Gabor expansions of signals: Computational aspects and open questions

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Abstract for readers, orientation

Although the theory of Gabor Analysis, i.e. the decomposition of distributions over locally compact groups in terms of TF-shifted copies of a Gabor atom along a lattice is theoretically well understood there are still many interesting open questions concerning the practical realization of such a program.
Aside from the classical case, where one-dimensional signals on the real line are expanded, using a separable TF-lattice of the form *aZxbZ* there is no easily applicable and computationally realizable approach to the actual determination of Gabor coefficients for more general lattices, or for multi-dimensional signals.
The talk will address computational issues, the theoretical justification of numerical approximations to the continuous problem, and mention a list of open problems in the field which would be relevant for further progress and real-world applications of Gabor analysis. We will also discuss aspects of the question, what makes a Gabor frame a good one.

Let us call it Computational Gabor Analysis

Although "we are doing it for quite a while" (e.g. at NuHAG) the term is almost inexistent. I was pleased to find it in Peter Sondergaards PhD Thesis (in 2007, at DTU, under Ole Christensen), which became the starting point for the socalled **LTFAT**: The *Linear Time Frequency Toolbox* (as opposed to the one promoted by P. Flandrin, containing quadratic TF-representations).
Meanwhile the community has to be grateful to Peter Balazs and his team at ARI for the promotion and continuation of the now **Large Time-Frequency Toolbox**!

`DOWNLOAD at:`
`http://ltfat.sourceforge.net/download.php`

What are the benefits of computational Gabor analysis

There are various reasons, why I am sitting down [as a matter of fact very regularly] in order to start e.g. MATLAB (or sometimes GEOGEBRA), when thinking about Gabor analysis:

- Visualization tasks; illustrate situations;
- Test hypothesis on a discrete model;
- Carry out some simulations;
- Compute actual figure (like the frame bounds);
- Formulate conjectures, based on numerical evidence;

Vizualization: the different classical spaces

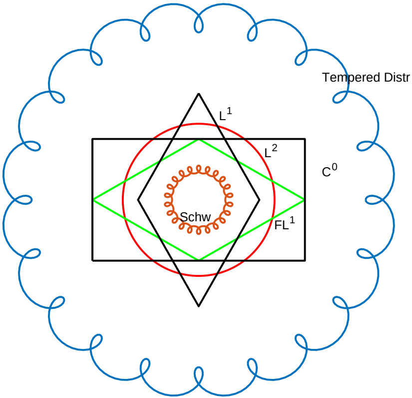


Figure: The classical setting, with Schwartz spaces

$L^1(\mathbb{R}^d)$ and the Fourier Algebra $\mathcal{F}L^1(\mathbb{R}^d)$

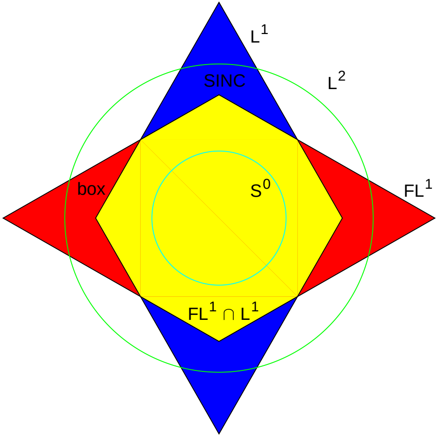


Figure: $\boldsymbol{L}^1(\mathbb{R}^d), \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ with $\boldsymbol{S}_0(\mathbb{R}^d)$ inside

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t, \omega);$$

TF-shifted Gaussians

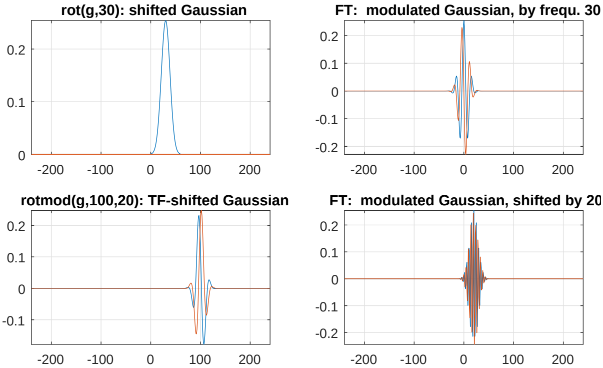


Figure: presentation using f2sp, rot, and rotmod

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, i.e. that $\|g\|_2 = 1$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h\rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th\rangle_{\mathcal{H}_2} = \langle h, T^*Th\rangle_{\mathcal{H}_1}, \, \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (1)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (2)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \quad (3)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (4)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (5)$$

with $\chi_s(t) = exp(2\pi i \langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" plane waves, resp. *characters* of $\mathbb{R}^d$).

Discrete reconstruction: Gabor Analysis

The continuous reconstruction formula suggests that the Riemannian sums to those reconstructing integrals provide a good approximation.
So instead of working with the full STFT $V_g(f)$ we work with a sampled version. A regular Gabor family is a family of the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, where $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is a lattice (a discrete subgroup).
Gabor analysis tells us that in the frame case there exists some $\tilde{g}$ (canonical dual Gabor atom) such that despite the discretization we still have perfect reconstruction:

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda\in\Lambda} \langle f, g_\lambda \rangle \tilde{g}_\lambda. \quad (6)$$

STFT-multiliers or Anti-Wick operators

The above reconstruction formula 4 suggests to define linear operators by applying a pointwise multiplication operator on the transform side (similar to Fourier multipliers). We denote by GM_m the Gabor (or STFT-) multiplier with the multiplier (or upper symbol) m, typically a real-valued function on $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$:

$$GM_m(f) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} m(\lambda) \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{7}$$

and in the discrete version we have *Gabor multipliers*

$$GM_m(f) = \sum_{\lambda \in \Lambda} m(\lambda) \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \tag{8}$$

Gabor Analyis: General Setting

Gabor Analysis is built on the use of *time- and frequency shifts*, thus we can do Gabor analysis over general LCA groups, not just $\mathbb{R}^d, \mathbb{Z}^d$ or finite groups.

Abstract Harmonic Analysis is teaching us how to transfer ideas understood in one setting to the other setting. There is always a commutative group G (of translation operators) and a *dual group* $\hat{G}$ of *"pure frequencies"* (or e.g. plane waves).

For *Finite Abelian groups*, such as $\mathbb{Z}_N$ or $\mathbb{Z}_M \times \mathbb{Z}_N$ (for image processing) can be treated exactly in a computer, while for the continuous, non-periodic setting a number of issues arise. Note that Gabor systems are typically non-orthogonal etc.

Standard Gaussian window, n=480, a=20, b=16

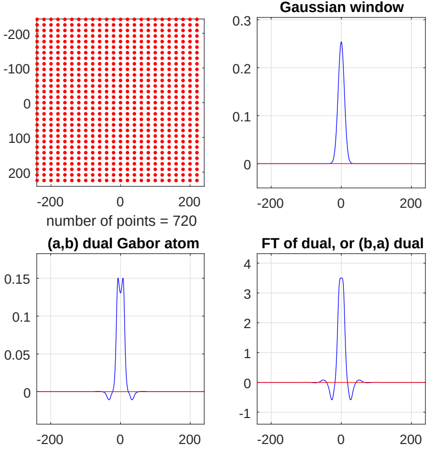


Figure: A separable Gabor lattice of redundancy 3/2

Window, tight window and dual window, time & frequency

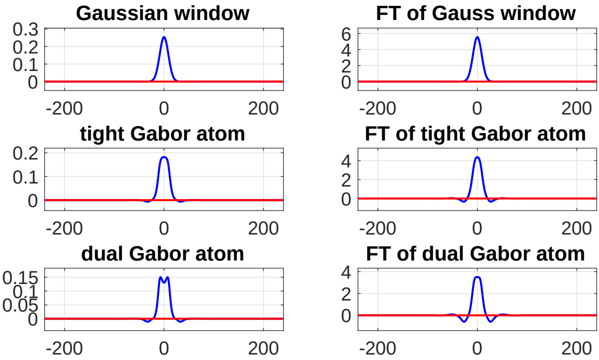


Figure: Showing the window, the tight Gabor window and dual Gabor window, for $a = 20, b = 16, n = 480$.

Canonical tight Gabor atoms for sheared lattices

The underlying lattices are obtained by applying to `LAM = lattp(n,a,b) = lattp(480,20,16);` the automorphism (of $\mathbb{Z}_n \times \mathbb{Z}_n$) `sidedigm` (reshuffling cyclic diagonals to row vectors).

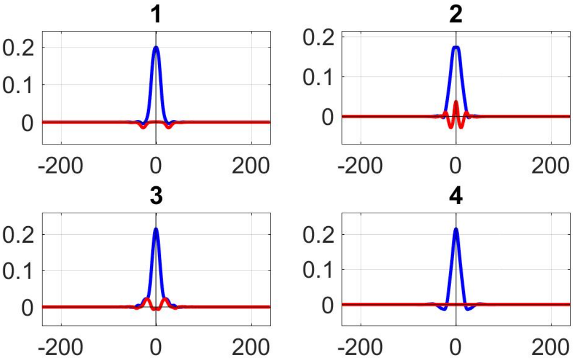


Figure: The last lattice is symmetric with respect to x-axis.

What are our goals? and state of the art

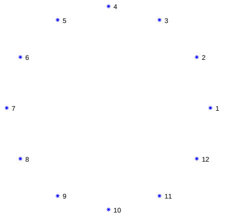
- Often the following question is addressed as the *key question* of Gabor Analysis: Given a TF-lattice $\Lambda \lhd G \times \widehat{G}$ and a *Gabor atom* $g \in \boldsymbol{L}^2(G)$. Can one show that the *Gabor family* $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a **frame** for $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$? (in view of Balian-Low asking for a Riesz basis would be asking too much!).
- Typically the discrete finite setting is studied in analogy, i.e. given a discrete Gauss-like function, and a pair of lattice constants we want to compute the dual Gabor atom, expand a test signal, invert a Gabor multiplier, etc.
- We might study the effect of an Anti-Wick operator (STFT-multiplier) either by effectively realizing it on some test signals, or by forming the matrix describing this operator and studying it then (e.g. from the point of view of spectral theory)

Some hints about pitfalls and tricks

Before going on with general considerations on *Computational Gabor Analysis* let me just share a few thoughts concerning elementary conventions which help to avoid *Pitfalls*, see

`Isaac Amidror: Mastering the Discrete Fourier Transform in One, Two or Several Dimensions: Pitfalls and Artifacts London: Springer, Vol.43 (2013).`

It all starts with the correct matching of finite vectors in $\mathbb{C}^N$ with function on $\mathbb{Z}_N$.



Convolution of box-functions

```
triang = real(ifft(fft(box).*fft(b0x));
```

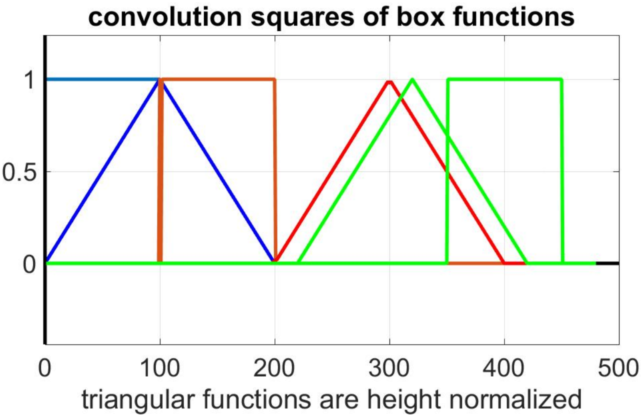


Figure: boxtriang1.jpg

Displayed more properly using PLOTC.M

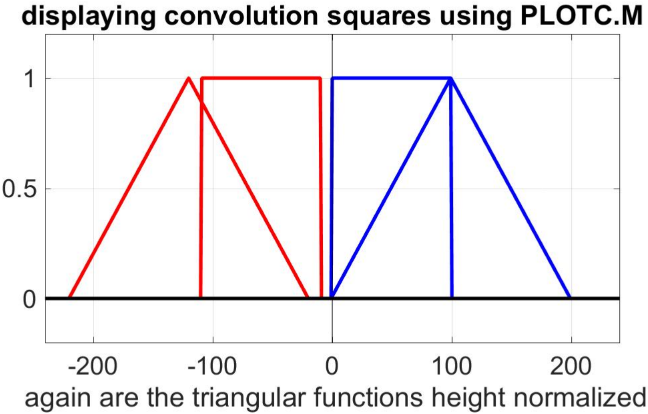


Figure: boxtriang2.jpg

How to plot a Fourier transform?

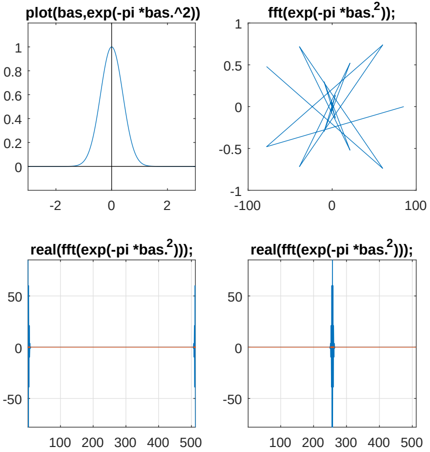


Figure: plotgaus1.eps

Problems with naive use of FFT

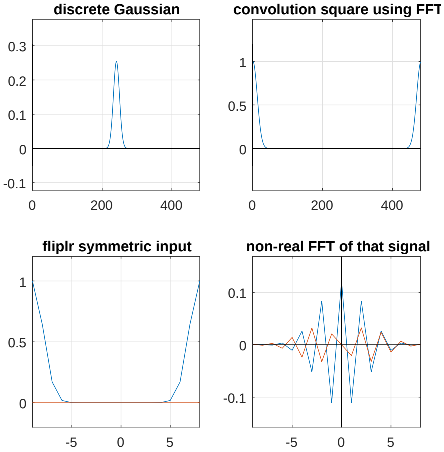


Figure: FFTproblems2.eps

Diagnosis of the previous examples

1. The first example: blue plot is a result of plotting in the complex plane! The second row shows that the Gauss function (with 512 samples) is far to broad, and even FFTSHIFT putting the spectrum in the middle does not show Gaussian shape!
2. In some applied books the readers are warned, that convolution using the FFT requires an FFTSHIFT, but this is just due to bad data structure (choice of origin in $\mathbb{Z}_N$);
3. The last example indicates that symmetric (FLIPLR) signals need not have real-valued DFT. So why does this happen!?

John Daugman's experiment

Figure: LENA256P.jpg: Gabor representation at critical density

John Daugman's experiment

John Daugman, one of the pioneers of Gabor Analysis (connecting Gabor Analysis with the human visual system) was one of the early pioneers of *Computational Gabor Analysis*, because he was - despite the non-orthogonality, actually computing Gabor expansions.

His explanatin for the missing information was the unfortunate numerical stability of the Gabor system, while in effect (this is how the simulation of his work has been realized) we have in fact a rank deficiency.

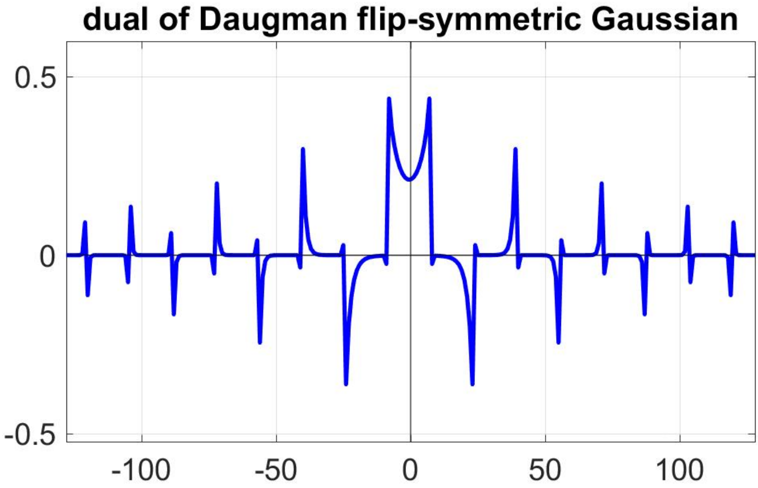
The Gabor system with Gaussian window of length 256 and critical density $a = b = 16$ has in fact only rank 255.

This is related to the zero of the Zak-transform of this Gauss function.

John Daugman's good luck

There was one factor that probably helped John Daugman to actually compute something between the "correct" case and perfect reconstruction. If one takes a "perfectly" (flip!)-symmetric Gaussian window one can find that the rank of the corresponding Gabor system with $a = 16 = b$ is now 256, and the condition number is not worse than 12.
One can even compute the dual Gabor window, which then looks as follows: Experts will easily see the similarity to the Bastiaans dual γ-function, which is not in $\boldsymbol{L}^p(\mathbb{R})$ for any $p < \infty$, but in $\boldsymbol{L}^\infty(\mathbb{R}^d)$.

John Daugman's good luck



The idea behind the term "**Conceptual Harmonic Analysis**" which I try to promote now for some time is to raise awareness for the fact that we ~~do not just want to make use of the analogies between the different settings, in the spirit of *Abstract Harmonic Analysis*, but that we~~ should take care of the *connections between the continuous and the discrete case*. Thus we enable ourselves to draw *reliable conclusions concerning the continuous limiting case* by carrying out computations using sufficiently "fine" discrete (but still finite) situations, which can be **realized** (finite time and finite precision) using appropriate mathematical software.

In this spirit we are thinking of the finite dimensional problems as *constructive but also computable approximations* to the continuous situation, but with *a priori control of errors*.

Gabor Analysis: Theoretical Aspects

This idea requires various considerations:

- What is the finite/discrete analogue of the continuous situation (e.g. use of FFT)?
- Which norms and which function spaces can be used to measure the distance and ensure correct approximation of the continuous situation (often not the $\boldsymbol{L}^2$-norm!)?
- Can we derive qualitative results at the most general level which allow us to show that for any required level of precision we can find a constructive (and realizable!) approximation?
- Once we have understood the general principle we should go for efficient numerical schemes and arguments showing optimality (up to constants).

Constructive Approximation Theory

Already a while ago people working in approximation theory started to make a difference between general approximation theoretic results and *constructive* ones.
The classical example is the *Weierstrass approximation Theorem*. It guarantees the uniform approximation of a continuous function over a compact interval $[a, b]$ without telling the user how to program it. A possible way out are Bernstein's polynomials.
Or, look at the approximation of continuous, periodic functions by Fourier series, and ask yourself when is it enough to know sufficiently many equidistant samples in order to compute (via FFT?) a good approximating trigonometric polynomial?
Constructive Approximation theory asks for a constructive description, which may however require an uncountable number of arithmetic steps!

Constructive Realizable Approximation

We propose a more realistic approach to approximation problems, requiring in fact **Constructive Realizability**.

In the current talk we will not discuss this issue, which is a separate topic, because it involves certain robustness considerations, function spaces, distributional convergence and other issues.

Instead we rather concentrate on the **needs** of the computational part for this concept, and the **possible benefits** that we can draw from e.g. computational experiments, even before the corresponding theoretical details are established. For example we can create *conjectures* or gain experience what *favourable conditions* might for the existence of a stable solution to a given problem.

Computational Challenges for Gabor Analysis

There is a number of issues that one has to take into account when one is thinking of *Computational Gabor Analysis*. In fact, concerns that I have been very well aware of during the last 25 years.

- It is not enough to have code that works for rather small problems, or that takes far to long to be used more frequently;
- in other words, we need fast and reliable code for the standard situations; and alternative code for special cases;
- software should be open and accessible within the community, which requires some standards (like the LTFAT toolbox) and shared conventions;
- not only computational costs are important, but also *memory requirements*;
- and many other issues, you name them ....

Gabor Analysis: Computational Aspects 2

Let us look more into the *psychological side* of Computational Gabor Analysis:

- We are doing it because we want to get numbers;
- We want to find numerical values for frame bounds;
- We want to invert the Gabor frame operator;
- We want to determine the dual Gabor atom;
- We want to apply a Gabor multiplier;
- We want to find the best approximation of a matrix by a Gabor multiplier;
- We want to know what a *good Gabor system* is!
- We look (?) for the best lattice for a given atom!?
- We want to **teach** Gabor analysis, **visualize** things

Recall the $S_0(\mathbb{R}^d)$-norm

The Segal algebra $S_0(\mathbb{R}^d)$ is defined as

$$S_0(\mathbb{R}^d) := \{f \in L^2(\mathbb{R}^d) \mid V_g(f) \in L^1(\mathbb{R}^{2d})\}$$

with

$$\|f\|_{S_0} := \|V_g(f)\|_{L^1(\mathbb{R}^{2d})}.$$

Usually one chooses $g(t) = e^{-pi|t|^2}$, with $\|g\|_2 = 1$.
This space is isometrically invariant under the Fourier transform and the TF-shifts.

Gabor Analysis: Computational Aspects

Even if we do not have theoretical results we can do a plausibility test, using finite dimensional computations only. For example we can carry out "analogous" experiments for different settings, which are related to the *same continuous setting*. Let us just compute the $\boldsymbol{S}_0(\mathbb{R})$-norm of a Gauss function (for $n = 480$ and then for $n = 4 * 480 = 1920$) or the condition number of the frame operator (for $a = 20, b = 16$, or *redundancy* $3/2$).

```
>> format long; >> sonorm(gaussnk(n))
ans = 1.999999999999982
>> sonorm(gaussnk(n*4))
ans = 1.999999999999960
>> cond(gabfrmat(gaussnk(n),a,b))
ans = 1.945436338386356
>> cond(gabfrmat(gaussnk(4*n),a*2,b*2))
ans = 1.945436338386379
```

Four different ways of describing an operator

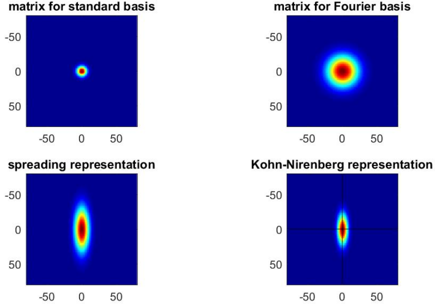


Figure: The orthogonal projection on the the third power of the Fourier invariant Gaussian, i.e. $g3 = g.^3$, $P = g3(:) * g3(:)'$, viewed in the four different domains.

Double preconditioning viewed in the spreading domain

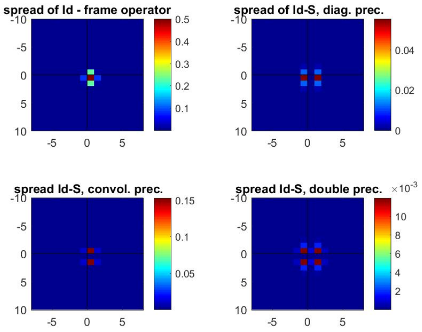


Figure: Looking closer into the difference between identiy operator and the spreading function of the Gabor frame operator, which according to the Janssen representation is concentrated on the lattice with lattice constants $(n/b, n/a)$. Hence only these contributions are presented!

Sorting a list of lattices according to Janssen test

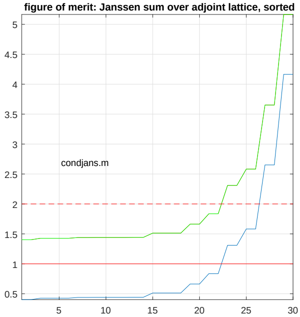


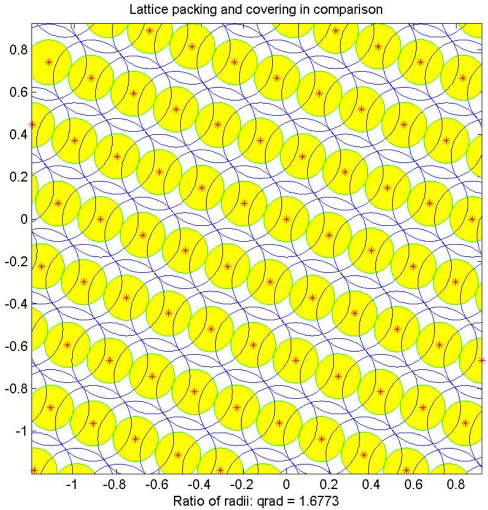
Figure: condjansdemo02.pdf: The green line represents the $\ell^1(\Lambda^\circ)$-norm of $V_g(g)(\lambda^c)$, with $V_g(g)(0) = \|g\|_2^1 = 1$, thus all the lattices with the blue line below the red threshold of 1 are good lattices

Comparing different figures of merit

When asking the question, what a good Gabor frames is, we can give a number of different answers:

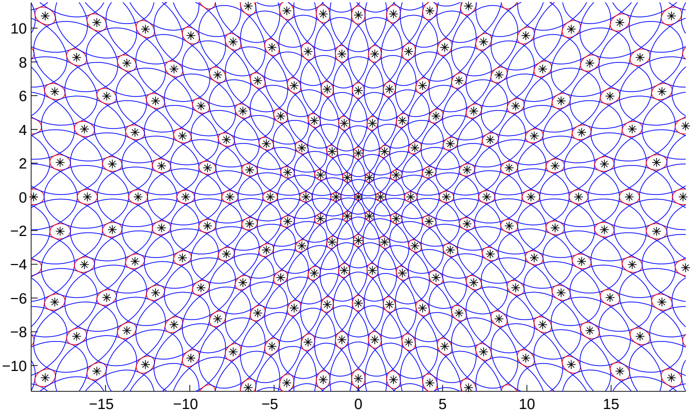
- the condition number of the Gabor frame operator;
- the concentration of the dual Gabor atom $\tilde{g}$;
- the $\boldsymbol{S}_0$-norm of the dual atom;
- the covering properties of the lattice Λ adapted to contour-lines of $V_g(g)$;
- etc. etc. etc.

Illustrating the covering properties



Illustrating the covering properties

MATLAB commands
A=[1 0.5;0 0.866];
warp=z.*(1+abs(z)).^0.4;
dil=1.4;
covellrd(A,warp,dil);



dilation = 1.4; ratio long/short axis = 1; warp function ... f(z) = z.*(1+abs(z)).^0.4

Comparing different figures of merit

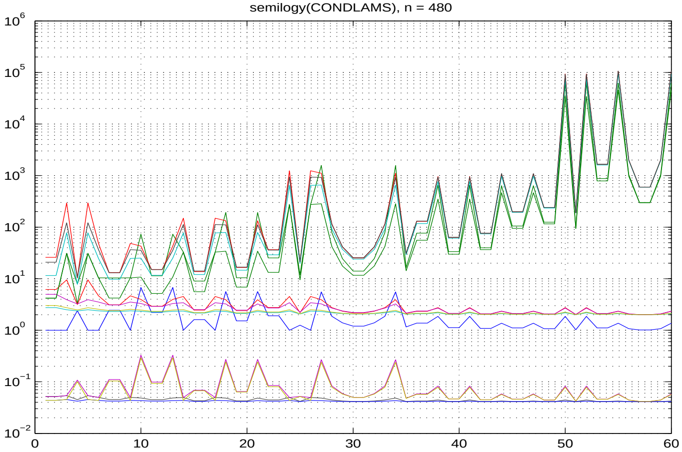


Figure: The covariant behavior if the different figures of merit as encouraging and suggest to use the most simple ones

The landscape of separable lattices for $n = 450$

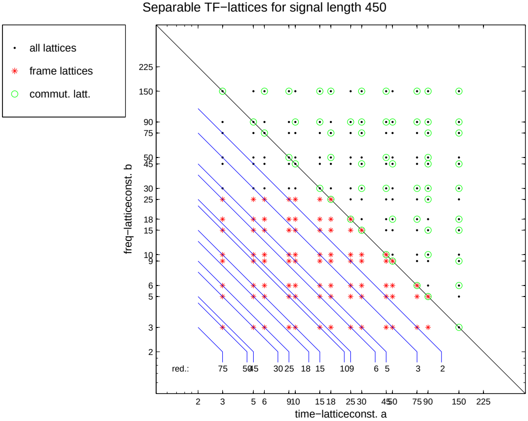
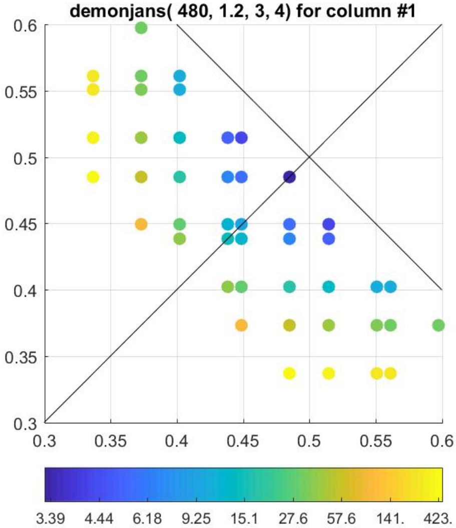


Figure: The landscape of separable lattices

Visualization of figures of merit



Sampling and periodization

One of the test, whether ideas concerning the "convergence" of a sequence of finite discrete models being convergent to some continuous limit is the following consideration:
Assume that we are sampling periodized versions of a continuous function at a given sampling rate, and then we take a multiple of the period, and a multiple of the sampling rate, then of course some of the old (more coarse) samples also belong to the new sampling set, and thus we can verify whether these values are convergent at adjacent levels.
We have seen this happen in many concrete cases.

Sampling and periodization II

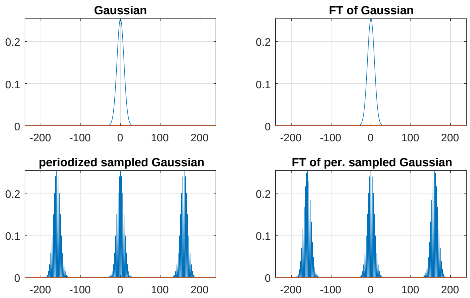


Figure: demsampfour1.eps

Sampling and periodization: TF picture

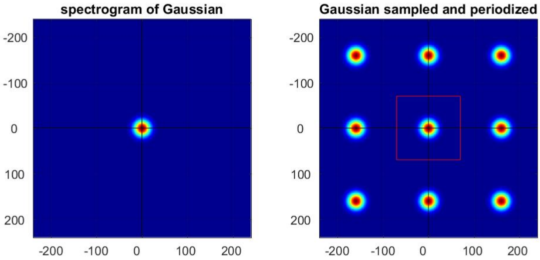


Figure: demsampfour1.jpg

Some of the findings during the UnlocX project (EU)

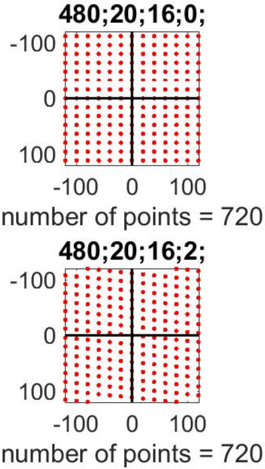
It was one of the main goals of WP3 (workpackage 3) of the FET network funded by the European Commission to describe "optimal Gabor families".
Since it is easy the verify that there is a positive correlation between a *good condition number* of the Gabor frame operator $S = \sum_{\lambda \in \Lambda} P_{g_\lambda}$ with the redundancy of the underlying lattice it is easy to verify (at least experimentally) that for e.g. a Gaussian the optimal choice, for a given redundancy $red > 1$ is the choice $a = \sqrt{n/red}$.
But there is another issue, namely the stability of approximation of slowly varying system (so-called *underspread operators* by Gabor multipliers, and here the high redundancy regime does no provide effectively better approximability, but rather more We are going to discuss this question in detail below.

The influence of the NABS-parameter s

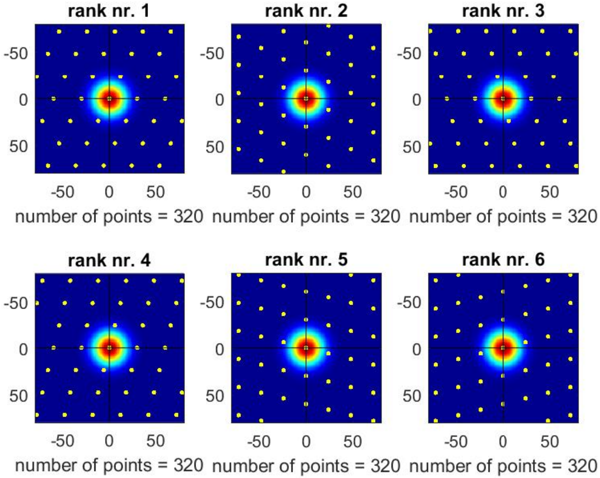
The most compact description of all possible lattices is the NABS format, which is $[n, a, b, s]$, where $[n, a, b]$ describe the usual lattices with lattice parameters (a, b) and signal length n. The s-parameter (shift!) tells us how much the columns at distance a move down in, going from one to the next nonzero column.
The MATLAB commnad `nabstolam.m` gives output like this

```
>> nabstolam(6,3,2,1); ans =
1 0 0 0 0 0
0 0 0 1 0 0
1 0 0 0 0 0
0 0 0 1 0 0
1 0 0 0 0 0
0 0 0 1 0 0
```

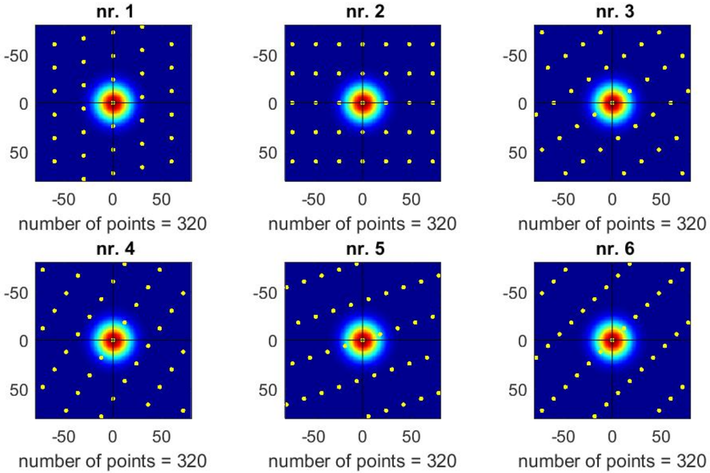
Illustration of different subgroups



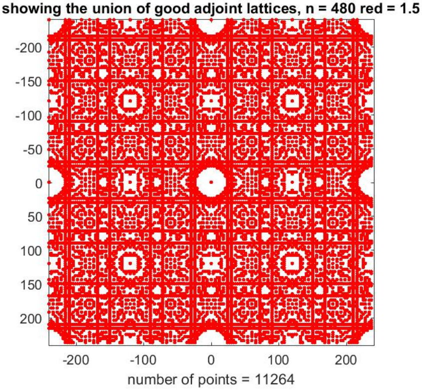
A collection of good adjoint lattices



A collection of OK adjoint lattices



Union of all good adjoint lattices!



Estimating the density of point sets

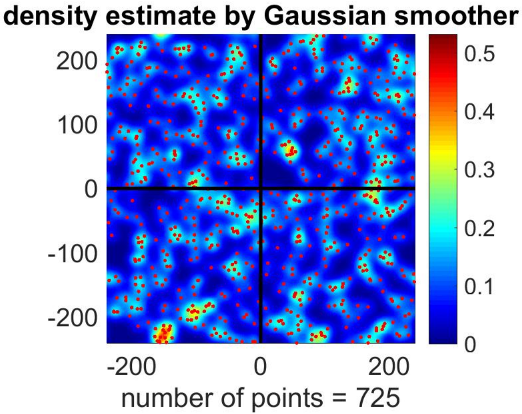


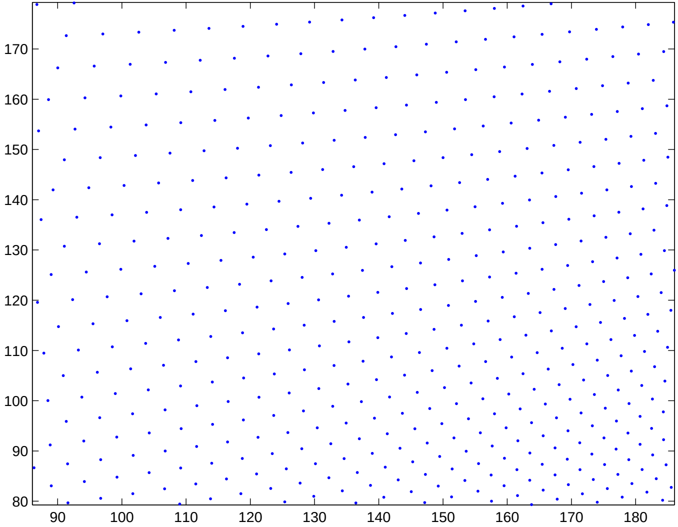
Figure: gaudens002.jpg

'Computational Gabor Analysis: Work ahead'

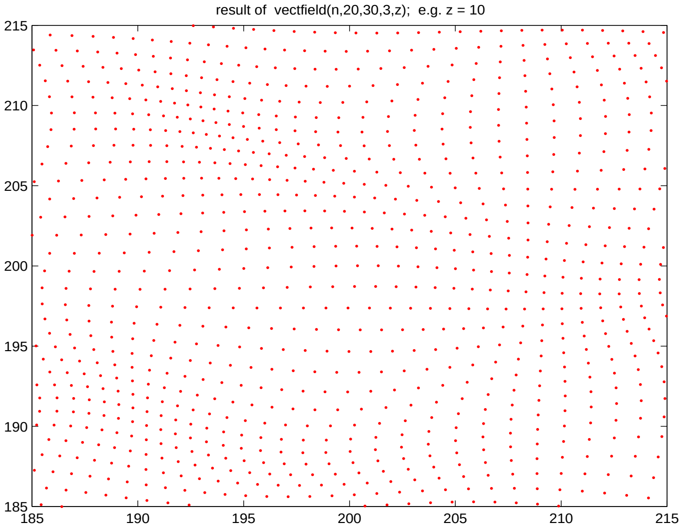
Although we know quite a bit, there is still much to be done, e.g.

1. 2D Gabor-analysis, *non-separable case*; in particular information about non-symplectic lattices?
2. slowly varying lattices or& and slowly varying atoms: find constructive methods via good approximate dual frames;
3. multiwindow setting, also in 2D, etc.
4. computing norms of functions or operators
5. deal with pseudo-differential and Fourier integral operators
6. go into numerical work concerning, e.g. the Schrödinger's equation

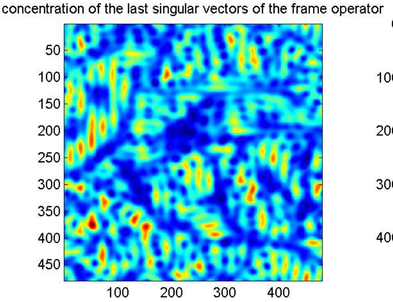
Gabor frames and vector fields

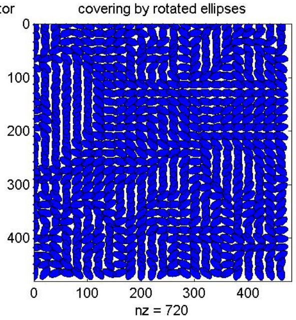


Gabor frames and vector fields II



Gabor frames and vector fields III





'And sometimes we just get a nice picture!'