol{S}_0(\mathbb{R}^d)$ (or weighted variants of this space, with STFT in some weighted $\boldsymbol{L}^1$-space over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$) in many places, e.g.

1. the fundamental identity of of Gabor analysis ([21]);
2. for the proof of the continuous dependency of the dual window on the window and the lattice ([18]);
3. for the approximate computation of dual Gabor windows ([15]);
4. for a discussion of Sjöstrand classes of pseudo-differential operators ([28, 30, 5];
5. Gabor multipliers, best approximation ([9, 16, 20]);

There is also a KERNEL THEOREM!

Theorem (Kernel Theorem for operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$)

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x, y)f(y)dy$$

with the understanding that $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ acts as:

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x, y)g(x)f(y)dxdy.$$

This is the "outer shell" of a Gelfand triple isomorphism.

The KERNEL THEOREM refined

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ *if and only if the corresponding operator* K *maps* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *into* $\boldsymbol{S}_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* w^*-*topology into the norm topology of* $\boldsymbol{S}_0(\mathbb{R}^d)$*.*

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ and the BGT of operator spaces

$$(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')).$$

Recent Books by Hans Triebel

We all know that Hans Triebel was all his life "writing books", and obviously he is continuing to do so, as the following publications show (let us put them here "for the records")
[31, 45, 46, 47, 48, 49, 50].

We all wish him that his health would allow him to pursue this activity, the willingness and ability to demonstrate his sharp mind and immense expert knowledge in a field, which he has truly shaped (as a writer and as a teacher) also in the years to come, and that it may continue to bring joy in his life, which is truly dedicated to "function spaces".

ALL THE BEST for the 80th birthday!

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