An Alternative Approach to Convolution and the Fourier Transform

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We are living in a time of changes!

I think it is easy to agree that we are living in a time of **changes**, be it our mode of work (using the internet via various devices), our mobility, our communication behaviour.
And how much has sience/mathematics changed? Should it change at all?? Why and how could it change to OUR best???

Think of mobility. Being mobile meant for a long time "owning a car". One had the choice between Otto motor and Diesel engine. But what it is now. We have (e.g. here in Copenhagen) bikes, E-bikes, good public transportation, and above all E-Cars (TESLA), without a central motor and all the technicalities (gears, etc.) required by this, but with completely new possibilities (and challenges, e.g. battery life, etc.).
We have seen automobile industry going to high standards, no question, but do we need it in this form, in the long run?

A WORD OF ORIENTATION!

Central questions in **Harmonic Analysis** are connected with properties of a variety of *Banach spaces of (generalized) functions*, bounded operators between them, but also Banach algebras, e.g. $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ with respect to convolution, or intertwining operators. In many cases one has by now rather good knowledge concerning unconditional bases for such spaces, or at least Banach frames or atomic decompositions.
Having a sufficiently broad basis in this field allows to ask (and answer!) more interesting questions (sometimes with less effort) compared to a mindset where "classical spaces are given and sacrosanct" (e.g. $\boldsymbol{L}^p$-spaces only), see [1, 3].

The Landscape of Function Space Theory

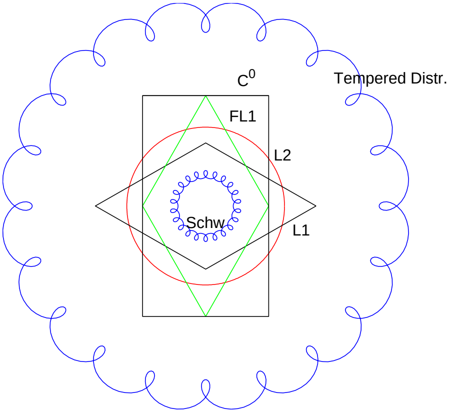
The territory of "**function spaces**" is vast, and even the term itself is subject to quite different interpretations. We would like to understand it in the spirit of Hans Triebel's "Theory of Function Spaces", which means Banach spaces of functions or (usually tempered) distributions (maybe ultra-distributions). Many of these function spaces have been introduced to allow a clean description of certain operators.

Function spaces are prototypical objects in *functional analysis* and many general principles have been first developed in the context of function spaces, while on the other hand the abstract principles of linear functional analysis can be quite nicely illustrated by applying them to (new and old) function spaces.

A listing of examples would be another talk.

The Landscape of Function Space Theory, II

Over the years I have developed a *"symbolic language"* for the different function spaces which should help to better understand the relative inclusion relations.



Choice of Spaces and Criteria

In my article [1] entitled "Choosing function spaces..." I argue, that - similar to real life - those function spaces which serve a purpose, which can be shown to be useful in different situations, the ones which are easy to use and/or help to derive strong results will gain popularity, should be taught and studied more properly, then those who are just "fancy" or which "can be constructed", because at the end the possible gain of using a very complicated function space to derive a statement which in practice is almost impossible to be applied is very modest.

Of course it is a *long way from the suggestion to discuss criteria of usefulness* comparable to what is in real-life a *consumer report*, but I am convinced that such an approach is important for a healthy development of the community. It will help us to keep contact with any kind of applications, and reduces the risk of abstract and finally complicated but close to useless theorems.

Good reasons to introduce the Lebesgue integral

Remember, that $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is the Banach space of all *Lebesgue* integrable function with the norm

$$\|f\|_1 := \int_{\mathbb{R}^d} |f(x)|dx$$

.

Well, strictly speaking we have to talk about *equivalence classes of measurable functions modulo functions vanishing only outside of some set of measure zeros*! We also "need it" in order to define the *convolution of two functions* via

$$f * g(x) := \int_{\mathbb{R}^d} g(x-y)f(y)dy, \quad f,g \in \boldsymbol{L}^1(\mathbb{R}^d).$$

Well, $f * g$ is defined in $\boldsymbol{L}^1(\mathbb{R}^d)$, since the integral only needs to exist a.e., but fine. We have $\|f * g\|_1 \leq \|f\|_1 \|g\|_1$.

Good reasons to introduce the Lebesgue integral II

Obviously one need Lebesgue integrability of $|f(t)e^{-2\pi i\omega\cdot t} = |f(t)|$ in order to define the (forward) Fourier tranasform

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i\omega\cdot t}\, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t\cdot\omega}\, d\omega, \tag{2}$$

Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$, which is *not* satifsied for arbitrary functions $f \in \boldsymbol{L}^1(\mathbb{R}^d)$. In the general case ($f \in \boldsymbol{L}^1$) one can obtain f from $\hat{f}$ using classical summability methods, convergent in the $\boldsymbol{L}^1$-norm. (cf. for example Chap. 1 of [5]).

Good reasons to introduce the Lebesgue integral III

Of course, the good properties of the Lebesgue integral help to introduce the *Hilbert space* $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ of "signals of finite energy", and is the basis to show the validity of *Plancherel's Theorem* which states:

Theorem

The Fourier transform can be adapted (in a way to be explained separately) to a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, *i.e. a linear and isometric bijection.*

The trouble here is, that in both directions one may have problems with the idea of a pointwise (a.e.) existing integral transform, because there are (important) functions in $\boldsymbol{L}^2(\mathbb{R}^d) \setminus \boldsymbol{L}^1(\mathbb{R}^d)$, as the SINC-function (Fourier transform of box-car function).

Goals of my presentation

In this talk I would like to indicate that it is possible to start *Harmonic Analyis* (even Abstract Harmonic Analysis over LCA = Locally Compact Abelian groups) from scratch **without Lebesgue integration**, and get surprisingly far with such an approach, which is in fact inspired by engineering terminology and thinking.

The separation of existing literature on Fourier Analysis into mathematical literature (using Lebesgue integration and proper version of distribution theory, specifically the Schwartz space) and the engineering literature, with similar words, but a completely different way of thinking and arguing is striking and could be the topic of another presentation!

Goals of my presentation

Quick recall of notations:
The **translation operator** T_z defined on arbitrary functions over an arbitrary *locally compact Abelian (LCA) group* G, mostly $G = \mathbb{R}^d$ for us, is defined by

$$T_z f(x) := f(x - z), \quad x, z \in G.$$

Of course $\mathrm{supp}(T_z f) = z + \mathrm{supp}(f)$, i.e. for positive $z \in \mathbb{R}$ the movement is to the right, and for negative $z \in \mathbb{R}$ to the left. Also for continuous functions the functional δ_v is the point evaluation: $f \mapsto f(u)$. Usually it is called the *Dirac measure* or *Dirac impuls at u*, engineers would often speak of the Dirac "function" (but we definitely try to avoid this view-point)!

Translation invariant linear systems

The main motivation to study convolution and the Fourier transforms for engineering applications comes from the study of so-called "translation invariant systems" via an *impulse response function* (or*convolution kernel*) and the so-called *transfer function*. Here T is a translation invariant operator iff it commutes with all time shifts $T \circ T_x = T_x \circ T$, for all $x \in G$.
Thanks to Plancherel's theorem one can prove this result:

Theorem

A bounded linear operator on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ which commutes with all translation operators $T_x, x \in \mathbb{R}^d$ is of the form

$$\widehat{Tf} = h \cdot \widehat{f}, \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d) \quad or? \quad Tf = \sigma * f, \sigma = \mathcal{F}^{-1}(h)?$$

for a uniquely determined $h \in \boldsymbol{L}^\infty(\mathbb{R}^d)$.

Translation invariant linear systems II

Most of the time the domain of such operators is not really specified, and one can argue that the most natural choice is $\mathbf{L}^2(\mathbb{R}^d)$. In this case the Fourier characterization is possible in an exact way, but what is the inverse Fourier transform on $\mathbf{L}^\infty(\mathbb{R}^d)$ and in which sense is $T(f)$ "represented" by the convolution?
In other cases there is also a clear answer.

Theorem (Wendel's Theorem)

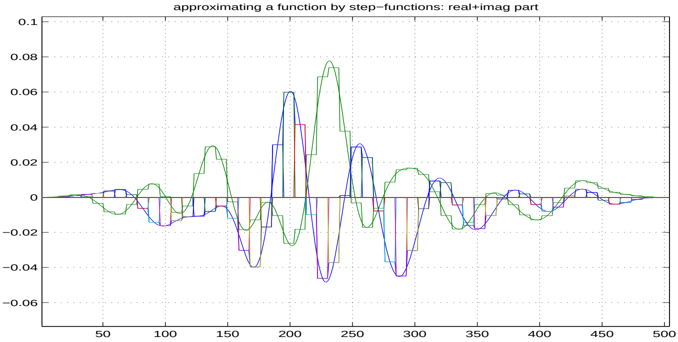
A bounded linear operator on $\left(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ *which commutes with all translation operators* $T_x, x \in \mathbb{R}^d$ *is of the form*

$$T(f) = \mu * f, \quad \forall f \in \mathbf{L}^1(\mathbb{R}^d),$$

for a uniquely determined (impulse response) $\mu \in \mathbf{M}_b(\mathbb{R}^d)$*, i.e. a bounded, regular Borel measure* μ *on* $\mathbb{R}^d$ *(and with a suitable definition of* $\mu * k$*, at least for* $k \in \mathbf{C}_c(\mathbb{R}^d)$*).*

Impulse response by pictures

It is this (and only) this situation where the usual argument shown in engineering books, can be applied. Given a smooth (complex-valued) input signal one approximates it by step functions. *OBSERVING* that the output of T to the box functions (normalized to area one!) have a limit on can show that this is some bounded measure μ (a w^*-limit in $\boldsymbol{M}_b(\mathbb{R}^d)$):



Impulse response by pictures: comments

Unfortunately such presentation of the topic practically never make the assumption explicit that go into this explanation:

- how are the approximations by the step functions determined
- for which input signals can one assume that one has convergence, and in which norm: we could argue: local averages converge in the $\boldsymbol{L}^1$- to $f \in \boldsymbol{L}^1(\mathbb{R}^d)$.
- The assumption, that the operator preserves this kind of convergence is also never justified or made explicit. We could justify it for bounded operators on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
- How can the "observed" fact be justified, that the output sequence to a Dirac sequence (e.g. a sequence of "compressed box-car functions", converging (in the w^*-sense) to the "impulse response", i.e. a Dirac delta δ_0) is also convergent, to "some", call it μ;
- And finally why (and how) is $T(f) = \mu * f$.

The scandal in linear system theory

Not describing the problem in the way I have done above Irving Sandberg describes the so-called "scandal in system theory" in a series of papers, which I also mention in more detail in my paper [2], e.g. [6, 7].
He models translation invariant linear systems (so-called TILS) as bounded linear operators on $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (with the sup-norm), the space of (complex-valued) bounded and continuous function on $\mathbb{R}^d$ (of course the scandal appears in all dimensions) and demonstrates (using the axiom of choice resp. Hausdorff's maximality principle, implying the existence of translation invariant means) that there are bounded linear operators which cannot be represented by convolution operators with a bounded measure, in fact not even with any distribution!
But (as we all know) $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is NOT a separable Banach space. Hence I suggest to use $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$!

Translation invariant operators are convolution operators A

I can now follow roughly the key points of my script.
First the definitions:

Definition

$\boldsymbol{C}_c(\mathbb{R}^d) := \{f : \mathbb{R}^d \to \mathbb{C}$, continuous and with compact support$\}$

Definition

The *support* of a continuous (!) function, in symbols $\mathrm{supp}(f)$, is defined as the closure of the set of "relevant points": [a]

$$\mathrm{supp}(f) := \{x \,|\, f(x) \neq 0\}^{-}$$

[a] The superscript bar stands for "closure" of a set. Hence $\mathrm{supp}(f)$ is by definition a closed set.

Translation invariant operators are convolution operators B

Definition

$$C_b(\mathbb{R}^d) := \{f : \mathbb{R}^d \mapsto \mathbb{C}, \text{ continuous and bounded,}\}$$

endowed with norm

$$\|f\|_\infty = sup_{x\in\mathbb{R}^d}|f(x)|$$

The closure of $C_c(\mathbb{R}^d)$ in $\left(C_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is just $C_0(\mathbb{R}^d)$ which can be defined as

$$C_0(\mathbb{R}^d) := \{f \in C_b(\mathbb{R}^d), lim_{|x|\to\infty}|f(x)| = 0\}.$$

Obviously this is a separable space, because the piecewise linear functions over nodes of the form $2^{-k}\mathbb{Z}, k \in \mathbb{N}$ with rational coefficients are dense in $\left(C_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (similar for $d > 1$).

Translation invariant operators are convolution operators B

As "convolution kernels" we will need bounded measures, which can be defined easily in a purely functional analytic way.

Definition

We denote the dual space of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Sometimes the symbol $\boldsymbol{M}_b(\mathbb{R}^d)$ is used in order to emphasize that one has "bounded" (regular Borel) measures. As a dual space $\boldsymbol{M}(\mathbb{R}^d)$ carries the natural functional norm

$$\|\mu\|_{\boldsymbol{M}} := \sup_{\|f\|_\infty \le 1} |\mu(f)| = \sup_{\|f\|_\infty = 1} |\mu(f)|.$$

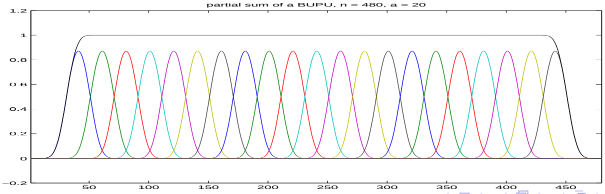
Of course discrete measures are important special case, i.e. $\mu = \sum_{k=1}^\infty c_k \delta_{x_k}$ with $\sum_{k=1}^\infty |c_k| < \infty$. They form a closed (!) subspace of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Definition

A (countable) *family* $\Phi = (T_\lambda \varphi)_{\lambda \in \Lambda}$, where φ is a compactly supported function (i.e. $\varphi \in \boldsymbol{C}_c(\mathbb{R}^d)$), and $\Lambda = A(\mathbb{Z}^d)$ a lattice in $\mathbb{R}^d$ (with $\det(A) \neq 0$) is called a **regular BUPU** if

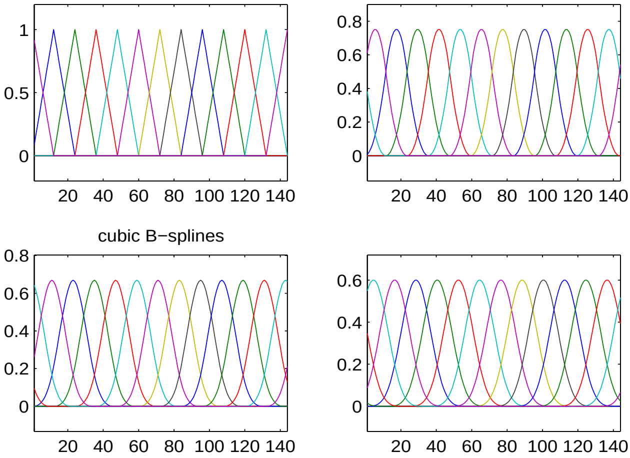
$$\sum_\lambda \varphi(x - \lambda) \equiv 1.$$

We say that $\operatorname{diam}(\Phi) \leq \gamma$ if $\operatorname{supp}(\varphi) \subset B_\gamma(0)$.



Different BUPUs of B-spline type

B-spline BUPUs of order 1 (pcw. linear), 2, 3 (cubic), and 4.



Translation invariant operators are convolution operators D

Two important ways to make use of such BUPUs are crucial for our approach. The first one is a replacement for the $\sigma-$additivity important in the context of measure theory. We formulate it for more general BUPUs $\Psi = (\psi_i)_{ii \in I}$.
We also use the usual convention

$$\mu \cdot h(f) := \mu(h \cdot f), \quad h \in \mathbf{C}_b(\mathbb{R}^d), f \in \mathbf{C}_0(\mathbb{R}^d), \mu \in \mathbf{M}(\mathbb{R}^d).$$

Lemma

Let $\Psi = (\psi_i)_{i \in I}$ be any non-negative BUPU, then

$$\sum_{i \in I} \|\mu \cdot \psi_i\|_{\mathbf{M}(\mathbb{R}^d)} = \|\mu\|_{\mathbf{M}(\mathbb{R}^d)}, \quad \mu \in \mathbf{M}(\mathbb{R}^d).$$

Translation invariant operators as convolution operators E1

BUPUs allow also to approximate bounded measures by discrete measures, in the vague or w^*-sense. Recall:

Definition

A bounded sequence (or also *net*) $(\mu_\gamma)_{\gamma\in\Gamma}$ of bounded measures is convergent to $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$ in the w^*-sense if and only if

$$\lim_\gamma \mu_\gamma(f) = \mu_0(f), \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d).$$

Definition

Given $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ and a BUPU Ψ with $\mathrm{supp}(\psi_i) \subseteq B_\gamma(x_i)$ we define the discretization operator D_Ψ as follows

$$D_\Psi(\mu) := \sum_{i\in I} \mu(\psi_i)\delta_{x_i}.$$

Translation invariant operators as convolution operators E1

Remark: This method of discretization is by far not as *abstract as it may look*. If we would choose a regular B-spline BUPU of of order zero, i.e. a sequence of indicator functions $(\mathbf{1}_{\alpha[\mathbf{k},\mathbf{k}+\mathbf{1})})_{\mathbf{k}\in\mathbf{Zst}}$ the visual representation of $D_{\Psi}(\mu)$ would be nothing else but a *histogram*. Just on our setting one has to think of a slightly smoothed version of a histogram, because we can apply measures only to continuous functions.

Remark: If one describes a probability measure in terms of a distribution function, i.e. an monotonously increasing function F, with $lim_{x\to-\infty}F(x) = 0$ and $lim_{x\to\infty}F(x) = 1$, then the kind of approximation we are looking at is a (pointwise, perhaps a.e.) approximation by increasing step functions.

Translation invariant operators are convolution operators F

Lemma

The net $D_\Psi(\mu)$ *is uniformly bounded, with*

$$\|D_\Psi(\mu)\|_{\boldsymbol{M}(\mathbb{R}^d)} \leq \|\mu\|_{\boldsymbol{M}(\mathbb{R}^d)}$$

and w^**-convergent to* μ *for any* $\mu \in \boldsymbol{M}(\mathbb{R}^d)$*, i.e. Given* $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ *and* $\varepsilon > 0$ *there exists* $\gamma_0 > 0$ *such that for* $0 < \gamma \leq \gamma_0$ *any BUPU* Ψ *with* $\operatorname{diam}(\Psi) \leq \gamma$ *satisfies*

$$|D_\Psi\mu(f) - \mu(f)| < \varepsilon.$$

The net is uniformly tight i.e. for any $\varepsilon > 0$ *there exists* $p \in \boldsymbol{C}_c(\mathbb{R}^d)$*, with* $0 \leq p(x) \leq 1$ *such that*

$$\sup_{\operatorname{diam}(\Psi)\leq 1} \|(1-p)\cdot D_\Psi\mu\|_{\boldsymbol{M}(\mathbb{R}^d)} \leq \varepsilon.$$

Translation invariant operators are convolution operators H

Definition

The Banach space of all “translation invariant linear systems” (TLIS) on $\boldsymbol{C}_0(\mathbb{R}^d)$ is denoted by

$$\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d)) = \{T : \boldsymbol{C}_0(\mathbb{R}^d) \to \boldsymbol{C}_0(\mathbb{R}^d),\ T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\}.$$

It is easy to show that $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is a closed subalgebra of the Banach algebra of $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ (in fact it is even closed with respect to the strong operator topology), hence it is a Banach algebra of its own right (with respect to composition as multiplication). We will see later that it is in fact a commutative Banach algebra.

Translation invariant operators are convolution operators G

Theorem

There is a natural isometric isomorphism between the Banach space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$, endowed with the operator norm, and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$, the dual of $\big(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\big)$, by means of the following pair of mutually inverse linear mappings:

1. *Given a bounded measure $\mu \in \boldsymbol{M}(\mathbb{R}^d)$ we define the operator C_μ (convolution operator with kernel μ) via:*

$$C_\mu f(x) = \mu(T_x f^\checkmark). \tag{3}$$

2. *Conversely we define $T \in \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ the linear functional $\mu = \mu_T$ by*

$$\mu_T(f) = [Tf^\checkmark](0). \tag{4}$$

The claim is that both of these mappings: $C : \mu \mapsto C_\mu$ and the mapping $T \mapsto \mu_T$ are linear, non-expansive, and inverse to each other. Consequently they establish an isometric isomorphism between the two Banach spaces with

$$\|\mu_T\|_{\mathbf{M}} = \|T\|_{\mathcal{L}(\mathbf{C}_0(\mathbb{R}^d))} \quad \text{and} \quad \|C_\mu\|_{\mathcal{L}(\mathbf{C}_0(\mathbb{R}^d))} = \|\mu\|_{\mathbf{M}}. \tag{5}$$

Definition

Recall the notion of a FLIP operator: $\check{f}(z) = f^{\checkmark}(z) = f(-z)$
We can also extend this operator to measures by setting

$$\mu^{\checkmark}(f) = \mu(f^{\checkmark}), \quad \mu \in \mathbf{M}(\mathbb{R}^d), f \in \mathbf{C}_0(\mathbb{R}^d).$$

INTRODUCING CONVOLUTION for measures

Since the space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is not only a Banach space, but also a Banach algebra (under composition, which is obviously associative) and $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is isometrically isomorphic, one can transfer the algebra structure to $\boldsymbol{M}(\mathbb{R}^d)$ by declaring:

Definition

The (new) measure, to be denoted by $\mu_1 * \mu_2$ (a NEW, internal multiplication for $\boldsymbol{M}(\mathbb{R}^d)$) is the unique functional on $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ which generates the composite TILS $C_{\mu_1} \circ C_{\mu_2}$.

$$\|\mu_1 * \mu_2\|_{\boldsymbol{M}(\mathbb{R}^d)} \leq \|\mu_1\|_{\boldsymbol{M}(\mathbb{R}^d)} \cdot \|\mu_2\|_{\boldsymbol{M}(\mathbb{R}^d)}.$$

Also the associative (mixing internal and external action)

$$\mu_1 * (\mu_2 * f) = (\mu_1 * \mu_2) * f, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d)$$

requires NO PROOF but has been turned into a definition.

Convolution with discrete measures: translation operators

In the correspondence of the main theorem it is important to see, what discrete measure do, and it is easy to verify that Dirac measures δ_z correspond to the translation operators, in other words (usual symbols) one has

$$\delta_z * f = T_z f, \quad f \in \boldsymbol{C}_0(\mathbb{R}^d), z \in \mathbb{R}^d.$$

This makes the following result even more useful, showing the approximation of a general TILS on $\boldsymbol{C}_0(\mathbb{R}^d)$ by finite sums of translation operators.

Theorem

Let $T = C_\mu$ be a TILS on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$.

$$D_\Psi \mu * f \to \mu * f, \quad \textit{for } \operatorname{diam}(\Psi) \to 0, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \qquad (6)$$

Approximation of convolution by discrete convolution

Lemma

Assume that $(\mu_n)_{n\geq 1}$ *is a bounded and tight sequence in* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. *Then* $\mu_0 = w^* - \lim_n \mu_n$ *if and only if*

$$\mu_n * f \to \mu_0 * f, \quad \forall f \in \boldsymbol{C}_b(\mathbb{R}^d),$$

uniformly over compact sets. The statement is equally valid if the convergence is valid for all $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ *only.*

It is also possible to show that in this situation $(\mu_1, \mu_2) \mapsto \mu_1 * \mu_2$ is continuous in the w^*-topology, and in particular

$$D_\Psi \mu_1 * D_\Psi \mu_2 \to \mu_1 * \mu_2$$

and consequently discrete convolution (the group law) dictates the form of convolution in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ as defined (in particular commuitativity).

Commutativity of convolution, Fourier Stieltjes transform

This fact also allows to establish commutativity of convolution:

$$\mu_1 * \mu_2 = \mu_2 * \mu_1.$$

The next topic is the definition of the *Fourier (Stieltjes) transform* which in the usual setting would be

$$\hat{\mu}(s) := \int_{\mathbb{R}^d} e^{-2\pi i \langle s,t \rangle} d\mu(t),$$

giving a bounded on continuous function in the frequency domain, with $\|\hat{\mu}\|_\infty \leq \|\mu\|_{\boldsymbol{M}(\mathbb{R}^d)}$.
Denoting the *characters* of $\mathbb{R}^d$ by

$$\chi_s(t) := e^{2\pi i \langle s,t \rangle}, t \in \mathbb{R}^d,$$

we would naturally define $\widehat{\mu}(\chi) := \mu(\overline{\chi}), \chi \in \widehat{G}$, but this is not directly possible, because a priori μ is only defined on $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$, and not (yet) on all of $\left(\boldsymbol{C}_b(G), \|\cdot\|_\infty\right)$.

Extending the action to $\left(C_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$

Another variation of this result is provided by the next proposition. The absolute convergence of the splitting $\mu = \sum_{i\in I} \mu\psi_i$ based on any given BUPU with $\sum_{i\in I} \|\mu\psi_i\|_{\boldsymbol{M}} = \|\mu\|_{\boldsymbol{M}}$ allows to define for any h (in a way which is independent of the BUPU Ψ!)

$$\mu(h) := \sum_{i\in I} \mu\psi_i(h) = \sum_{i\in I} \mu(\psi_i \cdot h),$$

obtaining in this way the unique (and norm-preserving) extension of the functional $\mu \in \left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)'$ which respects uniform convergence over compact sets (for bounded sequences).
The injectivity of the linear mapping $\mu \mapsto \widehat{\mu}$ is then not at all trivial (requiring a form of Stone Weierstrass theorem), but we can prove the convolution theorem on this basis (cf. [2]).

The convolution theorem for bounded measures

Theorem

The Fourier (Stieltjes) transform maps the Banach convolution algebra $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ *into the pointwise Banach algebra* $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$*, i.e.*

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2}, \quad \mu_1, \mu_2 \in \boldsymbol{M}(\mathbb{R}^d).$$

The extension of the action of course also allows to extend the convolution operators to all of $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and then obtain a TILS (but not! *all possible* TILS!) on $\left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right)$.

Theorem

The extended convolution operator $C_\mu : h \mapsto \mu * h, h \in \boldsymbol{C}_b(\mathbb{R}^d)$ *has the same norm as the operator on* $\boldsymbol{C}_0(\mathbb{R}^d)$*. One the has only* $D_\Psi \mu * h \to \mu * h$*, uniformly over compact set.*

Characters as eigenvectors for convolution operators

But now one can express the fact that the characters (pure frequencies) are joint eigenvectors to all of these (commuting!) convolution operators, with

$$\mu * \chi = \widehat{\mu}(\chi)\chi, \quad \chi \in \widehat{G}.$$

In this sense one can see the Fourier transform as the natural (and in the sense of the Gelfand transform, applied to the convolution algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$) *joint diagonalization* or *spectral representation* of this Banach algebra.
The verification of the convolution theorem combines the fact that it is almost trivial (by the defining properties of characters):

$$\chi(x+y) = \chi(x)\chi(y), \quad x, y \in G,$$

for discrete measuresn and can be pushed to general statements by the convergence just mentioned (for $h = \chi$).

How to introduce $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ in this setting ?

Note: For $k \in \boldsymbol{C}_c(G)$ on can prove that $\mu_k : f \mapsto \int_G f(x)k(x)dx$ is an element of $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ and that

$$\|\mu_k\|_{\boldsymbol{M}} = \|k\|_1 = \int_G |k(x)|dx.$$

Hence $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ "is" the closure of $\mu_{\boldsymbol{C}_c(G)}$ in $\boldsymbol{M}(\mathbb{R}^d)$.

Theorem

$\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ can be characterized within $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ as the subspace of elements with continuous shift, i.e.
$\mu \in (\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ belongs to $\boldsymbol{L}^1(\mathbb{R}^d)$ if it satisfies

$$\lim_{x \to 0} \|T_x\mu - \mu\|_{\boldsymbol{M}} = 0.$$

In particular, $\boldsymbol{L}^1(G)$ is a closed ideal within $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$, and the restriction of the Fourier transform to $\boldsymbol{L}^1(G)$ maps $\boldsymbol{L}^1(G)$ injectively into $(\boldsymbol{C}_0(\widehat{G}), \|\cdot\|_\infty)$ (Lemma of Riemann-Lebesgue).

Practical remarks

The approach presented also allows to show, avoiding the theory of integration with values in a Banach space (i.e. Bochner integration) that $\boldsymbol{M}(\mathbb{R}^d) * \boldsymbol{B} \subset \boldsymbol{B}$, for any so-called *homogeneous Banach space* $\boldsymbol{B}$ (such as $\boldsymbol{L}^p$, for $1 \leq p < \infty$).
It allows allows to explain how to otain the *integrated group representation* on a Banach space from an isometric, strongly continuous group representation or even projective representation of this type!
The approach presented here has been tested already few times in courses (for master students) at the University Vienna in the last years and I was mostly satisfied with the (oral) exams with the students, see

http://www.univie.ac.at/nuhag-php/login/skripten/data/AngAnal15Skript.pdf

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