Numerical Harmonic Analysis Group

The role of BANACH GELFAND TRIPLES for CONCEPTUAL HARMONIC ANALYSIS

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The KEY MESSAGES

A recent survey of the subject is in the PhD thesis of Mads Jakobsen (see also [4]).

- The prototypical (Schwartz) *Gelfand Triple* is

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow \boldsymbol{L}^2(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d).$$

  BANACH GELFAND TRIPLES are obtained by replacing $\mathcal{S}(\mathbb{R}^d)$ by a Banach space, the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

- This setting allows to deal with ordinary functions, Dirac measures, Dirac combs and their *Fourier transforms* at the same time, but also one has a *kernel theorem*!
- **CONCEPTUAL HARMONIC ANALYSIS** represents the idea that one should combine Abstract Harmonic Analysis (Fourier Analysis over LCA groups) and Num. Harmonic Analysis, going *beyond the analogy* between the settings.

Key messages expanded

Summarizing experiences also from this conference, with some **comments from a psychological or sociological view-point** I hope to contribute to the well-being of the mathematical community and thus hopefully also to the success of this ISAAC event in Växjö, and at least stimulate discussions!

1. We, as a community are *willing to do more and more complicated things*, refine existing techniques and invent new and complicated tools;
2. We are inventive in *asking questions that normal people do not understand nor would care for*!
3. We tend to forget to some extent what the true goals of our activities are, and what “practical people” really need (**not what they claim they need!**).

History and practical issues

There is no time to go into a discussion of historical considerations, but one can certainly point to the **200 years of history** of the subject, since the suggestion of J. B. Fourier that *"every function has a Fourier expansion"* to the modern, ubiquitous use of DSP (digital signal processing) methods in our daily life.

Irving Sandberg has (mathematically) correct pointed out in a series of papers that there is a **"scandal"** in the way how engineering students are exposed to translation invariant linear systems (in course on "systems theory"). Not all the operators on $\boldsymbol{C}_b(\mathbb{R})$ are convolution operators (due to the Hahn-Banach Theorem), but as far as I can say his critical voice was not appreciated very much among his colleagues.

The TRUE SCANDAL

What is for me the true scandal that we have to fight is the situation that Fourier Analysis is perceived and understood completely differently by Engineers and by Mathematicians.
And this is not the fault of either group, but the consequence of a natural development. But it is not fate and unavoidable, and we should do something about it!
Do we have answers for the engineering students, why

- why they should be careful with **integrals** in certain cases, and ignore those rules in other cases (!Dirac!)?
- why there are so different rules for **convolution**, depending on the context (e.g. periodic or non-periodic functions)?
- how the different types of **Fourier transforms** are related to each other?

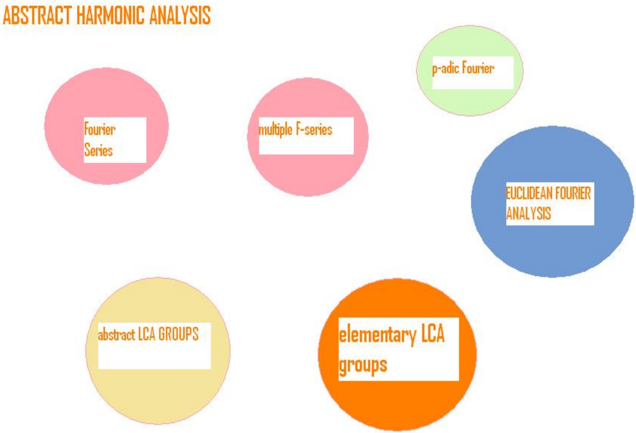
Why BANACH Gelfand Triples

- Banach spaces and linear functionals, even Banach algebras can be explained to engineering students (Experience!)
- The concept of w^*-convergence is not as abstract as one might think (and does not need a topology course);
- It is important to transport the idea that there are "ordinary" functions (maybe just relatively simple ones), and generalized functions, modelled as linear functionals.
- One needs clear rules, under which conditions certain operations (convolution, Fourier transform, pointwise multiplications) can be performed. ...
- That one can do a lot (except for PDEs!) in this setting might be a surprise... you will see ...!

Personal Background

- I begin my scientific career in *Abstract Harmonic Analysis* (AHA), working with LCA groups, dual groups, characters;
- My favorite topic (without much motivation) was *convolution and function spaces*, so I understood that *Harmonic Analysis* as a *branch within Functional Analysis*;
- Inspired by the *Theory of Function Spaces* (H. Triebel) I learned to view HA as a part of *Functional Analysis*;
- Later on I got interested Numerical Harmonic Analysis ($>$ **NuHAG**) and *real world applications*;
- I also liked distribution theory, and in fact the dual spaces for many function spaces (e.g. Sobolev spaces) are effectively Banach spaces of distributions.

The classical view of Abstract Harmonic Analysis



Personal Experiences

- Starting in 1989 I was doing more and more MATLAB experiments, e.g. in connection with our work on the irregular sampling problem for band-limited signals;
- In the last 25 years or so *Time-frequency Analysis*, in particular *Gabor Analysis* got into the focus of my work;
- Here it turned out that certain function spaces (*modulation spaces*, in particular $\boldsymbol{M}^1 = \boldsymbol{S}_0(\mathbb{R}^d)$) play a crucial role, comparable to the classical Besov-Triebel-Lizorkin spaces in the context of wavelet theory;
- Numerical simulations contributed a lot to the understanding of questions concerning Gabor analysis, and AHA was the guideline to transfer between the groups $\mathbb{R}^d$ and finite Abelian groups (at an *intuitive level*!).

Personal Work-Style

I can describe some crucial ingredients of my work as follows:

- Analyze a problem from the abstract point of view, but try to understand it properly in the continuous ($\mathbb{R}^d$) and finite (MATLAB) setting;
- Use theoretical methods to find good algorithms (e.g. compute the dual Gabor atom for a given lattice, determine the best approximation of a matrix by a Gabor multiplier,etc.);
- Get concrete numbers (e.g. condition number of a Gabor frame) from these computations;
- Get an idea about possible theoretical statements based on observations of finite-dimensional computations;

As a consequence there are some open questions!
How much can I trust my intuition?

The Idea of CONCEPTUAL HARMONIC ANALYSIS I

Abstract Harmonic Analysis is describing very well the *analogy* between different settings for convolution or the Fourier transform. It tells us that for every ABELIAN group G there is a dual group $\widehat{G}$ consisting of *characters* or *pure frequencies*. Using the Haar measure on G we can integrate any $\boldsymbol{L}^1(G)$-function f in order to obtain the Fourier transform $\hat{f}$ on $\widehat{G}$.

Lebesgue integration theory gives us that $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ is a Banach algebra with respect to convolution, and that the Fourier transform turns convolution into pointwise multiplication.

But *all this does not tell us* how to use the FFT (one of the nice routines coming along with MATLAB) in order to actually compute (at least approximately) this function $\hat{f} \in \boldsymbol{C}_0(\mathbb{R}^d)$.

The Idea of CONCEPTUAL HARMONIC ANALYSIS II

In contrast, the idea of *CONCEPTUAL HARMONIC ANALYSIS* (CHA) as proposed by the speaker recently is to go beyond this *analogy* of groups and try to find out in which sense and to which extent one could use those finite groups as *approximations to the non-periodic and continuous setting*, but not only in a heuristic way, but an *approximation theoretic spirit and numerical way.*

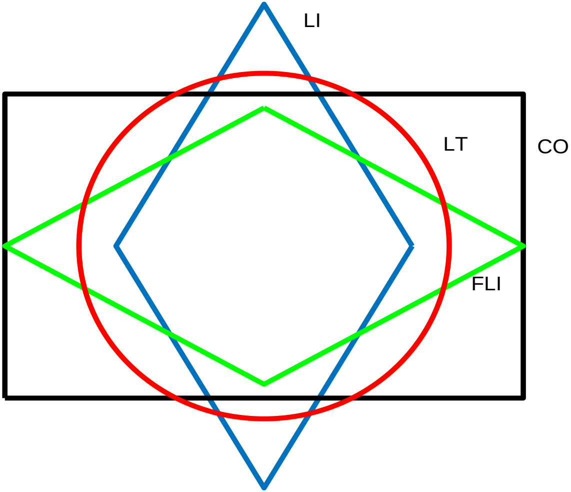
In principle, many of the questions that arise in the discussion and have to be addressed in the development of the CHA-idea can be formulated in the following way: Given an operator T from $(\boldsymbol{B}^1, \|\cdot\|^{(1)})$ to $(\boldsymbol{B}^2, \|\cdot\|^{(2)})$, and $\varepsilon > 0$, find a *computationally realizable* way to approximate $T(f)$ (for a given $f \in \boldsymbol{B}^1$) by some expression Tf_a, with $\|T(f) - Tf_a\|_{\boldsymbol{B}^2} < \varepsilon$.

The Idea of CONCEPTUAL HARMONIC ANALYSIS III

It is clear that a certain form of *distribution theory* will be needed for such a task (at least implicitly), because one has to switch between continuous functions (or distributions on $\mathbb{R}^d$) and finite vectors, typically equi-distant samples of a continuous function. The standard setting (if the Fourier transform should be one such linear operator) might be the *Gelfand triple* consisting of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidely decreasing function, densely embedded into $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ which in turn is embedded into the dual space $\mathcal{S}'(\mathbb{R}^d)$ (the space of *tempered distributions*).

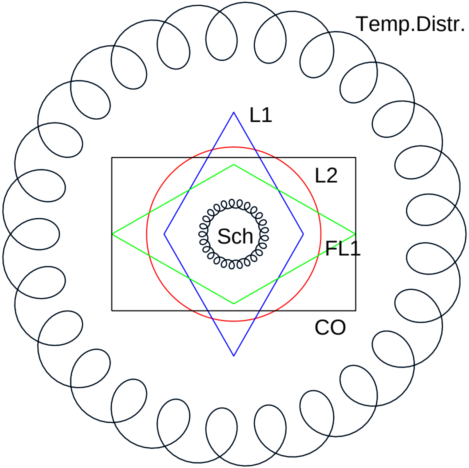
We will discuss an alternative, the **Banach Gelfand triple** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$. Also, because it is easy to define over LCA groups (this goes beyond this talk).

A schematic description of the situation: L^1, L^2

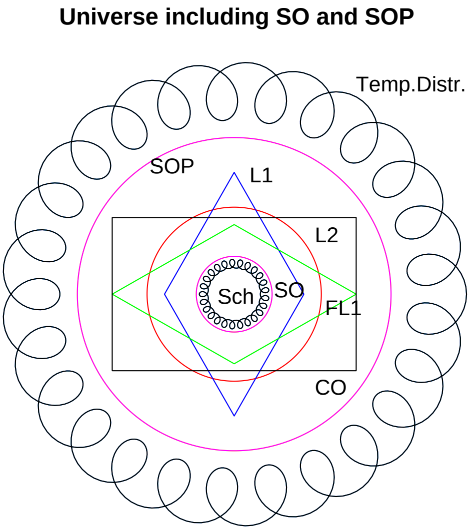


A schematic description of the situation: L^1, L^2, C_0

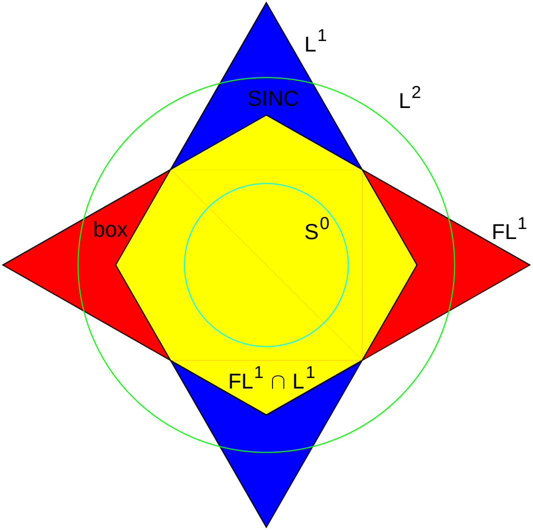
Universe of tempered distributions



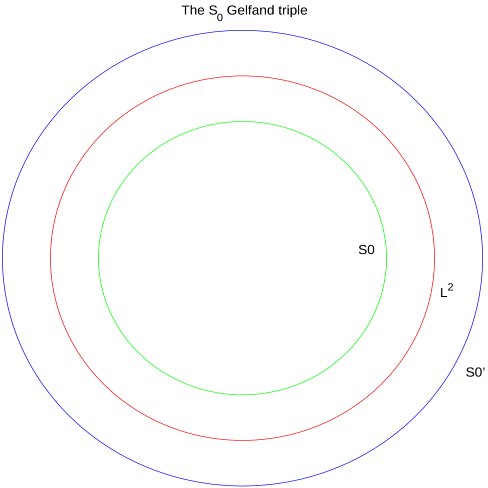
A schematic description that we are going for



A closeup on the known spaces

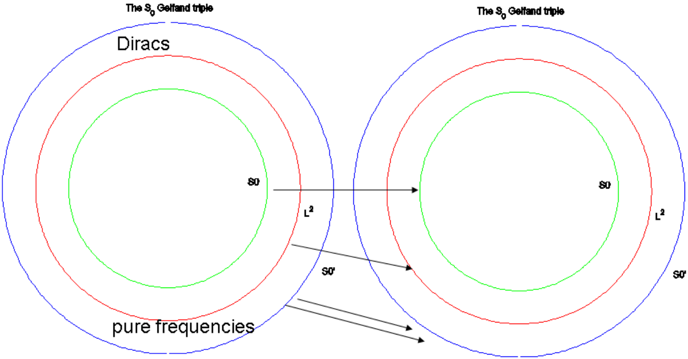


The Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$



A pictorial presentation

Gelfand triple mapping



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then IN ADDITION w^*-w^*--continuous!

Banach Gelfand Triples, the prototype

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$, 1979

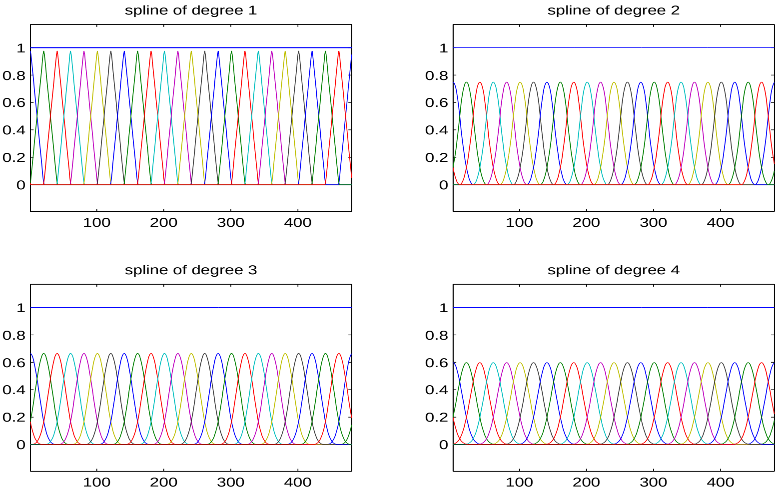
In the last 2-3 decades the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ (equal to the modulation space $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$) and its dual, $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ have gained importance for many questions of Gabor analysis or time-frequency analysis in general.

It can be characterized as the smallest (non-trivial) Banach space of (continuous and integrable) functions with the property, that time-frequency shifts acts isometrically on its elements, i.e. with

$$\|T_x f\|_B = \|f\|_B, \quad \text{and} \quad \|M_s f\|_B = \|f\|_B, \quad \forall f \in B,$$

where T_x is the usual translation operator, and M_s is the *frequency shift* operator, i.e. $M_s f(t) = e^{2\pi i s \cdot t} f(t), t \in \mathbb{R}^d$. This description implies that $S_0(\mathbb{R}^d)$ is also Fourier invariant!

Illustration of the B-splines providing BUPUs



The Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$: description

There are many different ways to describe $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Originally it has been introduced as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$, but the standard approach is to describe it via the STFT (short-time Fourier transform) using a Gaussian window given by $g_0(t) = e^{-\pi|t|^2}$.

A short description of the Wiener Amalgam space for $d = 1$ is as follows: Starting from the basis of B-splines of order ≥ 2 (e.g. triangular functions or cubic B-splines), which form a (smooth and uniform) partition of the form $(\varphi_n) := (T_n\varphi)_{n\in\mathbb{Z}}$ we can say that $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ if and only if

$$\|f\| := \sum_{n\in\mathbb{Z}} \|\widehat{f \cdot \varphi_n}\|_{\boldsymbol{L}^1} < \infty.$$

Using tensor products the definition extends to $d \geq 2$.

Banach Gelfand Triple (S_0, L^2, S_0'): BASICS

Let us collect a few facts concerning this Banach Gelfand Triple (BGTr), based on the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$:

- $S_0(\mathbb{R}^d)$ is dense in $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, in fact within any $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p < \infty$ (or in $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$);
- Any of the L^p-spaces, with $1 \leq p \leq \infty$ is continuously embedded into $S_0'(\mathbb{R}^d)$;
- Any translation bounded measure belongs to $S_0'(\mathbb{R}^d)$, in particular any Dirac-comb $\sqcup\!\sqcup_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$, for $\Lambda \lhd \mathbb{R}^d$.
- $S_0(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any $\sigma \in S_0'(\mathbb{R}^d)$ there exists a sequence of test functions s_n in $S_0(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} f(x) s_n(x) dx \to \sigma(f), \quad \forall f \in S_0(\mathbb{R}^d). \tag{1}$$

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

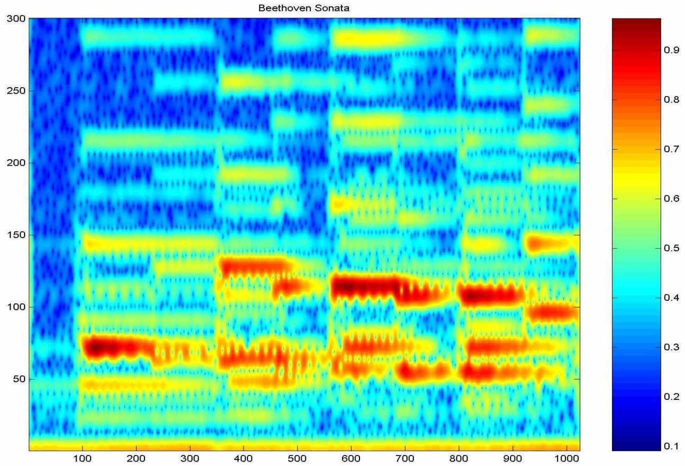
Behavior under Fourier transform

$$(T_x f)\widehat{} = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

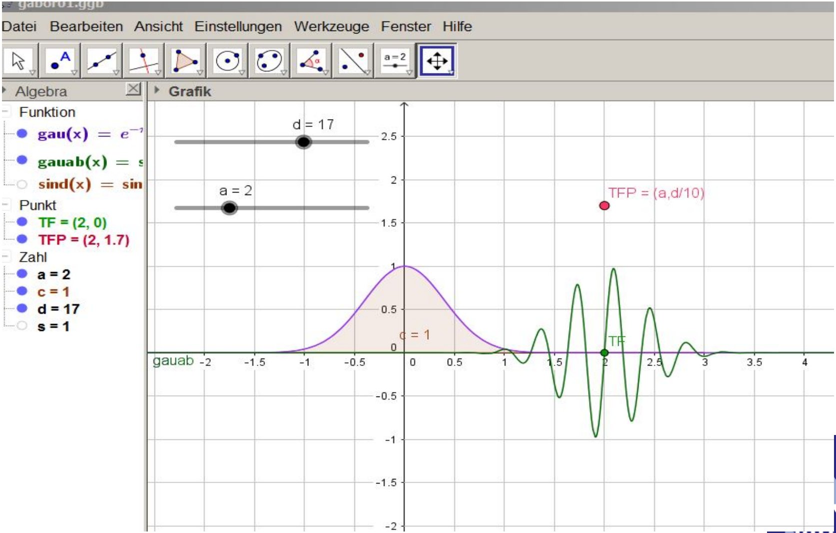
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a practical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. We have the **Moyal identity**

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^*: \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h\rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!)\, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th\rangle_{\mathcal{H}_2} = \langle h, T^*Th\rangle_{\mathcal{H}_1}, \; \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$

In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (2)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h\rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (3)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \quad (4)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (5)$$

This is quite analogous to the situation of the Fourier transform

$$(6) \quad f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, = \int_{\mathbb{R}^d} \hat{f}(x) e^{2\pi i s\cdot}\, ds \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (6)$$

with $\chi_s(t) = exp(2\pi i\langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Introducing the space $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

The Banach space (and actually Segal algebra) $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ has been introduced by the speaker in 1979, in a paper in Monatshefte f. Math., entitled "On a New Segal Algebra". Most of the basic properties if this space of test functions, including is minimality among Banach spaces of functions which are isometrically invariant under time-frequency shifts, and the Fourier invariance have been demonstrated already in that first paper. Modern approaches to this space, called $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$ in the book of Gröchenig, can be found it his book. It is now common practice to define $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ via the membership of the STFT (short time Fourier transform) with respect to a Gaussian window $g_0(t) = e^{-\pi|t|^2}$ and choose as a norm

$$\|f\|_{S_0} := \int_{\mathbb{R}^{2d}} |V_g(\lambda)| d\lambda = \|V_g(f)\|_1. \tag{7}$$

Characterization of $S_0'(\mathbb{R}^d)$ and w^*-convergence

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ if and only if its (continuous) STFT is a *bounded* function. Furthermore convergence corresponds to uniform convergence of the spectrogram (different windows give equivalent norms!).
We can also extend the Fourier transform form $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$ via the usual formula $\hat{\sigma}(f) := \sigma(\hat{f})$.
The weaker convergence, arising from the functional analytic concept of w^*-convergence has the following very natural characterization: A (bounded) sequence σ_n is w^*- convergence to σ_0 if and only if for one (resp. every) $S_0(\mathbb{R}^d)$-window g one has

$$V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda) \quad \text{for} n \to \infty,$$

uniformly over compact subsets of phase-space.

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (6) (Fourier inversion) and the new formula formula (5). While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [6]).
Gabor Analysis is the theory describing how one can get exact recovery while still using a not too dense lattice Λ.

Applications to Translation Invariant Systems

Engineers like to describe "translation invariant systems" as convolution operators by some *impulse response*, or equivalently by the pointwise multiplication of $\hat{f}$ (the input signal) by some *transfer function*. In sloppy terms:

$$T(f) = \mu * f, \quad \text{with} \quad \mu = T(\delta_0)$$

$$\widehat{Tf} = h \cdot \hat{f}.$$

Here we refer to the engineering terminology: A TILS is linear operator (often the domain is left undefined!) with the property that

$$T_x \circ T = T \circ T_x, \quad x \in \mathbb{R}^d.$$

Translation Invariant Systems II, TILS2

Theorem

$$\mathcal{H}_G(\boldsymbol{S}_0(\mathbb{R}^d), \boldsymbol{S}_0'(\mathbb{R}^d)) = \boldsymbol{S}_0'(\mathbb{R}^d)$$

i.e. for every $T \in \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ *commuting with translation there is a unique* $\sigma = \sigma_T$ *such that* $Tf(x) = \sigma(T_x f^{\checkmark})$ *where* $f^{\checkmark}(x) = f(-x)$. *Also the converse is true, and the operator norm of* T *is equivalent to the* $\boldsymbol{S}_0'$*-norm of* σ.

Corollary

Any translation invariant operator from $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ *to* $\left(\boldsymbol{L}^q(\mathbb{R}^d), \|\cdot\|_q\right)$, $1 \leq p, q < \infty$ *can be represented (on* $\boldsymbol{S}_0(\mathbb{R}^d)$*) as a convolution operator by* $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ *or with the transfer "function"* $h = \widehat{\sigma}$ *(Fourier multipliers).*

Classical Analysis and Summability

Among the functions typically used in Fourier analysis only the so-called BOX-function ($\mathbf{1}_Q$) (being discontinuous) and its Fourier transform, the SINC-function (not belonging to $\boldsymbol{L}^1(\mathbb{R}^d)$) are NOT elements of $\boldsymbol{S}_0(\mathbb{R})$, while (according to F. Weisz)

all the classical summability kernels belong to $\boldsymbol{S}_0(\mathbb{R}^d)$

.

The space $\boldsymbol{S}_0(\mathbb{R}^d)$ is also the natural domain for the *Poisson summation formula*, another important tool in Fourier analysis:

$$\sum_{k\in\mathbb{Z}^d} f(k) = \sum_{n\in\mathbb{Z}^d} \widehat{f}(n), \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{8}$$

There are counter-examples, but they all work only when one is using function spaces not contained in $\boldsymbol{S}_0(\mathbb{R}^d)$.

A First Application to the Fourier Transform

When it comes to the approximate realization of a Fourier related task we can point to joint work with N. Kaiblinger:

Theorem

Given $f \in \boldsymbol{S}_0(\mathbb{R})$ and $\varepsilon > 0$ it is enough to apply the FFT to a sequence of equi-distant samples, taken sufficiently fine and over a sufficiently long interval, then apply the FFT *to this sequence and regain a continuous (and compactly supported) function $\hat{f}_a$ in $\boldsymbol{S}_0(\mathbb{R})$ via (linear or) quasi-inteprolation, with*

$$\|\hat{f} - \hat{f}_a\|_{\boldsymbol{S}_0} < \varepsilon.$$

The Metaplectic Invariance of $S_0(\mathbb{R}^d)$

Aside from the convenient properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (including the possibility to use such a space over general LCA groups) the first important application of $\boldsymbol{S}_0(\mathbb{R}^d)$ is in the Lecture Notes of Hans Reiter, entitled *Metaplectic Groups and Segal Algebras* which appeared in the Springer Lect. Notes in Mathematics in 1989, and which provides a detailed description of the use of $\boldsymbol{S}_0(\mathbb{R}^d)$ for the analysis of the **metaplectic** group, which among others includes the *Fractional Fourier transforms*.

The invariance of $\boldsymbol{S}_0(\mathbb{R}^d)$ under general automorphisms of the group $\mathbb{R}^d$ as well as under all the metaplectic operator, e.g. convolution or pointwise multiplication by the chirp functions $t \to e^{i\alpha t^2}, 0 \neq \alpha \in \mathbb{R}$ is crucial in this setting.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis I

ONE of the key questions in Gabor analysis is the question, when a Gabor family $(G,\Lambda) = (g_\lambda)_{\lambda\in\Lambda}$, with some Gabor atom $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ is a Gabor frame, where $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ is some lattice.
Standard frame theory tells us the following things:

1. (g,Λ) defines a Gabor frame if and only if the frame operator

$$S_{g,\Lambda} : f \to \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda$$

is invertible on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$;

2. In such a case $\widetilde{g} := S^{-1}(g) \in \boldsymbol{L}^2(\mathbb{R}^d)$ generates the dual frame, i.e. the dual frame is of the form $(\widetilde{g}_\lambda)_{\lambda\in\Lambda}$.
3. This allows two kinds of representations of any $f \in \boldsymbol{L}^2$:

$$f = \sum_{\lambda\in\Lambda} \langle f, \widetilde{g}_\lambda\rangle g_\lambda = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle \widetilde{g}_\lambda. \quad (9)$$

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis II

The fact that it is impossible to find Gaborian Riesz bases with "good generators" (by the Balian-Low Theorem, i.e. for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ (g, Λ) never gives a Riesz basis for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$!) makes it important to control the window g as well as the dual window in terms of "of good quality".

The first step is already the boundedness of the frame operator $S_{g,\Lambda}$, which is relatively easy to show for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ (and most of the time not available for $g \notin \boldsymbol{S}_0(\mathbb{R}^d)$). More important is the following observation: Whenever $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ the operator $S_{g,\Lambda}$ is bounded on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Gröchenig/Leinert have shown:

Theorem

Whenever $S_{g,\Lambda}$ is invertible on $\boldsymbol{L}^2(\mathbb{R}^d)$ for some $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, it is also invertible on $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, hence $\widetilde{g} = S^{-1}(g) \in \boldsymbol{S}_0(\mathbb{R}^d)$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis III

This can be used to check that the representation formula (9) is also valid in the $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$-sense for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and can be extended (now with w^*-convergence) to general $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$.

For $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ it is true that a small jitter error, i.e. using instead of $V_g(\lambda) = \langle f, g_\lambda \rangle$ some nearby sampling value $V_g(\lambda + \gamma_\lambda)$ with $|\gamma_\lambda| \leq \gamma_0$ for some small constant γ_0. Then, e.g., the reconstruction of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ from these slightly perturbed samples will show error (in the $\boldsymbol{S}_0$-norm sense!).

Also for the computation of *approximate dual Gabor windows h* it is important to ensure a small error in the $\boldsymbol{S}_0$-norm sense, because otherwise it is *not possible* to control the error of the computable operator $\sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle h_\lambda$ in the operator norm sense (even on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$).

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis IV

In finite dimensions, e.g. over the group $\mathbb{Z}_N$, a Gabor family is a frame if and only if it is a generating system for $\mathbb{C}^N$, or in other words, if and only if every $\mathbf{x} \in \mathbb{C}^N$ can be represented as linear combination of elements from the Gabor family. Writing `GAB` for the Gabor family with atom $g \in Cst^n$ it is clear that we need $n \geq N$ such vectors for the spanning property, resp. we need that GAB is of maximal rank N.

The "optimal representation" for a redundant system is then of course the **MNLSQ** solution $\mathbf{y}_0$, i.e. the choice of those coefficients which represent the given signal as $\mathbf{x} = GAB * \mathbf{y}_0$ which minimize $\|\mathbf{y}\|_{\mathbb{C}^n}$ among all coefficient sequences with $\mathbf{x} = GAB * \mathbf{y}$. This sequence can be obtained via the pseudo-inverse matrix `pinv(GAB)` via $\mathbf{y}_0 = \text{pinv}(GAB) * \mathbf{x}$. The collection of (conjugate) rows of $\text{pinv}(GAB)$ or columns of $\text{pinv}(GAB)' = \text{pinv}(GAB)'$ in MATLAB notation is just the dual frame!

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis V

The important formula which applies in this situation (it can be derived easily from the SVD decomposition of a matrix $\mathbf{A}$)

$$\mathrm{pinv}(\mathbf{A}') = inv(\mathbf{A} * \mathbf{A}') * \mathbf{A}$$

shows that the *dual frame* can be obtained by applying the inverse of the frame matrix $\mathbf{S} = \mathbf{A} * \mathbf{A}'$ to the elements of the original frame (columns of $\mathbf{A}$).
But it is better to use a commutative diagram for this, showing that and how the signal $\mathbf{x}$ can be reconstructed from the set of scalar products with the frame elements, i.e. from $\mathbf{A}' * \mathbf{x}$ by multiplying from the left with $(\mathbf{A} * \mathbf{A}')^{-1} * \mathbf{A}$.

The Role of $S_0(\mathbb{R}^d)$ for Gabor Analysis VI

In the continuous setting and for Gabor frames with $\mathbf{S}_0(\mathbb{R}^d)$-atoms g we have the following situation:

Theorem

Given a Gabor frame (g, Λ) with $g \in \mathbf{S}_0(\mathbb{R}^d)$ one has:

- *the coefficient mapping $\mathbf{C} : f \to V_g(f)|_\Lambda$ is an BGTr homomorphism from $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.*
- *For $\widetilde{g} = S^{-1}(g) \in \mathbf{S}_0(\mathbb{R}^d)$ the Gabor synthesis mapping*

$$\mathbf{R} : (\mathbf{c}_\lambda) \to \sum_{\lambda \in \boldsymbol{\Lambda}} \mathbf{c}_\lambda \widetilde{\mathbf{g}}_\lambda$$

  is a BGTr homomorphism $(\ell^1, \ell^2, \ell^\infty)(\Lambda) \to (\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$;
- *$\mathbf{R}$ is a left inverse to $\mathbf{C}$: $\mathbf{R} \circ \mathbf{C} = \mathbf{Id}$ on $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$.*

RELEVANT APPLICATIONS

After a quick general description of Banach Gelfand Triples (BGTr) in an abstract setting and the foundations of the concrete BGTr, based on the Segal Algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ we indicate some of the many applications, e.g.

1. Fourier Transform as unitary BGTr Automorphism
2. The Kernel Theorem
3. The Spreading representation of Operators
4. The Kohn-Nirenberg Symbol of Operators
5. Gabor Analysis and Janssen Representation
6. Robustness Considerations in Gabor Analysis
7. Generalized Stochastic Processes

The Kernel Theorem

It is clear that such operators between functions on $\mathbb{R}^d$ cannot all be represented by integral kernels using locally integrable $K(x,y)$ in the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x,y \in \mathbb{R}^d, \tag{10}$$

because clearly multiplication operators should have their support on the main diagonal, but $\{(x,x) \,|\, x \in \mathbb{R}^d\}$ is just a set of measure zero in $\mathbb{R}^d \times \mathbb{R}^d$!

Also the expected "rule" to find the kernel, namely

$$K(x,y) = T(\delta_y)(x) = \delta_x(T(\delta_y)) \tag{11}$$

might not be meaningful at all.

The Hilbert Schmidt Version

There are two ways out of this problem

- restrict the class of operators
- enlarge the class of possible kernels

The first one is a classical result, i.e. the characterization of the class $\mathcal{HS}$ of Hilbert Schmidt operators.

Theorem

A linear operator T on $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$ is a Hilbert-Schmidt operator, i.e. is a compact operator with the sequence of singular values in ℓ^2 if and only if it is an integral operator of the form (10) with $K \in \boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$. In fact, we have a unitary mapping $T \to K(x,y)$, where $\mathcal{HS}$ is endowed with the Hilbert-Schmidt scalar product $\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*)$.

The Schwartz Kernel Theorem

The other well known version of the kernel theorem makes use of the *nuclearity* of the *Frechet space* $\mathcal{S}(\mathbb{R}^d)$ (so to say the complicated topological properties of the system of seminorms defining the topology on $\mathcal{S}(\mathbb{R}^d)$).
Note that the description cannot be given anymore in the form (10) but has to replaced by a "weak description". This is part of the following well-known result due to L. Schwartz.

Theorem

There is a natural isomorphism between the vector space of all linear operators from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, i.e. $\mathcal{L}(\mathcal{S},\mathcal{S}')$, and the elements of $\mathcal{S}'(\mathbb{R}^{2d})$, via $\langle Tf, g\rangle = \langle K, f\otimes g\rangle$, for $f,g\in\mathcal{S}(\mathbb{R}^d)$.

The S_0-KERNEL THEOREM

In the current setting we can describe the kernel theorem as a unitary Banach Gelfand Triple isomorphism, between operator and their (distributional) kernels, extending the classical Hilbert Schmidt version.

First we observe that $\mathbf{S}_0$-kernels can be identified with $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$, i.e. the *regularizing operators* from $\mathbf{S}_0'(\mathbb{R}^d)$ to $\mathbf{S}_0(\mathbb{R}^d)$, even mapping bounded and w^*- convergent nets into norm convergent sets. For those kernels also the recovery formula (11) is valid.

Theorem

The unitary Hilbert-Schmidt kernel isomorphisms extends in a unique way to a Banach Gelfand Triple isomorphism between $(\mathcal{L}(\mathbf{S}_0', \mathbf{S}_0), \mathcal{HS}, \mathcal{L}(\mathbf{S}_0, \mathbf{S}_0'))$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$.

The Spreading Representation

The **spreading representation** of operators interpretation. In some sense it can be viewed as a kind of Fourier Transform for operators. For the case of $G = \mathbb{Z}_N$ we have N^2 time-frequency shift operators (cyclic shifts combined with pointwise multiplication by pure frequencies), and in fact they form an orthonormal basis for the (Euclidean) space of $N \times N$-matrices (linear operators on $\mathbb{C}^N$), with the Frobenius scalar product.

Theorem

There is a unique (unitary) Banach Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$*, which maps the time frequency shift operators* $\pi(\lambda) := M_\omega T_t$ *to the Dirac measures* $\delta_{t,\omega} \in \boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Spreading Representation II

This also tells us, that an operator $T \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ is regularizing if it can be written as an operator-valued Riemannian integral

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(\lambda)\pi(\lambda) d\lambda. \tag{12}$$

Of course one can also write explicit formulas (involving various transformations and partial Fourier transform) for the transition between the kernel of an operator T and its spreading "function" $\eta(T)$ (cf. [3]) which are valid in the pointwise sense (using standard integration theory), while one has to extend it to the Hilbert space case by continuity (like the usual proofs of Plancherel's theorem) and then extend it to the *outer layer* via duality (or w^*-continuity). See also [1]

The Kohn-Nirenberg Symbol

For various applications in the area of *pseudo-differntial operators* and for applications in Gabor Analysis also the so-called **Kohn-Nirenberg Symbol** $\sigma(T)$ of an operator T is of interest. It is obtained from the spreading representation via the so-called *symplectic Fourier transform*.

Theorem

The (unitary)KNS Banach Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, $T \to \sigma(T)$ has the following covariance property:

$$\sigma[\pi(\lambda) \circ T \circ \pi(\lambda)'] = T_{t,\omega}\sigma(T).$$

Applications to Gabor Multipliers

This last property can be used to e.g. describe the best approximation of a given operator by a Gabor multiplier. The most important Gabor multipliers arise from tight regular Gabor frames, i.e. families of the form $(\pi(\lambda)g)_{\lambda\in\Lambda}$, with Λ being any lattice in $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, with the property (writing g_λ for $\pi(\lambda)g$) with the following reconstruction property:

$$f = \sum_{\lambda\in\Lambda} \langle f, g_\lambda\rangle g_\lambda, \quad f \in (\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0'). \tag{13}$$

We also can write $P_\lambda : f \to \langle f, g_\lambda\rangle g_\lambda$. Given a numerical sequence over the lattice (m_λ) the Gabor multiplier $G_m := \sum_{\lambda\in\Lambda} m(\lambda)P_\lambda$. The problem of best approximation of some $\mathcal{HS}$ operator by Gabor multipliers can be reformulated as an approximation problem using spline-type spaces via the Kohn-Nirenberg connection.

There is just one Fourier transform

As a colleague (Jens Fischer) at the German DLR (in Oberpfaffenhausen) puts it in his writing: "There is just one Fourier Transform" And I may add: and it is enough to know about $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ in order to understand this principle and to make it mathematically meaningful.

In **engineering courses** students learn about discrete and continuous, about periodic and non-periodic signals (typically on $\mathbb{R}$ or $\mathbb{R}^2$), and they are treated separately with different formulas. Finally comes the DFT/FFT for finite signals, when it comes to computations. The all look similar.

Mathematics students learning Abstract Harmonic Analysis learn that one has to work with different LCA groups and their dual groups. Gianfranco Cariolaro (Padua) combines the view-points somehow in his book Unified Signal Theory (2011).

w^*-convergence justifies the various transitions!

Periodicity and Fourier Support Properties

The world of distributions allows to deal with continuous and discrete, periodic and non-periodic *signals* at equal footing. Let us discuss how they are connected.

The general Poisson Formula, expressed as

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} \tag{14}$$

can be used to prove

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda * f) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} \cdot \mathcal{F}(f), \tag{15}$$

or interchanging convolution with pointwise multiplication:

$$\mathcal{F}(\sqcup\!\sqcup_\Lambda \cdot f) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\perp} * \mathcal{F}(f). \tag{16}$$

I.e.: Convolution by $\sqcup\!\sqcup$ (corresponding to *periodization*) corresponds to pointwise multiplication (i.e. *sampling*) on the Fourier transform domain and *vice versa*.

Approximation by discrete and periodic signals

The combination of two such operators, just with the assumption that the sampling lattice Λ_1 is a subgroup (of finite index N) of the periodization lattice Λ_2 implies that

$$\sqcup\!\sqcup_{\Lambda_2} * [\sqcup\!\sqcup_{\Lambda_1} \cdot f] = \sqcup\!\sqcup_{\Lambda_1} \cdot [\sqcup\!\sqcup_{\Lambda_2} * f], \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{17}$$

For illustration let us take $d = 1$ and $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = N\alpha\mathbb{Z}$ and hence $\Lambda_1^\perp = (1/\alpha)\mathbb{Z}$. Then the periodic and sampled signal arising from equ. 17 corresponds to a vector $\mathbf{a} \in \mathbb{C}^N$ and the distributional Fourier transform of the periodic, discrete signal is completely characterized is again discrete and periodic and its generating sequence $\mathbf{b} \in \mathbb{C}^N$ can be obtained via the DFT (FT of quotient group), e.g. $N = k^2, \alpha = 1/k$, and period k.

Approximation by discrete and periodic signals 2

It is not difficult to verify that in this way, by making the sampling lattice more and more refined and periodization lattice coarser and coarser the resulting discrete and periodic versions of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, viewed as elements within $\boldsymbol{S}_0'(\mathbb{R}^d)$, are approximated in a bounded and w^*-sense by discrete and periodic functions.
This view-point can be used as a justification of the fact used in books describing heuristically the continuous Fourier transform, as a limit of Fourier series expansions, with the *period going to infinity*.

Mutual w^*-approximations

The density of test functions in the dual space can be obtained in many ways, using so-called *regularizing operators*, e.g. combined approximated units for convolution and on the other hand for pointwise convolution, based on the fact that we have

$$(\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d), \quad \text{and} \tag{18}$$

$$(\boldsymbol{S}_0(\mathbb{R}^d) \cdot \boldsymbol{S}_0'(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d). \tag{19}$$

Alternatively one can take finite partial sums of the Gabor expansion of a distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ which approximate σ in the w^*-sense (boundedly), for Gabor windows in $\boldsymbol{S}_0(\mathbb{R}^d)$.

On the other hand one can approximate test functions (in the w^*-sense) by discrete and periodic signals!

Approximation of Distributions by Test Functions

These properties of *product-convolution operators* or *convolution-product operators* can be used to obtain a w^*-approximation of general elements $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$ by test functions in $\mathbf{S}_0(\mathbb{R}^d)$. For example, one can take a Dirac family obtained by applying the compression operator

$$\mathrm{St}_\rho(g) := \rho^{-d} g(x/\rho), \quad \rho \to 0$$

in order to approximate σ by bounded and continuous functions of the form $\mathrm{St}_\rho(g_0) * \sigma$.
For the localization one can use the dilation operator

$$\mathrm{D}_\rho(h)(z) = h(\rho z), \quad \rho \to 0,$$

so altogether

$$\sigma = w^* - \lim_{\rho \to 0} \mathrm{D}_\rho g_0 \cdot [(\mathrm{St}_\rho g_0) * \sigma]$$

where all the functions on the right hand side belong to $\mathbf{S}_0(\mathbb{R}^d)$.

Generalized Stochastic Processes

The space of test functions is also very useful to model **Generalized Stochastic Processes** (GSPs) simple as bounded linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to some (abstract, or concrete) Hilbert space (of random variables): $\rho : f \to \rho(f) \in \mathcal{H}$.

Such GSPs have a natural *autocorrelation distribution* $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$, and its invariance properties correspond to e.g. wide-sense stationarity of the process itself.

There is also a Fourier transform $\hat{\rho}$ of such a process, and the autocorrelation of the $\hat{\rho}$ is just (the $2d$) Fourier transform of σ! The inverse Fourier transform is a very natural replacement for the "spectral representation" of a process.

Details can be found in [2].

Fourier Analysis over LCA Groups

Following Andre Weil ([5]) Fourier Analysis (and consequently Time-Frequency Analysis) has LCA groups as its natural domain. With some modifications $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ and thus the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(G)$ can be defined over general LCA groups G (using the Haar measure and Pontryagin's duality Theorem).

Among others it allows to extend the *Fourier transform* for $\boldsymbol{L}^p(G)$, for $1 \leq p \leq \infty$, and thus define the spectrum of elements of $h \in \boldsymbol{L}^\infty(G)$ as $\text{supp}(\hat{h})$, allowing for a more natural discussion of *sets of spectral synthesis*.

But of course this setting is also most appropate for the discussion of *Gabor Analysis* over general LCA groups!

A recent survey of the subject is in the PhD thesis of Mads Jakobsen (see also [?]).

The weighted case, outlook on PDE

In the current talk we have restricted our attention to the unweighted case, and only isometrically time-frequency invariant Banach spaces of distributions have been considered, with $\boldsymbol{S}_0(\mathbb{R}^d)$ as minimal and $\boldsymbol{S}_0'(\mathbb{R}^d)$ as the maximal space in the family. This is quite suitable for applications in Abstract Harmonic Analysis or in Communication Theory (Linear Systems, etc.), or Theoretical Physics (Quantum Theory), but not for PDE or pseudo-differential operators.
For such a setting one has to resort for *families of modulation spaces*, which are defined by means of the behaviour of the short time Fourier transform.

Fourier Standard Spaces, the Idea

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, continuously embedded between $\boldsymbol{S}_0(G)$ and $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$, i.e. with

$$(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}) \quad (20)$$

is called a **Fourier Standard Space** on G (FSS of FoSS) if it has a double module structure over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\widehat{G}))$ with respect to pointwise multiplication.

Typically we just require that in addition to (20) one has:

$$\boldsymbol{L}^1 * \boldsymbol{B} \subseteq \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1 \cdot \boldsymbol{B} \subseteq \boldsymbol{B}. \quad (21)$$

Constructions within the FSS Family

1. Taking Fourier transforms;
2. Conditional dual spaces, i.e. the dual space of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
3. With two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take intersection or sum
4. forming amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. defining pointwise or convolution multipliers;
6. using complex (or real) interpolation methods, so that we get the spaces $\boldsymbol{M}^{p,p} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$ (all Fourier invariant);
7. Fractional invariant kernel and hull: For any given standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we could define the largest Banach space inside of $\boldsymbol{B}$ which is inveriant under all the fractional FTs, or the smallest such space which allows a continuous embedding of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

FOURIER STANDARD SPACES: II

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions and so on. Thus among others the space $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.
Within the family there are two subfamilies, namely the *Wiener amalgam spaces* and the so-called *modulation spaces*, among them the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$.

TF-homogeneous Banach Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \; f \in \boldsymbol{B}. \tag{22}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous one often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$).

TF-homogeneous Banach Spaces II

An important fact concerning this family is the minimality property of the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d)$.

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