FOURIER STANDARD SPACES
A new class of function spaces

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OVERVIEW

In this talk we will concentrate on the setting of the LCA group $G = \mathbb{R}^d$, although practically all the results could be formulated in the general setting of locally compact Abelian group as promoted by A. Weil. Classical Fourier Analysis pays a lot of attention to $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ because these spaces (specifically for $p \in \{1, 2, \infty\}$) are important to set up the Fourier transform as an integral transform which also respects convolution (we have the convolution theorem) and preserving the energy (meaning that it is a unitary transform of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$).

OVERVIEW II

In the last 2-3 decase the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (equal to the modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$ and its dual, $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ have gained importance for many questions of Gabor analysis or time-frequency analysis in general.

Fourier standard spaces are now a class of Banach spaces sandwiched in between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$, with *two module structures*, one with respect to the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, and the other one with respect to pointwise multiplication by elements of the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.

As we shall point out there is a huge variety of such spaces, and many questions of Fourier analysis find an appropriate description in this context.

OVERVIEW III

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions and so on. Thus among others the space $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.
Within the family there are two subfamilies, namely the *Wiener amalgam spaces* and the so-called *modulation spaces*, among them the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$.

Banach Module Terminology

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}, \tag{1}$$

bilinear and associative...

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}. \tag{2}$$

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach ideal* in (or within, or of) a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$, and if in addition (1) is valid with

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \,.$$

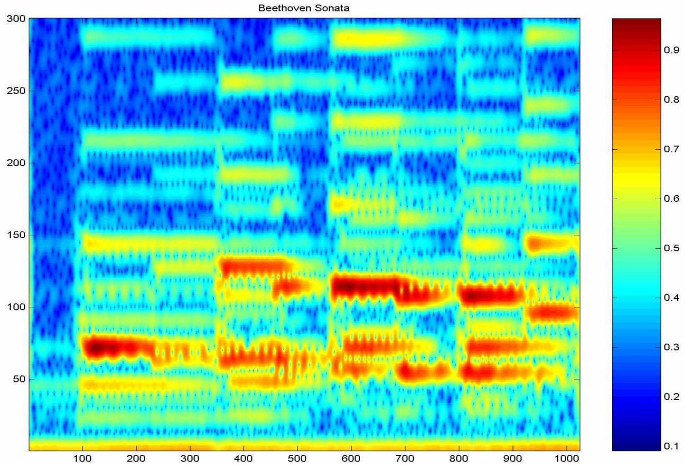
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

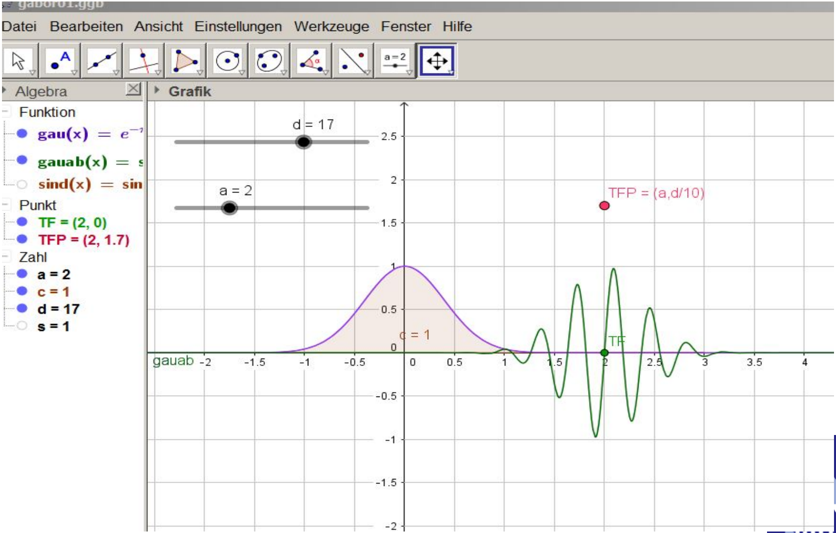
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \| Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^* Th \rangle_{\mathcal{H}_1}, \, \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^* T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \; d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (3)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (4)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \quad (5)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (6)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (7)$$

with $\chi_s(t) = exp(2\pi i \langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (7) (Fourier inversion) and the new formula formula (6). While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [6]).

Wendel's Theorem

Theorem

The space of $\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)$ *all bounded linear operators on* $\boldsymbol{L}^1(G)$ *which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with* $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$. *In terms of our formulas this means*

$$\mathcal{H}_{\boldsymbol{L}^1}(\boldsymbol{L}^1, \boldsymbol{L}^1)(\mathbb{R}^d) \simeq (\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b}),$$

$$\text{via } T \simeq C_\mu : f \mapsto \mu * f, \quad f \in \boldsymbol{L}^1, \mu \in \boldsymbol{M}_b(\mathbb{R}^d).$$

Lemma

$$B_{\boldsymbol{L}^1} = \{f \in \boldsymbol{B} \mid \|T_x f - f\|_{\boldsymbol{B}} \to 0, \ forx \to 0\}.$$

Consequently we have $(\boldsymbol{M}_b(\mathbb{R}^d))_{\boldsymbol{L}^1} = \boldsymbol{L}^1(\mathbb{R}^d)$, *the closed ideal of absolutely continuous bounded measures on* $\mathbb{R}^d$.

Pointwise Multipliers

Via the Fourier transform we have similar statements for the Fourier algebra, involving the *Fourier Stieltjes algebra*.

$$\mathcal{H}_{\mathcal{F}\boldsymbol{L}^1}(\mathcal{F}\boldsymbol{L}^1, \mathcal{F}\boldsymbol{L}^1) = \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d)), \quad \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))_{\mathcal{F}\boldsymbol{L}^1} = \mathcal{F}\boldsymbol{L}^1. \quad (8)$$

Theorem

The completion of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *(viewed as a Banach algebra and module over itself) is given by*

$$\mathcal{H}_{\boldsymbol{C}_0}(\boldsymbol{C}_0, \boldsymbol{C}_0) = \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right).$$

On the other hand we have $(\boldsymbol{C}_b(\mathbb{R}^d))_{\boldsymbol{C}_0} = \boldsymbol{C}_0(\mathbb{R}^d)$.

Essential part and closure

In the sequel we assume that $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a Banach algebra with bounded approximate units, such as $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (with convolution), or $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ or $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ with pointwise multiplication.

Theorem

Let $\boldsymbol{A}$ *be a Banach algebra with bounded approximate units, and* $\boldsymbol{B}$ *a Banach module over* $\boldsymbol{A}$*. Then we have the following general identifications:*

$$(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{9}$$

or in a slightly more compact form:

$$\boldsymbol{B}_{\boldsymbol{AA}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad \boldsymbol{B}_{\boldsymbol{A}}{}^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{AA}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{10}$$

Essential Banach modules and BAIs

The usual way to define the *essential part* $\boldsymbol{B_A}$ resp. $\boldsymbol{B_e}$ of a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to some Banach algebra action $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \bullet \mathbf{b}$ is the define it as the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ within $\left(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\right)$. However it is interesting that this subspace can be shown to have another nice characterization using BAIs (bounded approximate units in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$):

Lemma

For any Bounded Approximate Identity (BAI) $(\mathbf{e}_\alpha)_{\alpha \in I}$ *i in* $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ *one has:*

$$\boldsymbol{B_A} = \{\mathbf{b} \in \boldsymbol{B} \mid \lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b}\} \tag{11}$$

The Cohen-Hewitt Factorization Theorem

In particular one has: Let $(\mathbf{e}_\alpha)_{\alpha \in I}$ and $(\mathbf{u}_b)_{\beta \in J}$ be two bounded approximate units (i.e. bounded nets within $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$wa acting in the limit almost like an identity in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Then one has

$$\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b} \Leftrightarrow \lim_\beta \mathbf{u}_\beta \bullet \mathbf{b} = \mathbf{b}. \tag{12}$$

Theorem

(The Cohen-Hewitt factorization theorem, without proof, see [5]) Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI of size $C > 0$, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

Essential part and closure II

Having now Banach spaces of distributions which have two module structures, we have to use corresponding symbols. FROM NOW ON we will use the letter $\boldsymbol{A}$ mostly for pointwise Banach algebras and thus for the $\mathcal{F}\boldsymbol{L}^1$-action on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and we will use the symbol G (because convolution is coming from the integrated group action!) for the $\boldsymbol{L}^1$ convolution structure. We thus have

$$\boldsymbol{B}_{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad {\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad \boldsymbol{B}_{\boldsymbol{G}}^{\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}, \quad \boldsymbol{B}^{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}. \quad (13)$$

In this way we can combine the two operators (in view of the above formulas we can call them interior and closure operation) with respect to the two module actions and form spaces such as

$$ {\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{{\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}} \dots $$

or changes of arbitrary length, as long as the symbols $\boldsymbol{A}$ and $\boldsymbol{G}$ appear in alternating form (at any position, upper or lower).

Combining the two module structures

Fortunately one can verify (paper with W.Braun from 1983, J.Funct.Anal.) that any "long" chain can be reduced to a chain of at most two symbols, the *last occurence of each of the two symbols being the relevant one!* So in fact all the three symbols in the above chain describe the same space of distributions.
But still we are left with the follwoing collection of altogether eight two-letter symbols:

$$\boldsymbol{B}_{GA}, \boldsymbol{B}_{AG}, \boldsymbol{B}_{A}{}^{G}, \boldsymbol{B}^{G}{}_{A}, \boldsymbol{B}_{G}{}^{A}, \boldsymbol{B}^{A}{}_{G}, \boldsymbol{B}^{AG}, \boldsymbol{B}^{GA} \qquad (14)$$

and of course the four one-symbol objects

$$\boldsymbol{B}_{A}, \boldsymbol{B}_{G}, \boldsymbol{B}^{A}, \boldsymbol{B}^{G} \qquad (15)$$

Some structures, simple facts

Fortunately there is some natural structure
There are other, quite simple and useful facts, such as

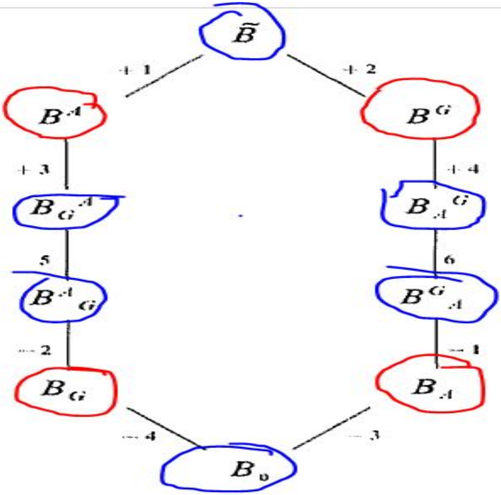
$$\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2) = \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2{}_{\boldsymbol{A}}) \tag{16}$$

which can easily be verified if $\boldsymbol{B}^1{}_{\boldsymbol{A}} = \boldsymbol{A} \bullet \boldsymbol{B}^1$, since then $T \in \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2)$ applied to $\mathbf{b}^1 = \mathbf{a} \bullet \mathbf{b}^{1'}$ gives

$$T(\mathbf{b}^1) = T(\mathbf{a} \bullet \mathbf{b}^{1'}) = \mathbf{a} \bullet T(\mathbf{b}^{1'}) \in \boldsymbol{B}^2{}_{\boldsymbol{A}}.$$

The Main Diagram

This diagram is taken from [1].



Fourier Standard Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, continuously embedded between $\boldsymbol{S}_0(G)$ and $\left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$, i.e. with

$$\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right)$$

is called a *Fourier Standard Space* on G (FSS of FoSS) if it has a double module structure over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\widehat{G}))$ with respect to pointwise multiplication.

TF-homogeneous Banach Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \ f \in \boldsymbol{B}. \tag{17}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous on often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$).

TF-homogeneous Banach Spaces II

An important fact concerning this family is the minimality property of the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$.

Discussion of the Diagram

For each of the Fourier Standard Spaces the discussion of the above diagram makes sense. One may see that it can collaps totally to a single space, or that it has in fact a rich (like $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$) or simple structure.

Theorem

A Fourier standard space is maximal, i.e. coicides with $\widetilde{\boldsymbol{B}} = \boldsymbol{B}^{\boldsymbol{AG}} = \boldsymbol{B}^{\boldsymbol{GA}}$ *if and only if* $\boldsymbol{B}$ *is a dual space.*
There is also a formula for the predual spaces, it is $((\boldsymbol{B}_0)')_0$*, where* $\boldsymbol{B}_0 = \boldsymbol{B}_{\boldsymbol{AG}} = \boldsymbol{B}_{\boldsymbol{GA}}$ *is just the closure of* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *resp.* $\boldsymbol{S}_0(\mathbb{R}^d)$ *in* $\boldsymbol{B}$.

Discussion of the Diagram II

Theorem

A Fourier standard space is reflexive if and only if both the space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and its dual are both minimal and maximal. In other words, for the space itself and its dual the diagram is reduced to a single space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

Constructions within the FSS Family

1. Taking Fourier transforms;
2. Conditional dual spaces, i.e. the dual space of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
3. With two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take intersection or sum
4. forming amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. defining pointwise or convolution multipliers;
6. using complex (or real) interpolation methods, so that we get the spaces $\boldsymbol{M}^p = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$ (which are in fact all Fourier invariant);
7. Fractional invariant kernel and hull: For any given standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we could define the largest Banach space inside of $\boldsymbol{B}$ which is inveriant under all the fractional FTs, or the smallest such space which allows a continuous embedding of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *(suitable norms, duality theory?)*

Tensor products

Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1) f^2(x_2), \ \ x_i \in \mathbb{R}^d, i = 1, 2.$$

Given two Banach spaces $\boldsymbol{B}^1$ and $\boldsymbol{B}^2$ embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, $\boldsymbol{B}^1 \hat{\otimes} \boldsymbol{B}^2$ denotes their *projective tensor product*, i.e.

$$\left\{ f \mid f = \sum f_n^1 \otimes f_n^2, \ \sum \|f_n^1\|_{\boldsymbol{B}^1} \|f_n^2\|_{\boldsymbol{B}^2} < \infty \right\}; \qquad (18)$$

It is easy to show that this defines a Banach space of tempered distributions on $\mathbb{R}^{2d}$ with respect to the (quotient) norm:

$$\|f\|_{\hat{\otimes}} := inf\{\sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, ..\} \qquad (19)$$

where the infimum is taken over all admissible representations.

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