

The Fuglede conjecture in dimension one

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Fuglede conjecture

Definition

Denote by $e_\lambda(x) = e^{2\pi i \lambda \cdot x}$. Let Ω be a Lebesgue measurable subset of $\mathbb{R}^d$ of finite measure. We say that Ω is a *spectral* set if there exists a set Λ in $\mathbb{R}^d$ such that $\{e_\lambda : \lambda \in \Lambda\}$ is an orthogonal basis in $L^2(\Omega)$. Ω is a *translational tile* if there exists a *tiling set* $\mathcal{T}$ such that $\bigcup_{t \in \mathcal{T}} (\Omega + t) = \mathbb{R}^d$ and $(\Omega + t) \cap (\Omega + t') = \emptyset$ zero of all $t \neq t'$ in $\mathcal{T}$, up to Lebesgue measure.

Conjecture (Fuglede, 1974)

Ω is a spectral set if and only if Ω is a translational tile.

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Conjecture (Fuglede, 1974)

Ω is a spectral set if and only if Ω is a translational tile.

We also say that a countable set Λ is *periodic* if there exists a *period* $a > 0$ such that $\Lambda + a = \Lambda$.

Definition

Let A be a finite subset of $\mathbb{Z}$. We say that A is a tile if there exists a set B in $\mathbb{Z}$ such that $A \oplus B = \mathbb{Z}$. We say that A is a spectral set in $\mathbb{Z}$ if there exists a finite set Λ in $\mathbb{R}$ (or $[0,1)$), with $\#\Lambda = \#A$, such that the matrix

$$\frac{1}{\sqrt{|A|}} \left(e^{2\pi i \lambda a} \right)_{\lambda \in \Lambda, a \in A}$$

is unitary (hence a Hadamard matrix).

Let A be a subset of $\mathbb{Z}_n$. We say that A is a tile if there exists a set B in $\mathbb{Z}_n$ such that $A \oplus B = \mathbb{Z}_n$. We say that A is a spectral set in $\mathbb{Z}_n$ if there exists a set Λ in $\mathbb{Z}_n$ (or $[0,1)$), with $\#\Lambda = \#A$, such that the matrix

$$\frac{1}{\#A} \left(e^{2\pi i \lambda a/n} \right)_{\lambda \in \Lambda, a \in A}$$

is unitary.

Some statements

We are interested in the validity and the relations among the following statements (for $G = \mathbb{R}$, $\mathbb{Z}$ or $\mathbb{Z}_n$):

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(USC(G)). If $\mathcal{B}$ is a finite family of bounded sets in G with the property that there exists $A \subset G$ such that $A \oplus B = G$ for all $B \in \mathcal{B}$, then the sets $B \in \mathcal{B}$ have a common spectrum Γ .

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(UTC(G)). If $\mathcal{A}$ is a finite family of bounded sets in G with the property that there exists a common spectrum Γ , then the sets $A \in \mathcal{A}$ have a common tiling set $\mathcal{T}$ so that $A \oplus \mathcal{T} = G$.

(Strong S-T($\mathbb{Z}$)). For every finite union of intervals $\Omega = A + [0, 1]$ with $A \subset \mathbb{Z}$, if Λ is a spectrum of Ω with minimal period $\frac{1}{N}$, then Ω tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset N\mathbb{Z}$.

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(Strong T-S($\mathbb{Z}$)). For every finite union of intervals $\Omega = A + [0, 1]$ with $A \subset \mathbb{Z}$, if Ω tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset k\mathbb{Z}$ for some integer $k > 0$, then Ω admits a spectrum Λ with period $\frac{1}{k}$.

Theorem

- (i) $(\mathbf{T-S}(\mathbb{Z}_n)) \iff (\mathbf{USC}(\mathbb{Z}_n)) \iff (\mathbf{T-S}(\mathbb{R})) \iff (\mathbf{T-S}(\mathbb{Z})).$
(ii) $(\mathbf{S-T}(\mathbb{Z}_n)) \iff (\mathbf{UTC}(\mathbb{Z}_n)),$
 $(\mathbf{S-T}(\mathbb{R})) \implies (\mathbf{S-T}(\mathbb{Z})) \implies (\mathbf{S-T}(\mathbb{Z}_n)).$ *If any spectral set Ω of Lebesgue measure 1 has a rational spectrum, then the converses hold.*
(iii) $(\mathbf{Strong\ T-S}(\mathbb{Z})) \iff (\mathbf{USC}(\mathbb{Z})) \iff (\mathbf{T-S}(\mathbb{R})).$
(iv) $(\mathbf{Strong\ S-T}(\mathbb{Z})) \iff (\mathbf{UTC}(\mathbb{Z})) \iff (\mathbf{S-T}(\mathbb{R})).$

Theorem (Lagarias-Wang)

Let Ω be a bounded measurable translational tile on $\mathbb{R}$ with tiling set $\mathcal{T}$. Then the tiling set $\mathcal{T}$ must be periodic with period an integral multiple of $|\Omega|$ and $\mathcal{T} \subset |\Omega| \cdot \mathbb{Q}$.

Moreover, if the period lattice contains $\mathbb{Z}$, then we have the following decomposition of Ω and $\mathcal{T}$: there exists a positive integer L and a finite collection of finite subsets of integers $\mathcal{B}$ such that

$$\Omega = \bigcup_{B \in \mathcal{B}} (\Omega_B + \frac{1}{L}B); \quad \mathcal{T} = \frac{A}{L} + \mathbb{Z}$$

where (i) $A \oplus B = \mathbb{Z}_L$ for all $B \in \mathcal{B}$, (ii) $\bigcup_{B \in \mathcal{B}} \Omega_B = [0, 1/L)$ with disjoint union.

Note here, that **(USC($\mathbb{Z}_n$))** implies **(T-S($\mathbb{R}$))**.

We say that Ω is a p -tile by $\frac{1}{p}\mathbb{Z}$ if

$$\sum_{k \in \mathbb{Z}} \chi_{\Omega}(x + \frac{k}{p}) = p. \text{ a.e..}$$

In other words, Ω covers the real line p times when it is translated by $\frac{1}{p}\mathbb{Z}$.

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Theorem (Iosevich, Kolountzakis)

Any spectra Λ of the spectral set Ω on $\mathbb{R}^1$ must be periodic with period an integral multiple of $|\Omega|^{-1}$. Moreover, Ω must be a p -tile by $\frac{1}{p}\mathbb{Z}$ where p is the period of Λ .

Since Ω is a p -tile by $\frac{1}{p}\mathbb{Z}$, we have the following natural decomposition of a spectral set Ω on $\mathbb{R}^1$. For $x \in \mathbb{R}$ and $S \subset \mathbb{Z}$ with $\#S = p$, where p is the period,

$$S_x = \{k \in \mathbb{Z} : x + \frac{k}{p} \in \Omega\}, \quad \Omega_S = \{x \in [0, \frac{1}{p}) : S_x = S\}$$
$$\Omega = \bigcup_{|S|=p} \left(\Omega_S + \frac{S}{p} \right), \quad \bigcup_{|S|=p} \Omega_S = [0, \frac{1}{p}). \quad (0.1)$$

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Proposition

Let Ω be a p -tile by $\frac{1}{p}\mathbb{Z}$. Then a set Λ of the form $\Lambda = \{0, \lambda_1, \dots, \lambda_{p-1}\} + p\mathbb{Z}$ is a spectrum of Ω if and only if $\{0, \lambda_1, \dots, \lambda_{p-1}\}$ is a spectrum for $\frac{1}{p}S_x$ for a.e. $x \in [0, \frac{1}{p})$. Moreover, if the sets S_x have a common tiling set $\mathcal{T}$, then Ω tiles $\mathbb{R}^1$ by the tiling set $\frac{1}{p}\mathcal{T}$.

Note that **(UTC($\mathbb{Z}_n$))** implies **(S-T($\mathbb{R}$))**, if the points λ_i are in some $\frac{1}{n}\mathbb{Z}$.

Proposition

[Matolcsi et al.] Let A be a finite set of integers. Let Λ be a spectrum for $\Omega = A + [0, 1)$, $0 \in \Lambda$. Then 1 is a period of Λ . Moreover, if we write $\Lambda = \Gamma + \mathbb{Z}$ with $\Gamma \subset [0, 1)$, then Γ is a spectrum for A if and only if Λ is a spectrum for $A + [0, 1)$.

Proposition

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Theorem

[Matolcsi et al.] Let $A \subset \mathbb{Z}_n$ and let $T(k) = n\{0, 1, \dots, k-1\}$. Define $B(k) = A + T(k)$. Then for large enough k ,

- (i) $B(k) + [0, 1)$ is a spectral set on $\mathbb{R}^1$ if and only if A is spectral in $\mathbb{Z}_n$.*
- (ii) $B(k) + [0, 1)$ is a tile on $\mathbb{R}^1$ if and only if A is tile in $\mathbb{Z}_n$*

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- (i) $(\mathbf{T-S}(\mathbb{Z}_n)) \iff (\mathbf{USC}(\mathbb{Z}_n)) \iff (\mathbf{T-S}(\mathbb{R})) \iff (\mathbf{T-S}(\mathbb{Z})).$
- (ii) $(\mathbf{S-T}(\mathbb{Z}_n)) \iff (\mathbf{UTC}(\mathbb{Z}_n))$ and $(\mathbf{S-T}(\mathbb{R})) \implies (\mathbf{S-T}(\mathbb{Z})) \implies (\mathbf{S-T}(\mathbb{Z}_n)).$ *If any spectral set Ω of Lebesgue measure 1 has a rational spectrum, then the converse holds.*

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Main issue with the reverse implication: we don't know if a spectrum for a set in $\mathbb{R}$ has to be rational.

Theorem

The following statements are equivalent

- 1 **(Strong T-S($\mathbb{Z}$)).** *For every set Ω of the form $\Omega = A + [0, 1]$ with $A \subset \mathbb{Z}$, A finite, if Ω tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset k\mathbb{Z}$, $k \in \mathbb{N}$, then it has a spectrum of period $\frac{1}{k}$.*
- 2 *For every set Ω of the form $\Omega = \cup_{i=1}^n (\alpha_i, \beta_i)$ with α_i, β_i rational and $|\Omega| = 1$, if Ω tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset \frac{1}{p}\mathbb{Z}$, $p \in \mathbb{N}$, then it has a spectrum of period p .*
- 3 **(USC($\mathbb{Z}$)).** *If $\mathcal{B}$ is a finite family of sets in $\mathbb{Z}$ with the property that there exists $A \subset \mathbb{Z}$ such that $A \oplus B = \mathbb{Z}$ for all $B \in \mathcal{B}$, then the sets $B \in \mathcal{B}$ have a common spectrum $\Gamma \subset [0, 1)$.*
- 4 **(T-S($\mathbb{R}$)).** *Every bounded measurable tile Ω in $\mathbb{R}$ is spectral.*

Theorem

The following statements are equivalent

- ❶ **(Strong S-T($\mathbb{Z}$))**. *For every finite union of intervals $\Omega = A + [0, 1]$ with $A \subset \mathbb{Z}$, if Λ is a spectrum of Ω with minimal period $\frac{1}{N}$, then Ω tiles $\mathbb{R}$ with a tiling set $\mathcal{T} \subset N\mathbb{Z}$.*
- ❷ **(UTC($\mathbb{Z}$))**. *If $\mathcal{A}$ is a finite family of bounded sets in $\mathbb{Z}$ with the property that there exists a common spectrum Γ , then the sets $A \in \mathcal{A}$ have a common tiling set $\mathcal{T}$ so that $A \oplus \mathcal{T} = \mathbb{Z}$.*
- ❸ **(S-T($\mathbb{R}$))**. *Every bounded spectral set in $\mathbb{R}$ is a tile.*

Theorem

Suppose the Fuglede conjecture is true (i.e. both $(\mathbf{S-T}(\mathbb{R}))$ and $(\mathbf{T-S}(\mathbb{R}))$ holds), then any bounded spectral sets Ω of $|\Omega| = 1$ will have a rational spectrum.

The Coven-Meyerowitz conjecture

Definition

Let $\Phi_s(x)$ be the cyclotomic polynomial corresponding to $s \in \mathbb{N}$.
Let A be a finite subset of $\mathbb{Z}^+ \cup \{0\}$ and let

$$A(x) = \sum_{a \in A} x^a.$$

Define

$$\mathcal{S}_A = \{s = p^\alpha : p \text{ is a prime, } \alpha \geq 1 \text{ an integer and } \Phi_s(x) \text{ divides } A(x)\}.$$

We say that A (or $A(x)$) satisfies the Coven-Meyerowitz property (CM-property) if $A(x)$ satisfies the following conditions:

(T1). $\#A = A(1) = \prod_{s \in \mathcal{S}_A} \Phi_s(1)$.

(T2). If $s_1, \dots, s_n \in \mathcal{S}_A$, then $\Phi_{s_1 \dots s_n}(x)$ divides $A(x)$.

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Theorem (Coven-Meyerowitz, Łaba)

(i) If $A \subset \mathbb{Z}^+ \cup \{0\}$ satisfies the CM-property, then A is a tile.

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- (i) If $A \subset \mathbb{Z}^+ \cup \{0\}$ satisfies the CM-property, then A is a tile.*
- (ii) If $A \subset \mathbb{Z}^+ \cup \{0\}$ satisfies the CM-property, then A is a spectral set of integers.*

The Coven-Meyerowitz conjecture

Theorem

- (i) *If every spectral set $A \subset \mathbb{Z}^+ \cup \{0\}$ satisfies the CM-property, then every bounded spectral subset of $\mathbb{R}$ tiles by translations.*
- (ii) *If every tile $A \subset \mathbb{Z}^+ \cup \{0\}$ satisfies the CM-property, then every bounded tile of $\mathbb{R}$ is a spectral set.*