GABOR ANALYSIS: Mathematical Overview with some Computational Aspects

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu
Otto Moenstedt Professor at DTU Copenhagen

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OVERVIEW

The GOAL of this presentation is to provide a basic understanding of Gabor Analysis, which is an important branch of time-frequency analysis.
We will indicate what kind of problems have to be addressed, which operators are of interest (e.g. Gabor multipliers) and which function spaces play a role in this context (namely *modulation spaces*, and in particular the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$).
We also have to talk about Gabor frames and Gabor Riesz sequences (bases for closed subspaces), i.e. not-so-usual concepts of functional analysis. Finally we will mention the corresponding numerical and computational questions and their realization in MATLAB, and the questions (about approximation quality) arising from the simulations.

Personal Background

I was doing first *Abstract Harmonic Analysis* for many years (advisor Hans Reiter, in Vienna), working with *function spaces* over locally compact (Abelian) groups, convolution, Fourier transforms (A. Weil, H. Triebel, J. Peetre,...)

Later I realized the close connections to signal processing, mostly through P. Butzer (sampling, Shannon's theorem, etc.) and also got interested in doing actualy numerical work (starting in 1989, J.J. Benedetto's group), which lead in 1992 to NuHAG.

In the 80th I got into contact with communication engineers (F. Hlawatsch, W. Kozek) who pointed out to me many connections between the abstract world and applications (filter banks). In particular: I had done "atomic decompositions" for the function space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, which can be interpreted as a simple form of Gabor expansions (D. Gabor, 1946, [4]).

How to present Gabor Analysis?

There are certainly many ways of presenting Gabor Analysis:

1. We could present it in the framework of Hilbert spaces (really: I am talking about "frames for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$!);
2. We could talk about Gabor analysis for discrete signals or pixel images first, avoiding some some troubles arising with continuous signals;
3. We could/will even reduce it further to finite Abelian groups, where we have unit vectors (instead of Dirac measures), and matrices instead of linear operators;
4. We can first provide some intuition via examples and pictures
5. We could give a course on the function spaces needed and the available results;

Since this is all to much we will mix these aspects!

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

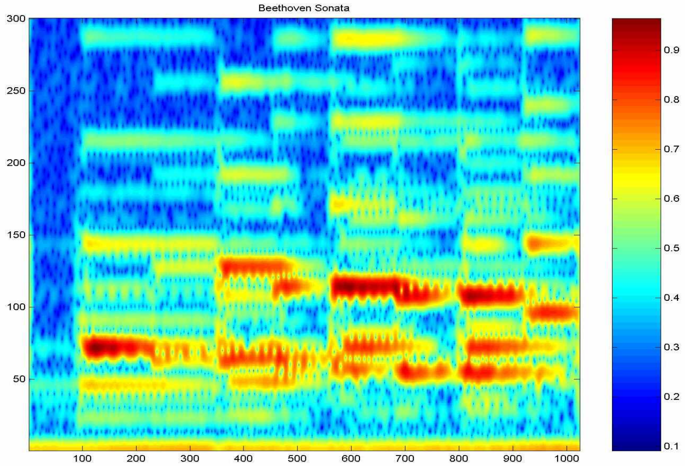
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

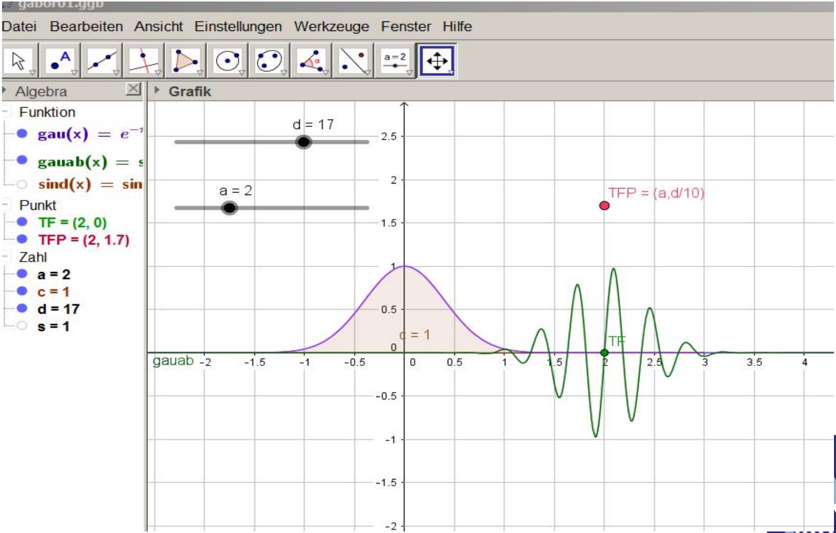
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

A series of questions

1. Which windows are useful for which purpose?
2. How dense does one have to sample, in order to reconstruct approximately/perfectly the *signal* f from the spectrogram?
3. What are the numerical procedures for perfect reconstruction?
4. What are the properties of general Gabor systems at all: $G = (\pi(\lambda)g)_{\lambda \in \Lambda}$, for some $\Lambda = \boldsymbol{A}(\mathbb{Z}^{2d})$, with $\det(\boldsymbol{A}) \neq 0$?

In a "modern interpretation" of his paper of 1946 D. Gabor was expressing the hope that for $\Lambda = \mathbb{Z}^{2d}$ the system could be a Riesz basis, with unique expansions of every $f \in \boldsymbol{L}^2(\mathbb{R}^d)$, thus giving the coefficients a particular meaning. They (would) describe the time-frequency content of f at the points of Λ.

The critical case: shortcomings!

As a more detailed mathematical analysis showed (only in the 80th, mostly by A.J.E.M. Janssen, [7]) the situation cannot be "saved", even in the context of distribution theory. There are various problems (which in fact have obscured the use of Gabor analysis for a long time, both practically and theoretically).

So let us describe the main problems of Gabor's system $(g_0, \mathbb{Z}^{2d})$:

1. Although the finite linear combinations are dense in $\boldsymbol{L}^2(\mathbb{R}^d)$ it is not possible to have a series representation of arbitrary $f \in \boldsymbol{L}^2(\mathbb{R}^d)$. Better approximation of f may require new sets of coefficients, or larger $\ell^2(\mathbb{Z}^{2d})$-norm.
2. Even allowing just bounded coefficients in $\ell^\infty(\mathbb{Z}^{2d})$ would not allow (distributional) series expansions of all the $\boldsymbol{L}^2(\mathbb{R}^d)$-functions, but already create non-uniqueness of representation! (chess-board signs represent zero).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \, \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (1)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (2)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h \rangle = \langle V_g^*(V_g(f)), h \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)} \, d\lambda. \quad (3)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (4)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \, \chi_s \, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (5)$$

with $\chi_s(t) = exp(2\pi i \langle s, t \rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (5) (Fourier inversion) and the new formula formula (4). While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [8]).

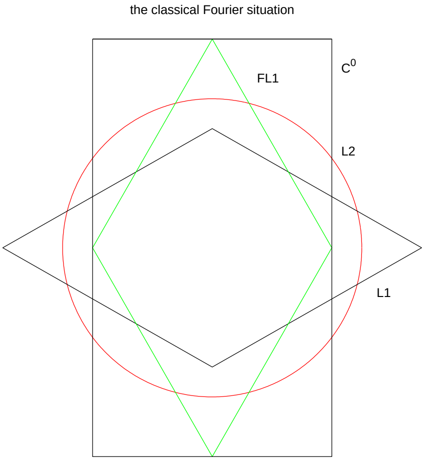
A number of questions

So we have a number of questions:

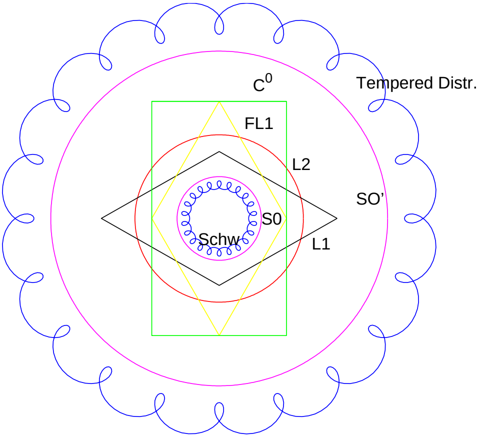
- What is the maximal grid size for a given (e.g. Gaussian window)?
- Which windows are 'good windows'?
- How can one stably reconstruct f from the sampled STFT $(V_g(f)(\lambda))_{\lambda\in\Lambda}$.
- Furthermore: How can one operate on the Gabor coefficients, i.e. the study of so-called **Gabor multipliers**!

The first answer can be given right away: For $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, with $0 < ab < 1$ one has stable reconstruction. Similar statements have been found only very recently (K. Gröchenig, J. Stöckler) for a class of *totally positive* functions ([6]).

A schematic description of the situation: *Lisp*, *Ltsp*

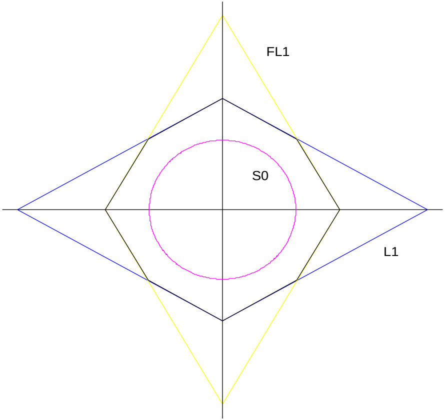


A better schematic description of the situation

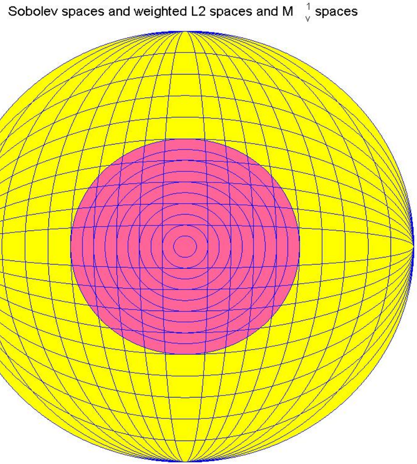


A better schematic description of the situation

The domain of the Fourier inversion formula: $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1$:



A better schematic description of the situation



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $W(\mathcal{F}L^1, \ell^1) = M^0_{1,1}(\mathbb{R}^d)$.

Basic properties of $M^\infty(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$

It is probably no surprise to learn that the dual space of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, i.e. $S_0'(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. As an amalgam space one has $S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)' = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in S_0(\mathbb{R}^d)$).
Consequently norm convergence in $S_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain atomic characterizations of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT. – Hifi recordings!

Important messages concerning $(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0})$

The space (of test functions) has all the good properties one knows from $\mathcal{S}(\mathbb{R}^d)$, but it is a Banach space.
It is Fourier invariant (even under the fractional Fourier transform), it is contained in all the $\boldsymbol{L}^p$-spaces, for $1 \leq p \leq \infty$ (and contains $\mathcal{S}(\mathbb{R}^d)$ as a dense subspace).
All the classical summability kernels (used for the Fourier inversion theorem) are in this class (this is why they are useful), and also Poisson's formula is valid in the strict sense

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad \forall f \in \mathbf{S}_0(\mathbb{R}^d). \tag{6}$$

Important messages concerning $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$

The Banach space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is the *largest* Banach space of distributions which is isometric under TF-shifts, hence contains all the $\boldsymbol{L}^p$-spaces. It is Fourier invariant via $\hat{\sigma}(f) := \sigma(\widehat{f}), f \in \boldsymbol{S}_0(\mathbb{R}^d)$.

It also contains any Haar measure of a subgroup, in particular $⊔⊔_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$, for an arbitrary discrete subgroup $\Lambda \lhd \mathbb{R}^d$ (often called Dirac Comb). Moreover $\mathcal{F}(⊔⊔_\Lambda) = C_\Lambda \cdot ⊔⊔_{\Lambda^\perp}$.

Since pointwise multiplication goes to convolution this provides a proof of the fact that "sampling on the time side" $(f \mapsto f⊔⊔_\Lambda)$ corresponds to periodization of $\widehat{f}$ on the Fourier transform side $(\widehat{f} \mapsto ⊔⊔_{\Lambda^\perp} * \widehat{f})$.
It allows to define spec(f), for $f \in \boldsymbol{L}^\infty$, as supp$(\widehat{f})$.

Further properties of $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ II

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ if and only its spectrogram for one non-zero window/atom $g \in S_0(\mathbb{R}^d)$ is bounded (and then for all such windows, due to the atomic characterizations of $S_0(\mathbb{R}^d)$).

Norm convergence in $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ corresponds to *uniform convergence* of the spectrograms.

The w^*-convergence in $S_0'(\mathbb{R}^d)$ corresponds to *uniform convergence over compact subsets of the TF plane*. A good HiFi recording covers the range of $0 - 20kHz$ for the duration of a song!

Elements of $S_0'(\mathbb{R}^d)$ have a *support* and a *Fourier transform*. Spectral synthesis implies that a distribution supported by a subgroup "comes from the subgroup" (adjoint of restriction).

Analogies to the Schwartz space

For most applications (!!except for PDEs, see Hörmander Theory) $\boldsymbol{S}_0(G)$ is a more simple space than the Schwartz-Bruhat space, also defined over general LCA (locally compact Abelian groups).

First of all it is Fourier invariant, resp. $\mathcal{F}(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0'(\tilde{G})$ for LCA groups.
One can regularize distributions from $\boldsymbol{S}_0'(\mathbb{R}^d)$ using *Wiener amalgam* convolution and pointwise multiplier results:

$$\boldsymbol{S}_0 \cdot (\boldsymbol{S}_0' * \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0, \quad \boldsymbol{S}_0 * (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0 \tag{7}$$

Although it is NOT a *nuclear Frechet space* there is a kernel theorem, which extends the usual kernel theorem for Hilbert Schmidt operators (which are exactly the operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ with *integral kernels* in $\boldsymbol{L}^2(\mathbb{R}^{2d})$!

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\boldsymbol{B} = \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$.

Theorem

There is a natural BGT-isomorphism between $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$. *This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. *Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators* $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

BANACH GELFAND TRIPLES: a new category

Definition

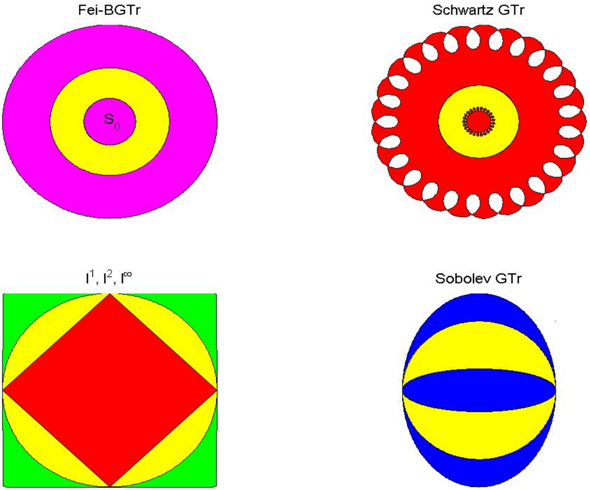
A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then automatically w^*-w^*--continuous!

Various Gelfand Triples



Banach Gelfand Triples, ctc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{U}) = \boldsymbol{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The BGT (S_0, L^2, S_0') and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.

For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the "ax+b"-group) on finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler's formula) to cos and sin functions in order to build a Wilson ONB for $\boldsymbol{L}^2(\mathbb{R}^d)$.

In this way another BGT-isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.

Guide to the literature

Most of our relevant papers at NuHAG are found at

`www.nuhag.eu/bibtex`

A good survey about the state of the art in Gabor analysis around 2000 is given in Charly Gröchenig's book [5], or [3].
The purely algebraic part of Gabor analysis is described in the paper [2]. The linear algebra aspects (overcomplete systems, etc.) is fully described in [1].

The role of $\mathbf{S}_0(\mathbb{R}^d)$ for Gabor Analysis

We will call $(\pi(\lambda)g)_{\lambda\in\Lambda}$ a Gabor family with Gabor atom g.

Theorem

Given $g \in \mathbf{S}_0(\mathbb{R}^d)$. Then there exists $\gamma > 0$ such that any γ-dense lattice Λ (i.e. with $\cup_{\lambda\in\Lambda} B_\gamma(\lambda) = \mathbb{R}^d$) the Gabor family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame. Hence there exists a linear mapping (the unique MNLSQ solution) $f \mapsto (c_\lambda) = \langle f, \tilde{g}_\lambda \rangle, \lambda\in\Lambda$, for a uniquely determined function $\tilde{g} \in \mathbf{S}_0(\mathbb{R}^d)$, thus

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda, \quad \forall f \in \mathbf{L}^2(\mathbb{R}^d).$$

In other words, the minimal norm representation of any $f \in \mathbf{L}^2(\mathbb{R}^d)$ can be obtained by just sampling the STFT with respect to the *dual window* $\tilde{g}$.

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

The dual Gabor atom $\tilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ provides not only the minimal norm coefficients, but also $\ell^1(\Lambda)$-coefficients for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and is well defined on $\boldsymbol{S}_0$, $\sigma \mapsto \sigma(\tilde{g}_\lambda)$ and defines representation coefficients in $\ell^\infty(\Lambda)$.
So in fact $f \mapsto (\langle f, \tilde{g}_\lambda \rangle)$ defines a Banach Gelfand triple morphism from the triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ to $(\ell^1, \ell^2, \ell^\infty)$. The (left) inverse mapping is the synthesis mapping

$$(c_\lambda) \mapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda,$$

with norm convergence for $c \in \ell^1$ or ℓ^2, and still w^*-sense in $\boldsymbol{S}_0'(\mathbb{R}^d)$ for $c \in \ell^\infty(\Lambda)$.

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

The good properties of g and $\tilde{g}$ also imply that not only is the range of spaces to which the Gabor expansions can be applied (more generally one could talk about the modulation spaces $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$), but also a certain robustness with respect to the choice of the Gabor atom.
Similar (in the sense of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$) atoms g have similar *dual windows*, but we also have a continuous dependence of $\tilde{g}$ on the lattice: If (g, Λ) with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $\Lambda = \boldsymbol{A}(\mathbb{Z}^{2d})$ define a Gabor frame with dual atom $\tilde{g}$, then for all lattice $\Lambda' = \boldsymbol{B}(\mathbb{Z}^{2d})$ with $|||\boldsymbol{A} - \boldsymbol{B}||| < \delta$ the families (g, Λ') also define Gabor frames and $\widetilde{g}'$ (dual with respect to Λ') are close to $\tilde{g}$.
This principle is important for the approximation of irrational lattices by rational lattices, because these are better suited for discrete and finite-dimensional approximation.

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

As one changes the lattices one observes the following fact: As one approximates the continuous case (the density of the lattice is getting higher and higher, in a uniform way), the (a normalized) version of $\tilde{g}$ tends (in the $\boldsymbol{S}_0$-norm) to g, i.e. for the *highly redundant case* one does not have to make any distinction between $\tilde{g}$ and g. Acoustic experiments speak of a redundancy of ca. $5-6$ as the change-point.

As the density (size of the fundamental domain of Λ, or $\det(\boldsymbol{A})$, i.e. ab for the case $\Lambda = a\mathbb{Z} \times b\mathbb{Z}$) tends to the critical case, the quality of $\tilde{g}$ deteriorates and we know a lot about the speed of deterioration.

For the case $a = 1 = b$ (Gabor's case) no such $\tilde{g}$ exists, not even in all of $\boldsymbol{L}^2(\mathbb{R}^d)$.

An important property of $S_0(\mathbb{R}^d)$ is the fact that one can compute an S_0-approximation of $\tilde{g}$ for $q_1\mathbb{Z} \times q_2\mathbb{Z}$, $q_1, q_2 \in \mathbb{Q}$, using just (fine regular) samples of g, by solving a positive definite operator equation of the form $S(\tilde{g}) = g$, where now g, gd are the sampled version of the true continuous Gabor atom g and gd respectively (at least up to some small error, which tends to zero, as the sampling rate tends to zero!)
Having the approximate samples of $\tilde{g}$ over grid it suffices to apply piecewise linear interpolation (or better quasi-interpolation, using cubic B-splines, i.e. form $\sum_{n\in\mathbb{Z}} f(\alpha n) T_{\alpha n}\psi$).

REMARK: The same is true for computing $\widehat{f}$ from the samples of f using just the discrete version of the Fourier transform, namely the FFT, followed by piecewise linear interpolation!

$S_0(\mathbb{R}^d)$ and Generalized Stochastic Processes

The joint generalization of stochastic processes (mapping from points to a Hilbert space $\mathcal{H}$ of probability measures) and distribution theory (linear mappings from a space of test functions to the complex numbers $\mathcal{H} = \mathbb{C}$) is of course the concept of *Generalized Stochastic Processes*, viewed as linear operators from $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ to a Hilbert Space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$.
Such a theory has been developed together with my PhD student Wolfgang Hörmann a while ago.
Key points are the existence of a *Fourier transform* of a process (the *spectral process*), a spectral representation, the existence of an autocorrelation distribution in $\boldsymbol{S}_0'(G \times G)$. The autocorrelation of the spectral process is the 2D-Fourier transform (in the $\boldsymbol{S}_0'$-sense) of the autocorrelation of the process.

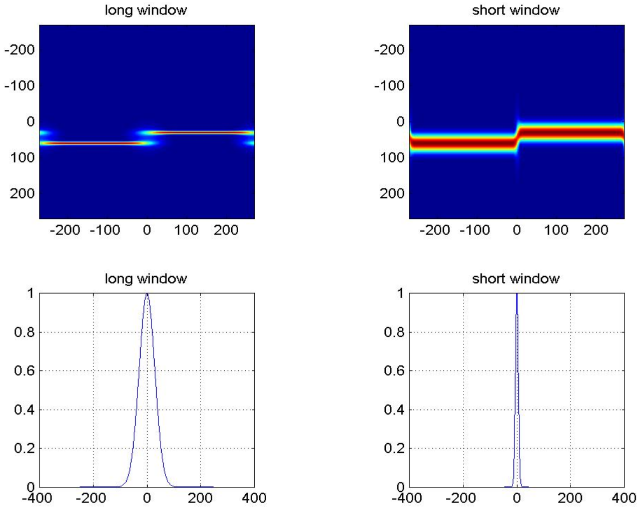
Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem. It uses $\boldsymbol{B} = \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ and $\boldsymbol{B}'$ coincides with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ (correctly: the linear operators which are w^* to norm continuous!).

Theorem

There is a natural BGT-isomorphism between $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$. *This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. *Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators* $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

Some Illustrations



Real World Gabor Multipliers

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Ferenc Weisz.

Further information, reading material

The NuHAG webpage offers a large amount of further information, including talks and MATLAB code:

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www.nuhag.eu
www.nuhag.eu/bibtex (all papers)
www.nuhag.eu/talks (all talks)
www.nuhag.eu/matlab (MATLAB code)
www.nuhag.eu/skripten (lecture notes)
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