FOURIER STANDARD SPACES
A comprehensive class of function spaces

Hans G. Feichtinger
hans.feichtinger@univie.ac.at
www.nuhag.eu

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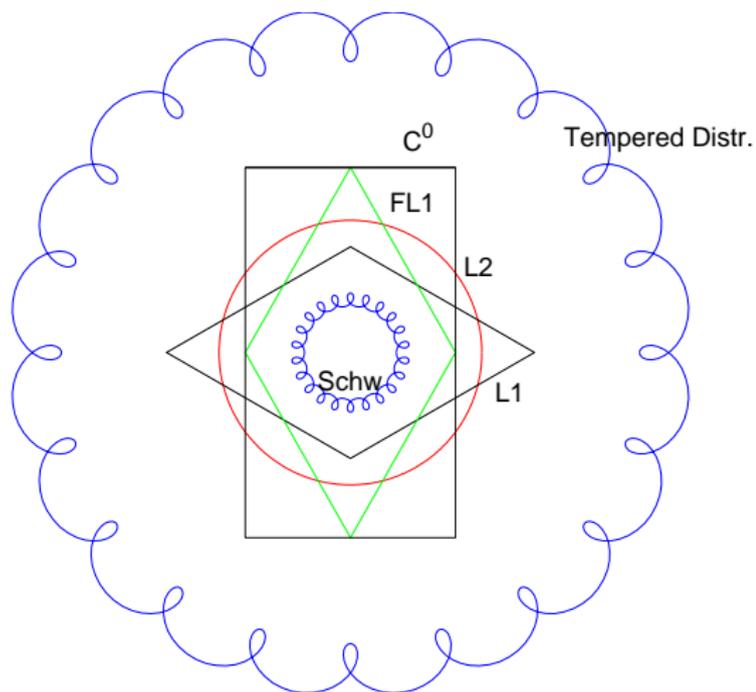
$L^1(\mathbb{R}^d)$ and the Fourier Algebra $\mathcal{F}L^1(\mathbb{R}^d)$
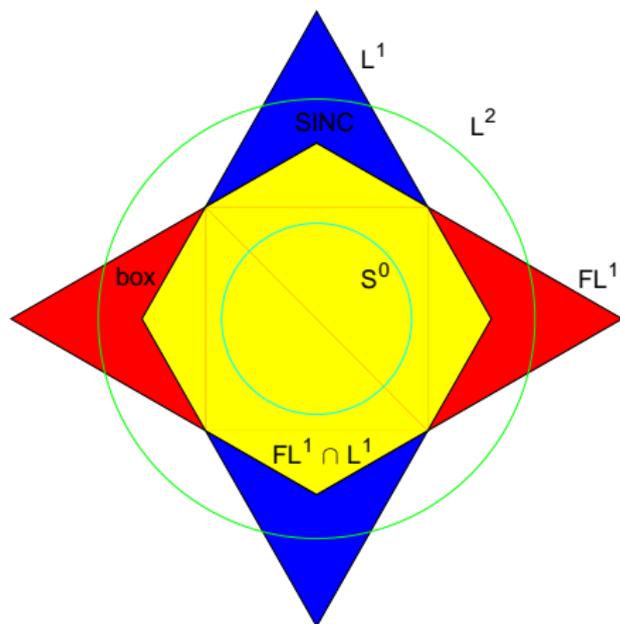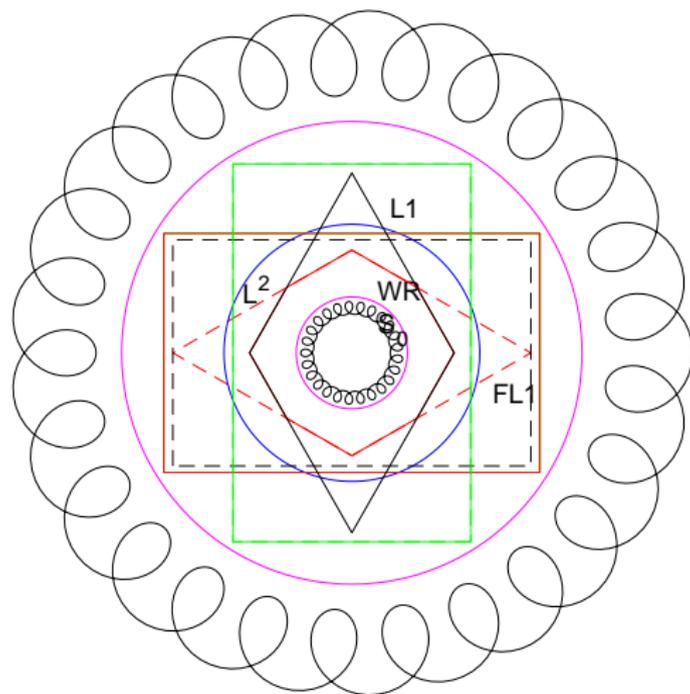

Figure: soplLIFLI3.jpg generated by soplot5.m

Figure: The classical setting

In this talk we will concentrate on the setting of the LCA group $G = \mathbb{R}^d$, although practically all the results could be formulated in the general setting of locally compact Abelian group as promoted by A. Weil. Classical Fourier Analysis pays a lot of attention to $(L^p(\mathbb{R}^d), \|\cdot\|_p)$ because these spaces (specifically for $p \in \{1, 2, \infty\}$) are important to set up the Fourier transform as an integral transform which also respects convolution (we have the convolution theorem) and preserving the energy (meaning that it is a unitary transform of the Hilbert space $(L^2(\mathbb{R}^d), \|\cdot\|_2)$).

OVERVIEW II

In the last 2-3 decase the Segal algebra $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$ (equal to the modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$) and its dual, $\big(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'}\big)$ have gained importance for many questions of Gabor analysis or time-frequency analysis in general.

Fourier standard spaces are now a class of Banach spaces sandwiched in between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$, with *two module structures*, one with respect to the Banach convolution algebra $\big(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\big)$, and the other one with respect to pointwise multiplication by elements of the Fourier algebra $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$.

As we shall point out there is a huge variety of such spaces, and many questions of Fourier analysis find an appropriate description in this context.

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions and so on. Thus among others the space $L^1(\mathbb{R}^d) \cap \mathcal{F}L^1(\mathbb{R}^d)$.

Within the family there are two subfamilies, namely the *Wiener amalgam spaces* and the so-called *modulation spaces*, among them the Segal algebra $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ or Wiener's algebra $(W(C_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_W)$.

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ with

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall a \in \boldsymbol{A}, b \in \boldsymbol{B}, \tag{1}$$

bilinear and associative...

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}. \tag{2}$$

Definition

A Banach space $(B, \|\cdot\|_B)$ is a *Banach ideal* in (or within, or of) a Banach algebra $(A, \cdot, \|\cdot\|_A)$ if $(B, \|\cdot\|_B)$ is continuously embedded into $(A, \cdot, \|\cdot\|_A)$, and if in addition (1) is valid with respect to the internal multiplication inherited from A.

An important class of such Banach ideals are the so-called *Segal algebras* (in the sense of Hans Reiter). By definition they are just translation invariant (isometrically!) Banach spaces, densely embedded into $(L^1(G), \|\cdot\|_1)$, and with continuous shift, but via vector-valued integration one can show that they are (essential, as we will see) Banach ideals within $(L^1(G), \|\cdot\|_1)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t).$$

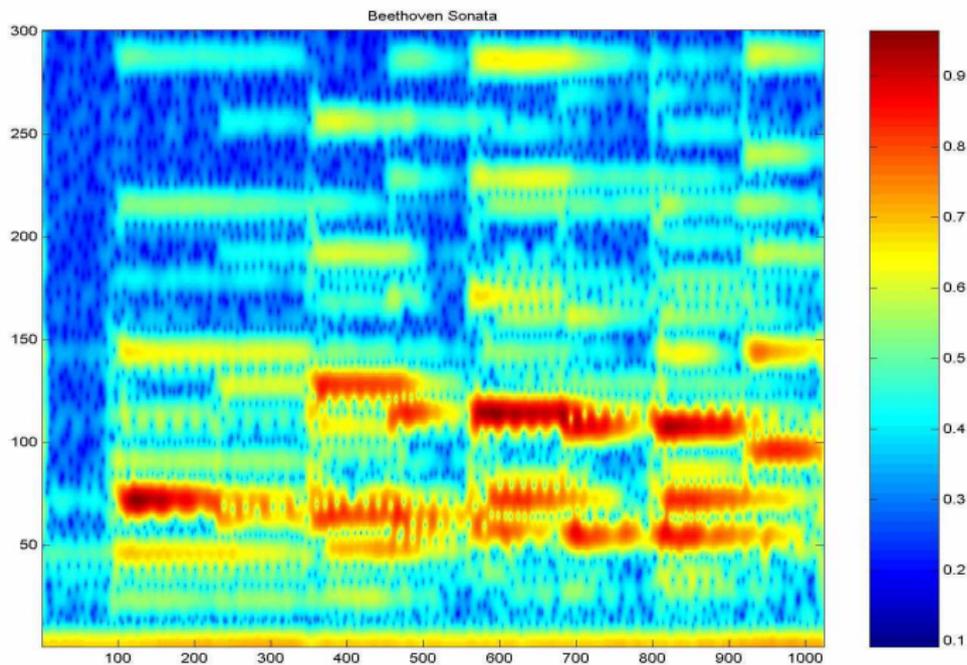Behavior under Fourier transform

$$(T_x f)\hat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\hat{\ } = T_\omega \hat{f}$$
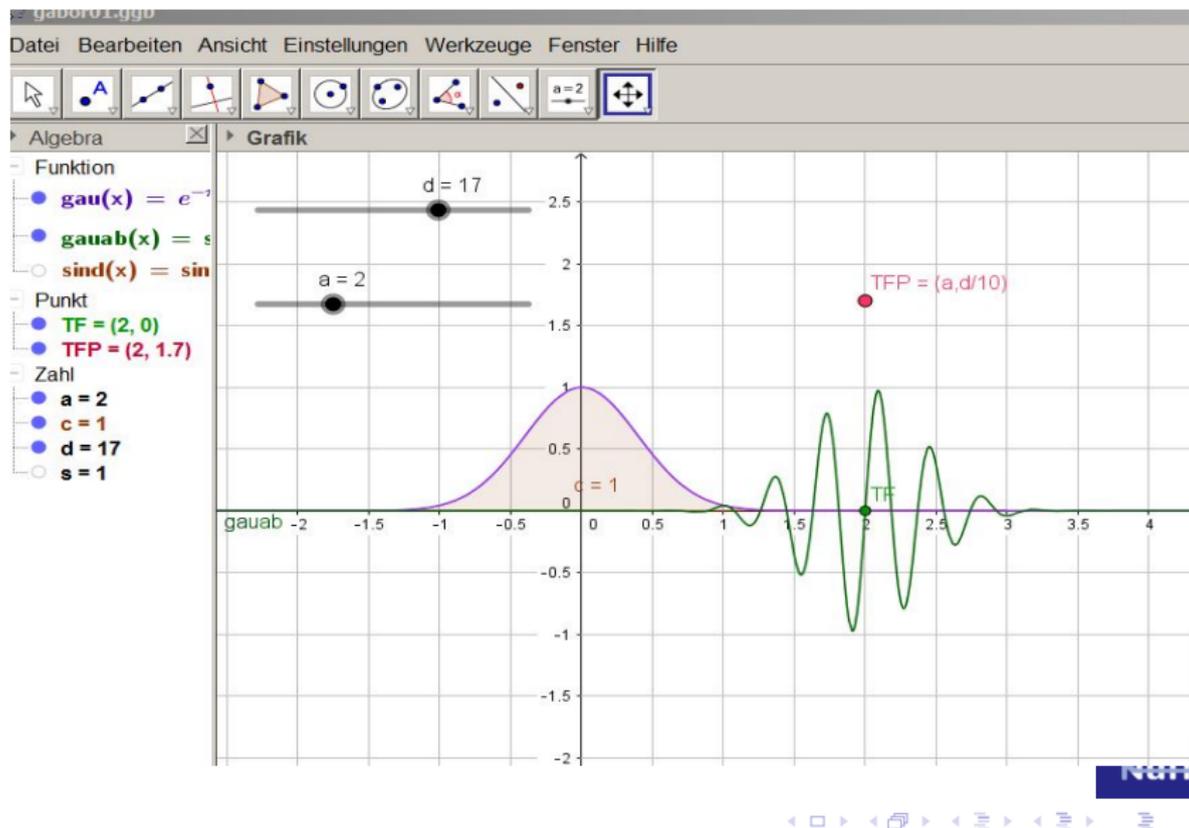
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda)g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

Beethoven Sonata

Demonstration using GEOGEBRA (very easy to use!!)

Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in L^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in L^2(\mathbb{R}^d),$$

in fact $V_g(f) \in C_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in L^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $L^2(\mathbb{R}^d)$, i.e. that $\|g\|_2 = 1$ is no problem we will assume this from now on.

Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|_{\mathcal{H}_1}^2 = (!) \|Th\|_{\mathcal{H}_2}^2 = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \, \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that
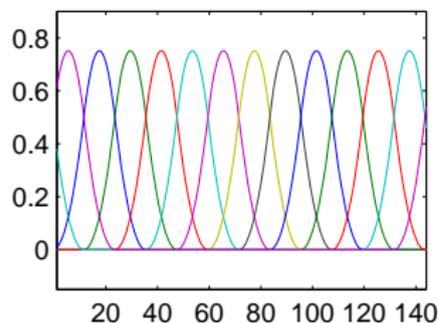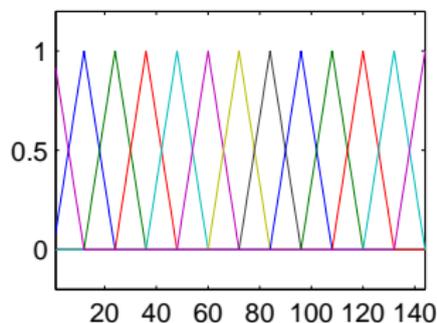
$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda) \pi(\lambda) g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \quad (3)$$

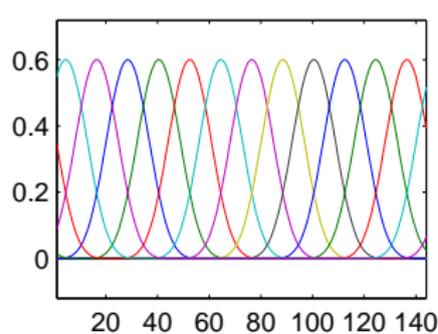understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:
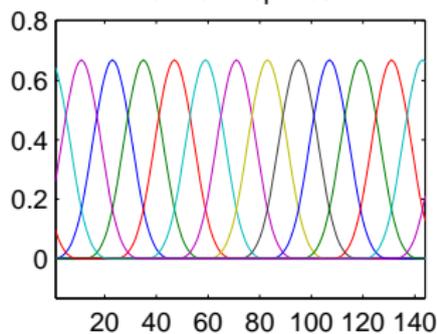
$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda) g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \quad (4)$$
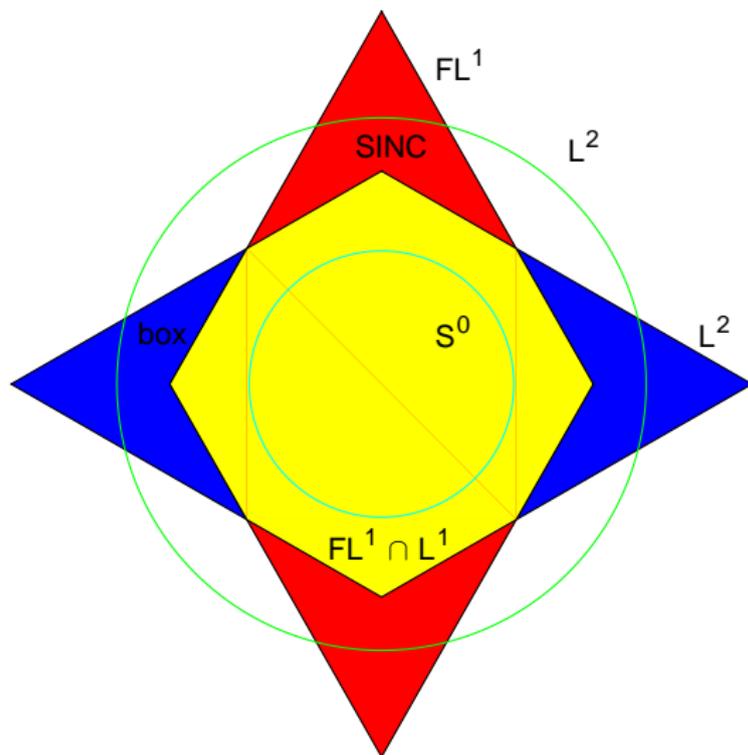
cubic B–splines

Figure: HFI Hsomething and I ... I ... 5

Putting things together we have

$$\langle f, h \rangle = \langle V_g^*(V_g(f)), h \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)} \, d\lambda. \quad (5)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle \, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (6)$$

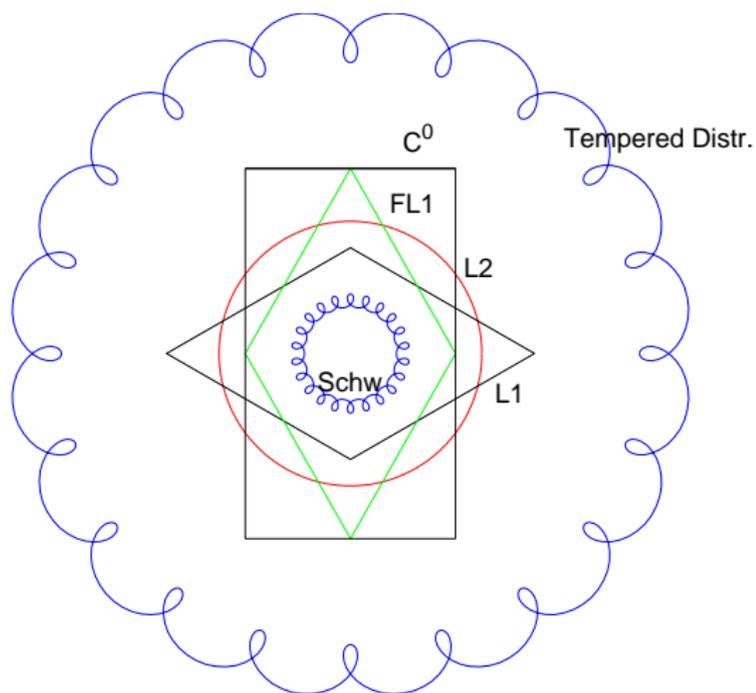This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle \, \chi_s \, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (7)$$

with $\chi_s(t) = exp(2\pi i \langle s, t \rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).
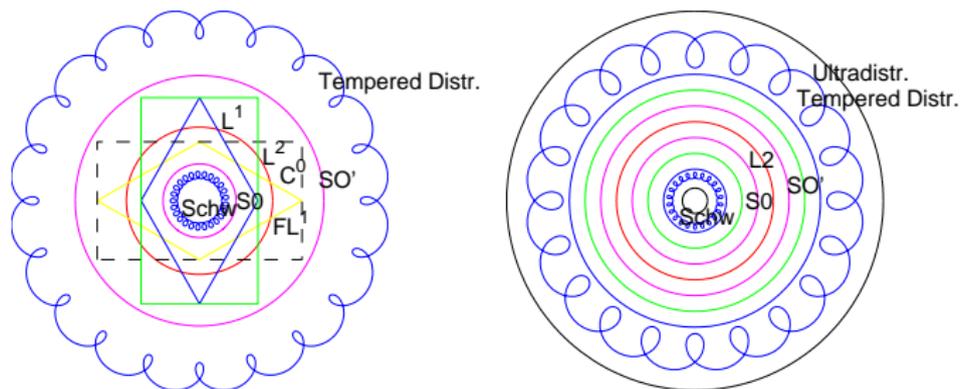
Figure: comparison of two collections, also ultra-distributions

Note the crucial difference between the classical formula (7) (Fourier inversion) and the new formula formula (6). While the building blocks g_λ belong to the Hilbert space $L^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in L^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $L^2(\mathbb{R}^d)$-sense.

The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [6]).

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u, \eta)f \in S_0(\mathbb{R}^d)$ for $(u, \eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, and $\|\pi(u, \eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, and $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0} \right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $W(\mathcal{F}L^1, \ell^1) = M^0_{1,1}(\mathbb{R}^d)$.

Basic properties of $M^\infty(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$

It is probably no surprise to learn that the dual space of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$, i.e. $S_0'(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. As an amalgam space one has $S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)' = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in S_0(\mathbb{R}^d)$).

Consequently norm convergence in $S_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain atomic characterizations of $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT. – Hifi recordings!

Wendel's Theorem

Theorem

The space of $\mathcal{H}_{L^1}(L^1, L^1)$ all bounded linear operators on $L^1(G)$ which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with $(M_b(G), \|\cdot\|_{M_b})$. In terms of our formulas this means

$$\mathcal{H}_{L^1}(L^1, L^1)(\mathbb{R}^d) \simeq (M_b(\mathbb{R}^d), \|\cdot\|_{M_b}),$$

via $T \simeq C_\mu : f \mapsto \mu * f, \quad f \in L^1, \mu \in M_b(\mathbb{R}^d).$

Lemma

$$B_{L^1} = \{f \in B \mid \|T_x f - f\|_B \to 0, \ for x \to 0\}.$$

Consequently we have $(M_b(\mathbb{R}^d))_{L^1} = L^1(\mathbb{R}^d)$, the closed ideal of absolutely continuous bounded measures on $\mathbb{R}^d$.

Pointwise Multipliers

Via the Fourier transform we have similar statements for the Fourier algebra, involving the *Fourier Stieltjes algebra*.

$$\mathcal{H}_{\mathcal{F}L^1}(\mathcal{F}L^1, \mathcal{F}L^1) = \mathcal{F}(M_b(\mathbb{R}^d)), \quad \mathcal{F}(M_b(\mathbb{R}^d))_{\mathcal{F}L^1} = \mathcal{F}L^1. \quad (8)$$

Theorem

The completion of $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (viewed as a Banach algebra and module over itself) is given by

$$\mathcal{H}_{C_0}(C_0, C_0) = \left(C_b(\mathbb{R}^d), \|\cdot\|_\infty\right).$$

On the other hand we have $\left(C_b(\mathbb{R}^d)\right)_{C_0} = C_0(\mathbb{R}^d)$.

Essential part and closure

In the sequel we assume that $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a Banach algebra with bounded approximate units, such as $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ (with convolution), or $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ or $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$ with pointwise multiplication.

Theorem

Let $\boldsymbol{A}$ be a Banach algebra with bounded approximate units, and $\boldsymbol{B}$ a Banach module over $\boldsymbol{A}$. Then we have the following general identifications:

$$(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{9}$$

or in a slightly more compact form:

$$\boldsymbol{B}_{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad \boldsymbol{B}_{\boldsymbol{A}}{}^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{10}$$

The usual way to define the *essential part* B_A resp. B_e of a Banach module $(B, \|\cdot\|_B)$ with respect to some Banach algebra action $(a, b) \mapsto a \bullet b$ is the define it as the closed linear span of $A \bullet B$ within $(B, \|\cdot\|_B)$. However it is interesting that this subspace can be shown to have another nice characterization using BAIs (bounded approximate units in $(A, \|\cdot\|_A)$):

Lemma

For any bounded approximate identity (BAI) $(e_\alpha)_{\alpha \in I}$ *i in* $(A, \|\cdot\|_A)$ *one has:*

$$B_A = \{b \in B \mid \lim_\alpha e_\alpha \bullet b = b\} \tag{11}$$

The Cohen-Hewitt Factorization Theorem

In particular one has: Let $(\mathbf{e}_\alpha)_{\alpha \in I}$ and $(\mathbf{u}_b)_{\beta \in J}$ be two bounded approximate units (i.e. bounded nets within $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$wa acting in the limit almost like an identity in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Then one has

$$\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b} \Leftrightarrow \lim_\beta \mathbf{u}_\beta \bullet \mathbf{b} = \mathbf{b}. \tag{12}$$

Theorem

(The Cohen-Hewitt factorization theorem, without proof, see [5])
Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI of size $C > 0$,
then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there
exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In
fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, continuously embedded between $\boldsymbol{S}_0(G)$ and $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$, i.e. with

$$(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$$

is called a *Fourier Standard Space* on G (FouSSp) if it has a double module structure over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\widehat{G}))$ with respect to pointwise multiplication.

Essential part and closure II

Having now Banach spaces of distributions which have two module structures, we have to use corresponding symbols. FROM NOW ON we will use the letter A mostly for pointwise Banach algebras and thus for the $\mathcal{F}L^1$-action on $(B, \|\cdot\|_B)$, and we will use the symbol G (because convolution is coming from the integrated group action!) for the L^1 convolution structure. We thus have

$$B_{GG} = B_G, \quad B^G{}_G = B_G, \quad B_G{}^G = B^G, \quad B^{GG} = B^G. \quad (13)$$

In this way we can combine the two operators (in view of the above formulas we can call them interior and closure operation) with respect to the two module actions and form spaces such as

$$B^G{}_A, \quad B_A{}^G{}_A, \quad B^G{}_A{}^G{}_A \cdots$$

or changes of arbitrary length, as long as the symbols A and G appear in alternating form (at any position, upper or lower).

Fortunately one can verify (paper with W.Braun from 1983, J.Funct.Anal., [1]) that any "long" chain can be reduced to a chain of at most two symbols, the *last occurrence of each of the two symbols being the relevant one!* So in fact all the three symbols in the above chain describe the same space of distributions.

But still we are left with the following collection of altogether eight two-letter symbols:

$$B_{GA}, B_{AG}, B_A{}^G, B^G{}_A, B_G{}^A, B^A{}_G, B^{AG}, B^{GA} \qquad (14)$$

and of course the four one-symbol objects

$$B_A, B_G, B^A, B^G \qquad (15)$$

Fortunately there is some natural structure
There are other, quite simple and useful facts, such as

$$\mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2) = \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2{}_{\boldsymbol{A}}) \tag{16}$$

which can easily be verified if $\boldsymbol{B}^1{}_{\boldsymbol{A}} = \boldsymbol{A} \bullet \boldsymbol{B}^1$, since then
$T \in \mathcal{H}_{\boldsymbol{A}}(\boldsymbol{B}^1{}_{\boldsymbol{A}}, \boldsymbol{B}^2)$ applied to $\mathbf{b}^1 = \mathbf{a} \bullet \mathbf{b}^{1'}$ gives

$$T(\mathbf{b}^1) = T(\mathbf{a} \bullet \mathbf{b}^{1'}) = \mathbf{a} \bullet T(\mathbf{b}^{1'}) \in \boldsymbol{B}^2{}_{\boldsymbol{A}}.$$

This diagram is taken from [1].
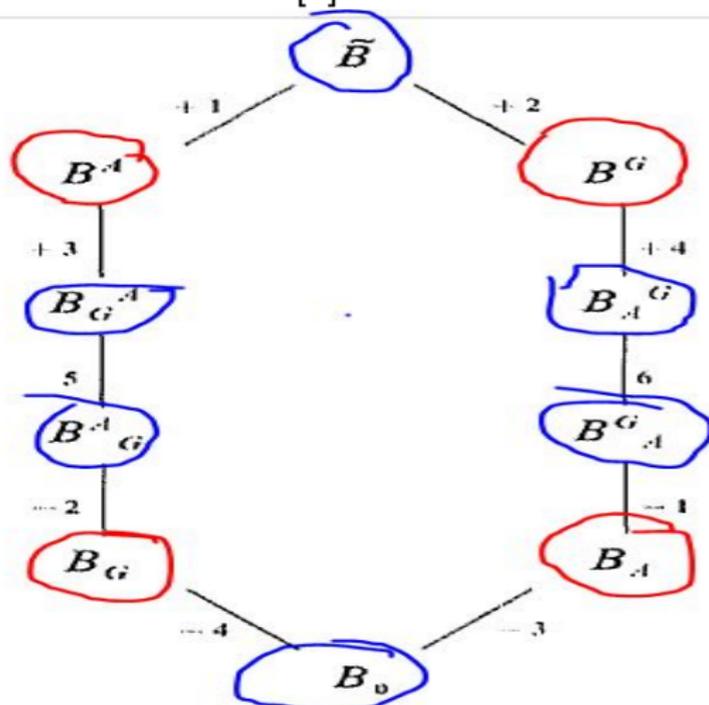
TF-homogeneous Banach Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \, f \in \boldsymbol{B}. \tag{17}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous on often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $(\boldsymbol{L}^{\infty}(\mathbb{R}^d), \|\cdot\|_{\infty})$).

An important fact concerning this family is the minimality property of the Segal algebra $\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big)$.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra
$\big(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\big) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d).$

For each of the Fourier Standard Spaces the discussion of the above diagram makes sense. One may see that it can collaps totally to a single space, or that it has in fact a rich (like $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$) or simple structure.

Theorem

A Fourier standard space is maximal, i.e. coicides with
$\widetilde{B} = B^{AG} = B^{GA}$ *if and only if B is a dual space.*
There is also a formula for the predual spaces, it is $((B_0)')_0$, where
$B_0 = B_{AG} = B_{GA}$ *is just the closure of $\mathcal{S}(\mathbb{R}^d)$ resp. $S_0(\mathbb{R}^d)$ in B.*

Theorem

A Fourier standard space is reflexive if and only if both the space $(B, \|\cdot\|_B)$ and its dual are both minimal and maximal. In other words, for the space itself and its dual the diagram is reduced to a single space $(B, \|\cdot\|_B)$.

1. Taking Fourier transforms;

2. Conditional dual spaces, i.e. the dual space of the closure of $S_0(G)$ within $(B, \|\cdot\|_B)$;

3. With two spaces B^1, B^2: take intersection or sum (preserves minimality and maximility);

4. Convolution kernels and Fourier multipliers between such spaces;

5. forming amalgam spaces $W(B, \ell^q)$; e.g. $W(\mathcal{F}L^p, \ell^q)(\mathbb{R}^d)$;

6. for example modulation space $M^{p,q}(\mathbb{R}^d)$ as inverse Fourier image of $W(\mathcal{F}L^p, \ell^q)(\mathbb{R}^d)$.

1. using complex (or real) interpolation methods, so that we get the spaces $M^{p,p} = W(\mathcal{F}L^p, \ell^p)$ (which are in fact all Fourier invariant);

2. translation invariant hulls of weighted space $L^p_w(\mathbb{R}^d)$;

3. restrictions to subgroups and periodizations (whenever it makes sense), e.g. $L^p(\mathbb{R}^d) \widehat{\otimes} L^q(\mathbb{R}^d)$ along $\Delta = (x, x) \,|\, x \in \mathbb{R}^d$: the so-called Herz algebra $A_p(G)$;

4. pointwise multipliers, e.g. $B_p(G)$, the pointwise multipliers of $A_p(G)$, or from $L^\infty(\mathbb{R}^d)$ to $\mathcal{F}L^p(\mathbb{R}^d)$ (the largest solid space such that $\hat{f} \in L^q(\mathbb{R}^d)$;

5. The algebra of tempered elements in $L^p(G)$, i.e. those which also define bounded convolution operators on $C_c(G)$, with respect to L^p-norm;

6. Images of FouSSp under metaplectic deformations, such as the *fractional Fourier transform*;

Now we talk about Fourier standard spaces over $\mathbb{R}^{2d}$ resp. $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$

1. Spaces of *kernels of operators* between FouSSps;
2. *Spreading functions* of such spaces
3. *Kohn-Nirenberg* symbols of such operator spaces;
4. Kernels of *Operator Ideals* in the sense of A. Pietsch, e.g. Hilbert-Schmidt or Schatten Classes;
5. symplectic Fourier transforms of such things
6. Schur multipliers
7. bilinear multipliers
8. Varopoulos algebra and Bimeasures

Tensor products

Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1)f^2(x_2), \ x_i \in \mathbb{R}^d, i = 1, 2.$$

Given two Banach spaces $\boldsymbol{B}^1$ and $\boldsymbol{B}^2$ embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, $\boldsymbol{B}^1 \hat{\otimes} \boldsymbol{B}^2$ denotes their *projective tensor product*, i.e.

$$\left\{ f \mid f = \sum f_n^1 \otimes f_n^2, \ \sum \|f_n^1\|_{\boldsymbol{B}^1} \|f_n^2\|_{\boldsymbol{B}^2} < \infty \right\}; \qquad (18)$$

It is easy to show that this defines a Banach space of tempered distributions on $\mathbb{R}^{2d}$ with respect to the (quotient) norm:

$$\|f\|_{\hat{\otimes}} := \inf\{\sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, ..\} \qquad (19)$$

where the infimum is taken over all admissible representations.

References

W. Braun and Hans G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

Elena Cordero, Hans G. Feichtinger, and Franz Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

Hans G. Feichtinger and Georg Zimmermann.
A Banach space of test functions for Gabor analysis.
In Hans G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.

Karlheinz Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Edwin Hewitt and Kenneth A. Ross.
Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups.
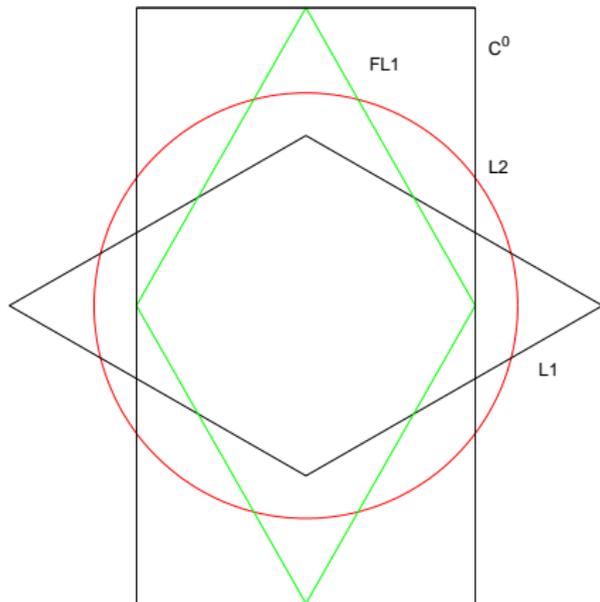Springer, Berlin, Heidelberg, New York, 1970.

Ferenc Weisz.
Inversion of the short-time Fourier transform using Riemannian sums.
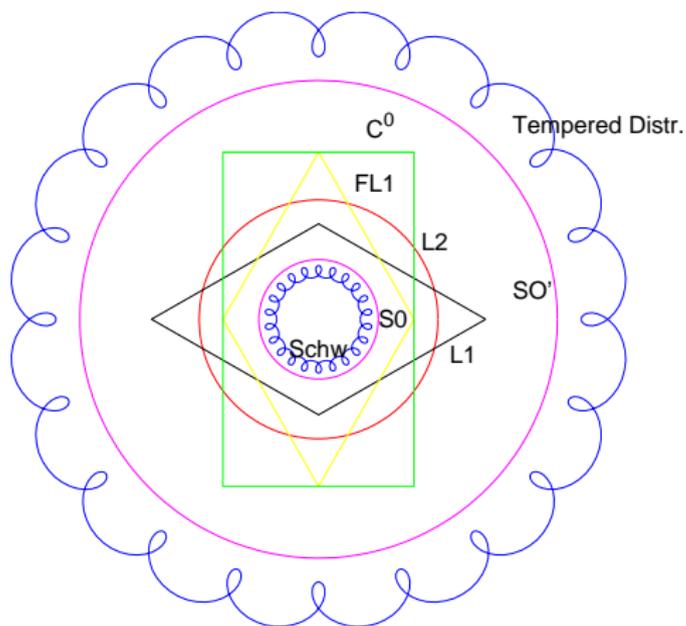*J. Fourier Anal. Appl.*, 13(3):357–368, 2007.

the classical Fourier situation

C^0

FL1

L2

L1

A better schematic description of the situation

The domain of the Fourier inversion formula: $L^1 \cap \mathcal{F}L^1$:
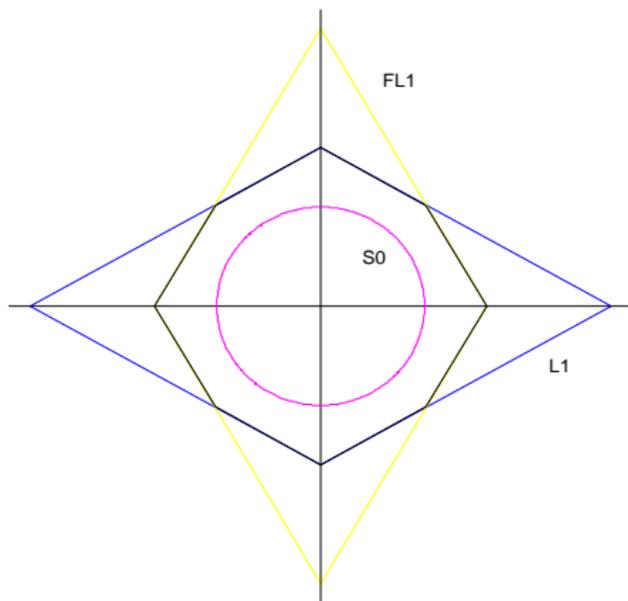
A better schematic description of the situation

Sobolev spaces and weighted L2 spaces and M $_\vee^1$ spaces