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The usefulness of certain Banach Gelfand Triples

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Serbian Academy of Sciences

ONLY PARTS of this material was presented in talks !!

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March 2006

Outline of the TALK

- Families of Banach spaces of functions/distributions
- Wiener Amalgam spaces;
- Modulation spaces (characterized by uniform decompositions);
- The Banach Gelfand Triples and their use;
- various (unitary Gefand triple isomorphism involving $(\mathcal{S}_0, \boldsymbol{L}^2, {\mathcal{S}_0}')$

What are function spaces good for?

- Describe the smoothness or variation/oscillation of functions;
- Describe (rate of) decay of functions, summability properties;
- Describe the mapping properties of linear operators, domains of unbounded operators;

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There is a huge zoo of Banach spaces of functions or distributions used in the literature:

- the classical L^p-spaces, but also Lorentz or Orlicz spaces (typically defined by the distribution of their values, hence rearrangement invariant);

- Lipschitz spaces, Besov spaces, Bessel potential spaces, Triebel Lizorkin spaces (smoothness);

- weighted spaces, mixed norm spaces;

- spaces describing bounded variation, Morrey-Campanato spaces;

- Hardy spaces, characterized by atomic decompositions;

- Herz spaces, defined by decompositions;

Complex Interpolation of Weighted L^p-spaces

Complex interpolation methods (at least from a user's point of view) are quite well known. Let us give as an typical example the Hausdorff-Young inequality, showing that for $1 \leq p \leq 2$ one has (with $1/p + 1/q = 1$):

$$\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subseteq \boldsymbol{L}^q(\mathbb{R}^d) \quad \text{and} \quad \|\hat{f}\|_q \leq \|f\|_p,$$

as a consequence of the validity of this estimate for the cases $p = 1$ (Riemann Lebesgue Lemma) and $p = 2$ (Plancherel).

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The "scale" of space obtained by complex interpolation of weighted L^p spaces are identified as weighted L^p-spaces themselves, according to the rule $1/p = (1-\theta)/p_1 + \theta/p_2$, and $w = w_1^{1-\theta} \cdot w_2^{\theta}$:

$$(L^{p_1}_{w_1}, L^{p_2}_{w_2})_{[\theta]} = L^p_w$$

Definition 1. *A triple, consisting of a Banach space* $\mathbf{B}$, *which is dense in some Hilbert space* $\mathcal{H}$, *which in turn is contained in* B *is called a Banach Gelfand triple.*

Definition 2. *If* $(B_1, \mathcal{H}_1, B_1')$ *and* $(B_2, \mathcal{H}_2, B_2')$ *are Gelfand triples then a linear operator* T *is called a [unitary] Gelfand triple isomorphism if*

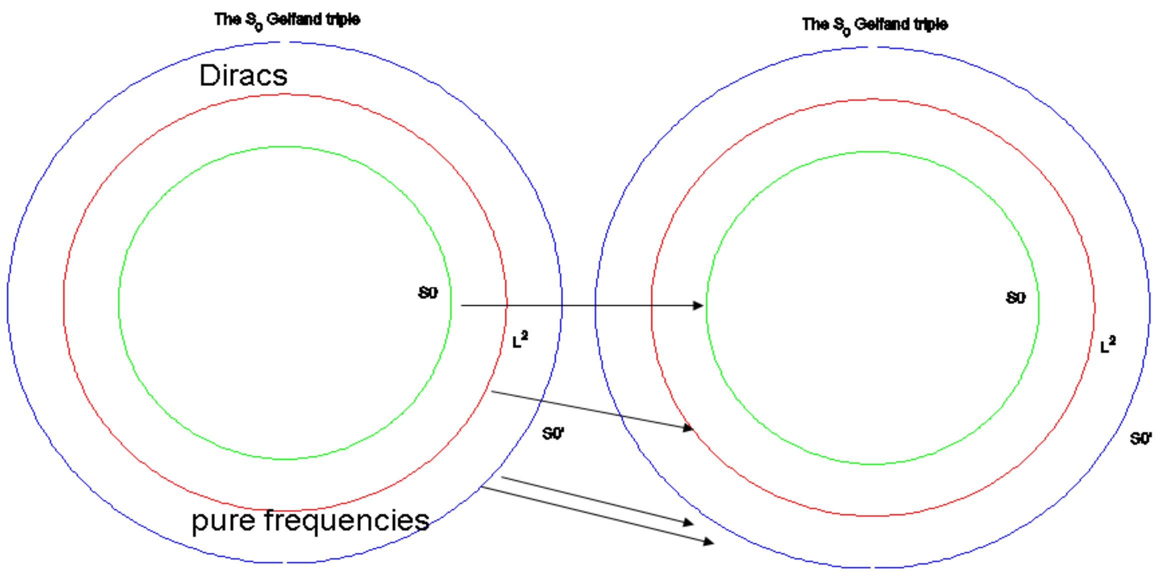
1. *A is an isomorphism between* B_1 *and* B_2.

2. *A is a [unitary operator resp.] isomorphism between* $\mathcal{H}_1$ *and* $\mathcal{H}_2$.

3. *A extends to a weak** *isomorphism as well as a norm-to-norm continuous isomorphism between* B_1' *and* B_2'.

The prototype is $(\ell^1, \ell^2, \ell^\infty)$, where the w^*-convergence corresponds to coordinate convergence in ℓ^∞.

Gelfand triple mapping



Realization of a GT-homomorphism

Very often a Gelfand-Triple homomorphism T can be *realized with the help of some kind of "summability methods"*. In the abstract setting this is a sequence [1] A_n, having the following property:

- each of the operators maps B_1' into B_1;
- they are a uniformly bounded family of Gelfand-triple isomorphisms $(B_1, \mathcal{H}_1, B_1')$ to $(B_2, \mathcal{H}_2, B_2')$
- $A_n f \to f$ in B_2 for any $f \in B_1$;

It then *follows* that the limit $T(A_n f)$ exists in $\mathcal{H}_2$ respectively in B_2' (in the w^*-sense) for $f \in \mathcal{H}_1$ resp. $f \in B_1'$ and thus describes concretely the prolongation to the full Gelfand triple.

[1] or more generally a net

Wiener Amalgams: Wiener's Role

First appearance in Norbert Wiener's theory of generalized harmonic analysis ("The Fourier Transform and Certain of its Applications") and Tauberian Theorems around 1929-1932 : [$W(L^1, L^2)$ and $W(L^2, L^1)$, $W(L^1, L^\infty)$ and a bit later $W(L^\infty, L^1)$], using the discrete norm for these spaces:

$$\|f\|_{W(L^p,\ell^q)} = \left(\sum_{n\in\mathbb{Z}} \left(\int_n^{n+1} |f(t)|^p \, dt\right)^{q/p}\right)^{1/q}, \quad (1)$$

with the usual adjustments if p or q is infinity. Advantage over ordinary L^p-spaces: natural inclusions, in the local component as over the torus, while globally on has the natural inclusions between sequence spaces, with opposite orientation. Hence $W(L^\infty, \ell^1)$ is the smallest within *this* family and $W(L^1, \ell^\infty)$ is the largest.

CLASSICAL Wiener Amalgams: Basic Properties

The use of amalgam spaces (cf. e.g. the survey article by Fournier and Stewart, Bull. Amer. Math. Soc., 1980) shows their usability in a wide range of problems of analysis. In most cases one can just argue, that one has to think *coordinatewise*.

CLASSICAL Wiener Amalgams: Basic Properties

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For example, with respect to duality, pointwise multiplication, or a Hausdorff-Young type statement for the Fourier transform:

$$W(L^p, \ell^q)' = W(L^{p'}, \ell^{q'}), \quad 1 \leq p, q, < \infty$$

$$\mathcal{F}W(L^p, \ell^q) \subseteq W(L^{q'}, \ell^{p'}), \quad 1 \leq p, q, \leq 2$$

Wiener Amalgam Space: Genera local/global components

Definition 3. *A Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *of tempered distributions is called a* **standard space** *if it satisfies the following conditions:*

1. $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \hookrightarrow \boldsymbol{B} \hookrightarrow \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$,
2. $\boldsymbol{B}$ *is translation and modulation invariant* $T_x\boldsymbol{B} = \boldsymbol{B}$ *and* $M_y\boldsymbol{B} = \boldsymbol{B}$ *for all* $x, y \in \mathbf{R}^d$.

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3. *The Banach algebra* $\boldsymbol{A}$ *of pointwise multipliers of* $\boldsymbol{B}$ *contains* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

4. *There is some Beurling algebra* $L^1_w(\mathbf{R}^d)$ *which acts boundedly on* $\boldsymbol{B}$ *through convolution, i.e.*

$$\|g * f\|_{\boldsymbol{B}} \leq \|g\|_{1,w}\|f\|_{\boldsymbol{B}} \quad \forall f \in \boldsymbol{B}, g \in L^1_w.$$

Selective, Continuous Description of Wiener Amalgam Spaces

Definition 4. *(Wiener Amalgam spaces) Let* $(\mathbf{B}, \|.\|_{\mathbf{B}})$ *be a standard space and* $(\mathbf{C}, \|.\|_{\mathbf{C}})$ *a solid and translation invariant Banach space of functions, i.e., a complete space of measurable functions, such that* $f \in \mathbf{C}$*, g measurable and* $|g(x)| \leq |f(x)|$ *for all* X*, implies* $g \in \mathbf{C}$ *and* $\|g\|_{\mathbf{C}} \leq \|f\|_{\mathbf{C}}$ *as well as* $T_x\mathbf{C} = \mathbf{C}$*.*

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Then we define for $f \in \mathbf{B}_{loc}$ *and some compactly supported "window"* $k \in A$ *the so-called control function with respect to the* $\mathbf{B}$*-norm:*

$$K(f, k) : x \mapsto \|(T_x k) \cdot f\|_{\mathbf{B}}.$$

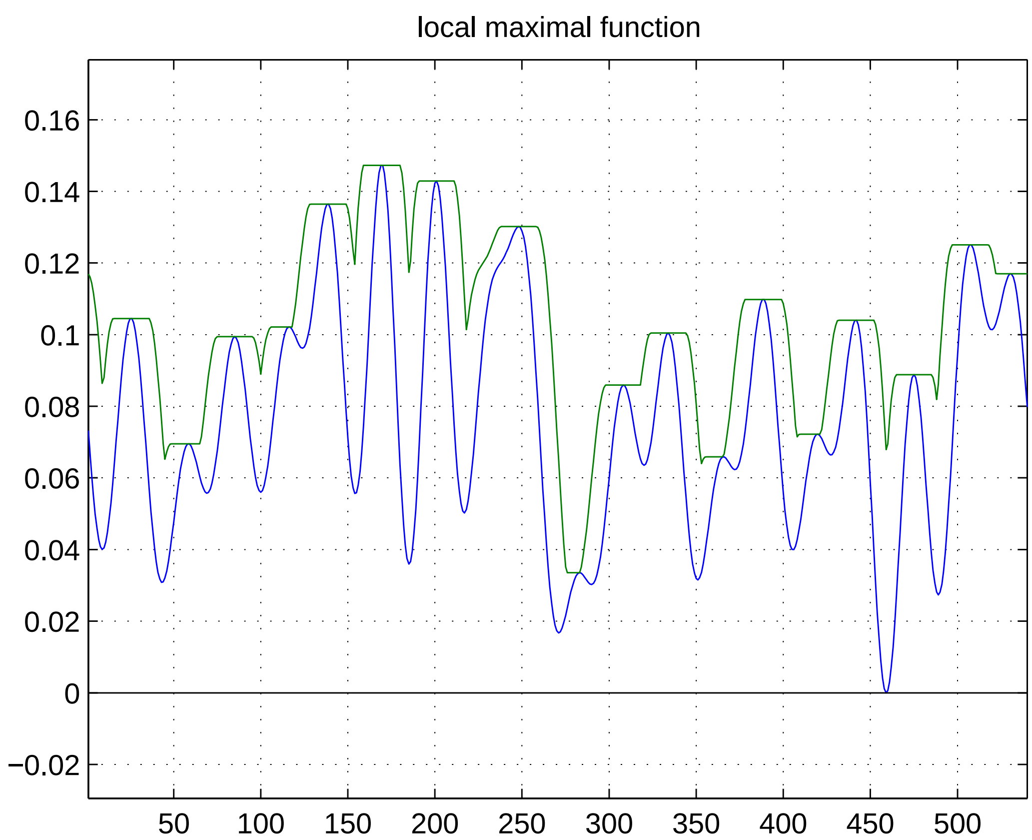
On the basis of this control function a linear space, the Wiener amalgam space with local component $\mathbf{B}$ *and global component* $\mathbf{C}$, *denoted by* $\boldsymbol{W}(\mathbf{B},\mathbf{C})$ *is defined as follows:*

$$\boldsymbol{W}(\mathbf{B},\mathbf{C}) := \{f \in \mathbf{B}_{loc} | K(f,k) \in \mathbf{C}\}.$$

Different windows k *define the* same *space and equivalent norms.*

These spaces are Banach spaces, and if $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ *is dense in* $\boldsymbol{W}(\boldsymbol{B},\boldsymbol{C})$ *then the dual space can be calculated coordinate-wise.*

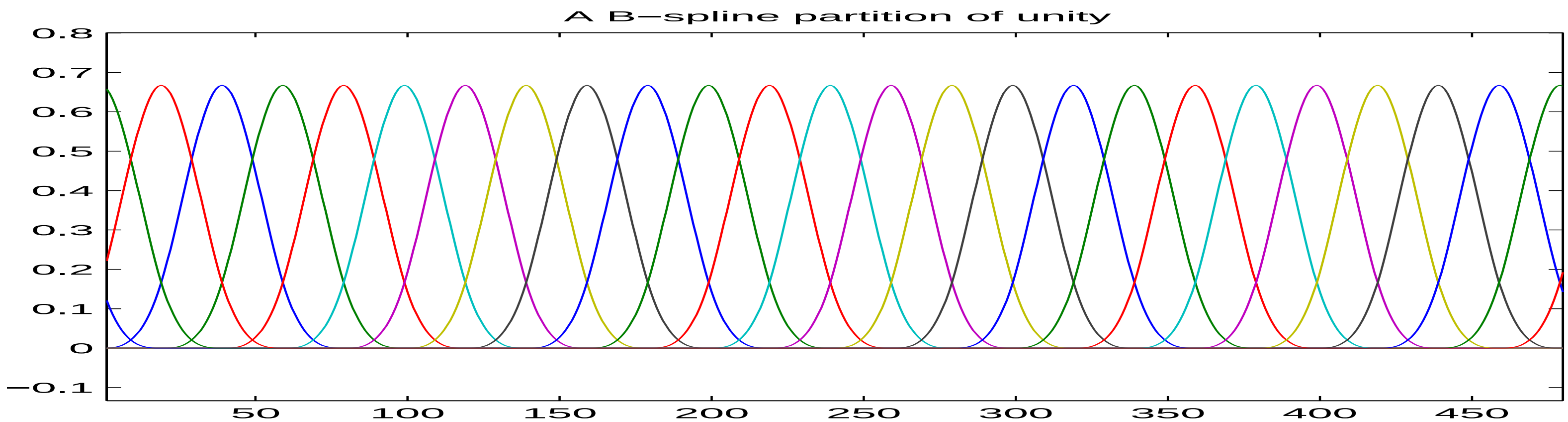
A Typical Control Function



Bounded Uniform Partitions of Unity

Definition 5. *A bounded family* $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ *in a Banach algebra* $(\mathbf{A}, \|\cdot\|_A)$ *is a regular A-Bounded Uniform Partition of Unity if*

$$\sum_{n\in\mathbb{Z}^d} \psi(x-n) = 1 \quad \textit{for all} \quad x \in \mathbf{R}^d$$



Selective, Discrete Description of Wiener Amalgam Spaces

Theorem 1. *Assume that* $\mathbf{A} \cdot \mathbf{B} \subseteq \mathbf{B}$, *with* $\|h \cdot f\|_B \leq \|h\|_A \|f\|_B$ *for all* $h \in \mathbf{A}, f \in \mathbf{B}$. *Then* $f \in W(\mathbf{B}, L^q_w)$, $1 \leq q < \infty$, *if and only if for each (or just for one individual)* $\mathbf{A}$*-BUPU* Ψ *one has*

$$\|f\|'_W = \left(\sum_{i \in I} \|f \psi_i\|_B^q w^q(x_i) \right)^{1/q} < \infty$$

Modulation Spaces (HF: around 1983)

Definition 6.

$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) = \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}L^p, \ell^q_s)\right)$$

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$$T_x f(t) = f(t-x)$$

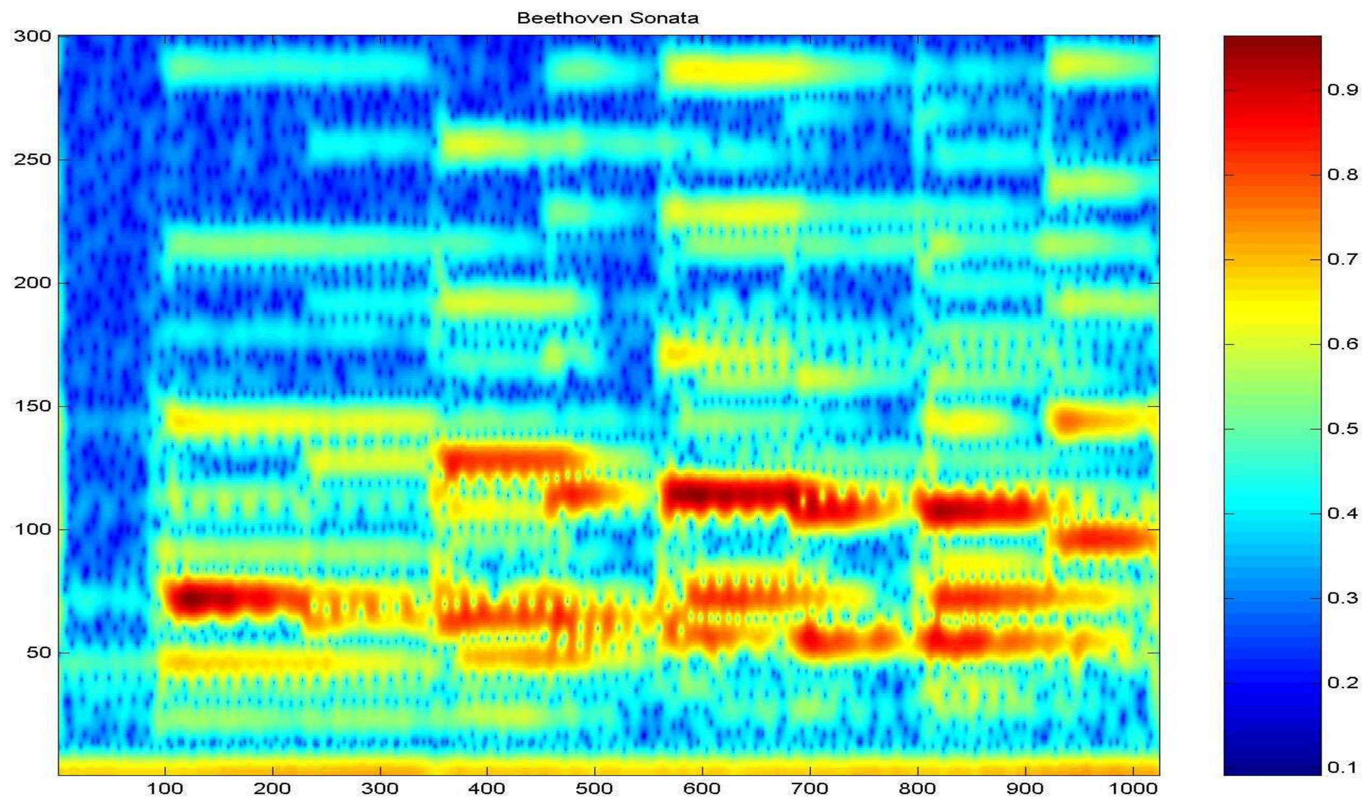
and $x, \omega, t \in \mathbf{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

$$V_g f(t,\omega) = \langle f, M_\omega T_t g\rangle$$


Beethoven Sonata
300
250
200
150
100
50
100 200 300 400 500 600 700 800 900 1000
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1

Modulation Spaces

The modulation spaces occur in the study of the concentration of a function in the time-frequency plane. They are defined in the following way: Let $g \in \mathcal{S}$ be a Schwartz function, $1 \leq p, q < \infty, s \in R$, then

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i.e. for which $V_g f$ belongs to some weighted mixed norm space over phase space. In the "classical" case the weight depends only on frequency, hence the spaces are isometrically translation invariant. The only important facts about the constraint imposed on $V_g f$ is the membership in some $solid\ and$ $translation\ invariant$ Banach space of functions.

The modulation space $M^s_{pq}(\mathbf{R})$ is a Banach space of tempered distributions, the definition is independent of the analyzing function g, and different g's yield equivalent norms on these spaces.

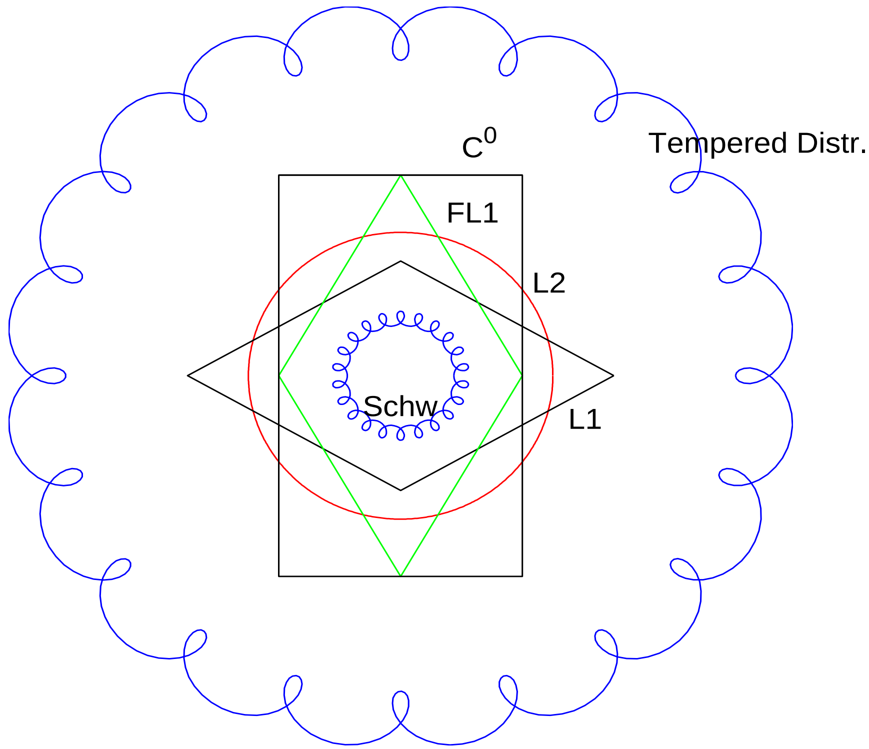
Among the modulation spaces are the following important function spaces:

(a) the Segal algebra $S_0(R)$ as $S_0 = M^0_{1,1}$.

(b) $L^2(R) = M^0_{2,2}$, and

(c) the Bessel potential spaces as $M^s_{2,2}$:

The classical view on the Fourier Transform



$\mathcal{S}_0(\mathbb{R}^d) = M^1(\mathbb{R}^d) := M^0_{1,1}(\mathbb{R}^d)$

A function in $f \in L^2$ is (by definition) in the subspace $S_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{L^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}^d}} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ is a Banach space, for any fixed, non-zero $g \in S_0(\mathbb{R}^d)$, and different windows g define the same space and equivalent norms. Since $S_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable windows. Often it is convenient to use the Gaussian as a window.

Lemma 1. *Let* $f \in S_0(\mathbb{R}^d)$*, then the following holds:*

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$*, and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

Basic properties of $\mathcal{S}_0(\mathbb{R}^d)$ resp. $\mathcal{S}_0(G)$

THEOREM:

- For any automorphism α of G the mapping $f \mapsto \alpha^*(f)$ is an isomorphism on $S_o(G)$; $[with(\alpha^* f)(x) = f(\alpha(x))], x \in G$.

- $\mathcal{F} S_o(G) = S_o(\hat{G})$; (Invariance under the Fourier Transform)

- $T_H S_o(G) = S_o(G/H)$; (Integration along subgroups)

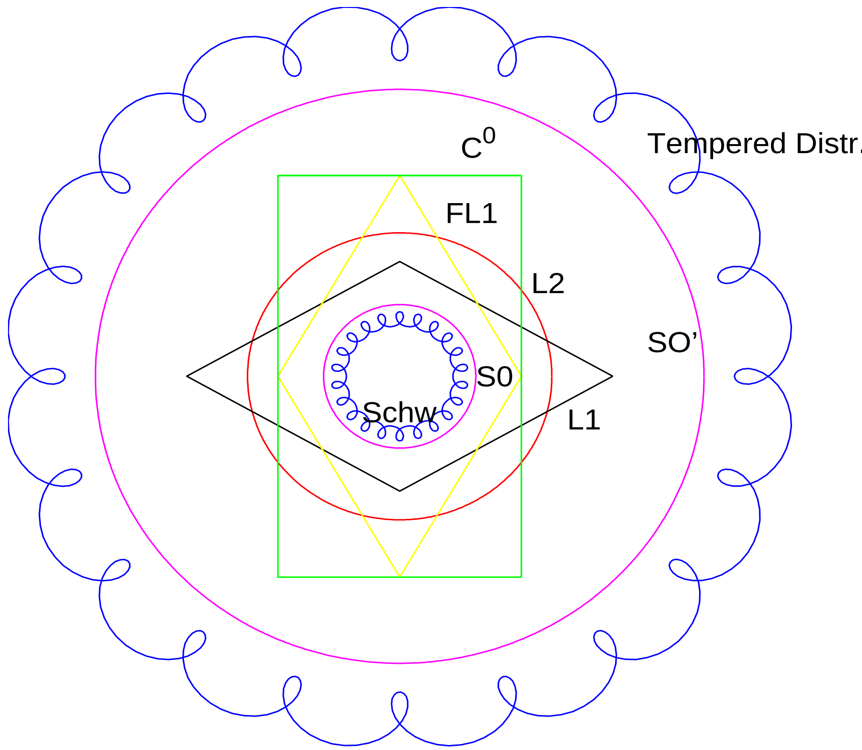
- $R_H S_o(G) = S_o(H)$; (Restriction to subgroups)

- $S_o(G_1) \hat{\otimes} S_o(G_2) = S_o(G_1 \times G_2)$. (tensor product stability);

THEOREM: (Consequences for the dual space)

- $S_o'(G)$ is a Banach space with a translation invariant norm;
- $S_o'(G) \subseteq \mathcal{S}'(G)$, i.e. $S_o'(G)$ consists of tempered distributions;
- $P(G) \subseteq S_o'(G) \subseteq Q(G)$; (sits between pseudo- and quasimeasures)
- $T(G) = W(G)' \subseteq S_o'(G)$; (contains translation bounded measures);
- $\mathcal{M}_T(G) \subseteq S_o'(G)$ (contains "transformable measures" by Gil-de-Lamadrid).

Schwartz space, S_0, L^2, S_0', tempered distributions



Basic properties of $S_0(\mathbb{R}^d)$ continued

THEOREM:

- the Generalized Fourier Transforms, defined by transposition
$$\langle \hat{\sigma}, f \rangle = \langle \sigma, \hat{f} \rangle,$$
for $f \in S_o(\hat{G}), \sigma \in S'_o(G)$, satisfies $\mathcal{F}(S'_o(G)) = S'_o(\hat{G})$.

- $\sigma \in S'_o(G)$ is H-periodic, i.e. $\sigma(f) = \sigma(T_h f)$ for all $h \in H$, iff there exists $\dot{\sigma} \in S'_o(G/H)$ such that $\langle \sigma, f \rangle = \langle \sigma, T_H f \rangle$.

- $S'_o(H)$ can be identified with a subspace of $S'_o(G)$, the injection i_H being given by
$$\langle i_H \sigma, f \rangle := \langle \sigma, R_H f \rangle.$$
For $\sigma \in S'_o(G)$ one has $\sigma \in i_H(S'_o(H))$ iff $\mathrm{supp}(\sigma) \subseteq H$.

The Usefulness of $\mathcal{S}_0(\mathbb{R}^d)$

Theorem 2. *(***Poisson's formula***) For $f \in \mathcal{S}_0(\mathbb{R}^d)$ and any discrete subgroup H of $\mathbf{R}^d$ with compact quotient the following holds true: There is a constant $C_H > 0$ such that*

$$\sum_{h \in H} f(h) = C_H \sum_{l \in H^\perp} \hat{f}(l) \tag{2}$$

with absolute convergence of the series on both sides.

By duality one can express this situation as the fact that the Comb-distribution $\mu_{\mathbb{Z}^d} = \sum_{k \in Z^d} \delta_k$, as an element of $S_0'(\mathbf{R}^d)$ is invariant under the (generalized) Fourier transform. Sampling corresponds to the mapping $f \mapsto f \cdot \mu_{\mathbb{Z}^d} = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k$, while it corresponds to convolution with $\mu_{\mathbb{Z}^d}$ on the Fourier transform side = periodization along $(\mathbb{Z}^d)^\perp = \mathbb{Z}^d$ of the Fourier transform $\hat{f}$. For $f \in \mathcal{S}_0(\mathbb{R}^d)$ all this makes perfect sense.

Regularizing sequences for $(S_0, \boldsymbol{L}^2, S_0')$

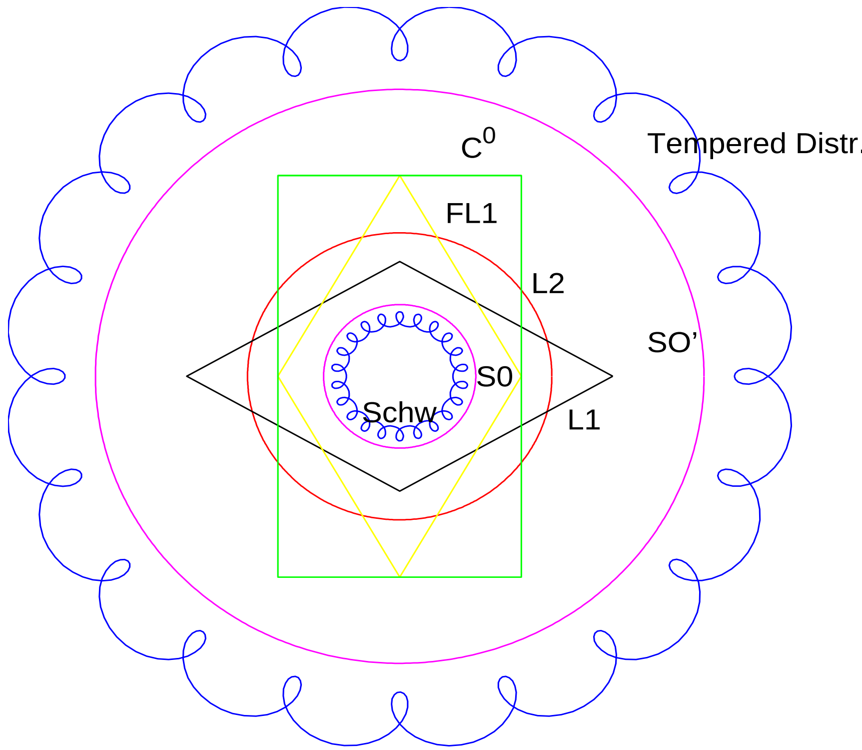
Wiener amalgam convolution and pointwise multiplier results imply that

$$S_0(\mathbb{R}^d) \cdot (S_0'(\mathbb{R}^d) * S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d) \quad S_0(\mathbb{R}^d) * (S_0'(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d)) \subseteq S_0(\mathbb{R}^d)$$

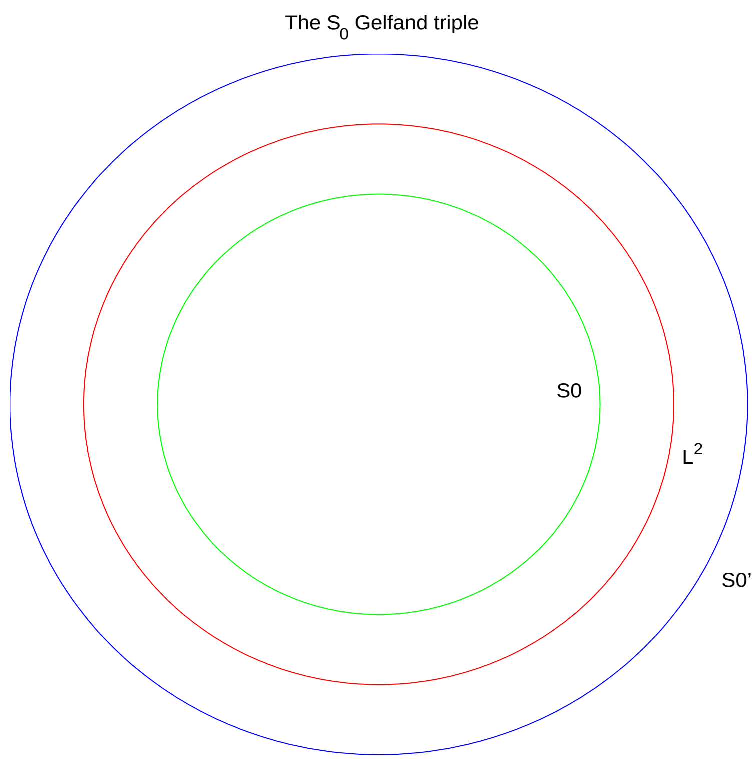
e.g. $S_0(\mathbb{R}^d) * S_0'(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}L^1, \ell^1) * \boldsymbol{W}(\mathcal{F}L^\infty, \ell^\infty) \subseteq \boldsymbol{W}(\mathcal{F}L^1, \ell^\infty)$.

Let now $h \in \mathcal{F}L^1(\mathbb{R}^d)$ be given with $h(0) = 1$. Then the dilated version $h_n(t) = h(t/n)$ are a uniformly bounded family of multipliers on $(S_0, \boldsymbol{L}^2, S_0')$, tending to the identity operator in a suitable way. Similarly, the usual Dirac sequences, obtained by compressing a function $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$ are showing a similar behavior: $g_n(t) = n \cdot g(nt)$
Following the above rules the combination of the two procedures, i.e. product-convolution or convolution-product operators of the form provide suitable regularizers: $A_n f = g_n * (h_n \cdot f)$ or $B_n f = h_n \cdot (g_n * f)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



The Gelfand Triple $(S_0, L^2, S_0{}')$



The Fourier transform is a prototype of a Gelfand triple isomorphism.

The Fourier transform as Gelfand Triple Automorphism

Theorem 3. *Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:*

(1) *$\mathcal{F}$ is an isomorphism from $S_0(\mathbb{R}^d)$ to $S_0(\widehat{\mathbb{R}^d})$,*

(2) *$\mathcal{F}$ is a unitary map between $L^2(\mathbb{R}^d)$ and $L^2(\widehat{\mathbb{R}^d})$,*

(3) *$\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $S_0'(\mathbb{R}^d)$ onto $S_0'(\widehat{\mathbb{R}^d})$.*

Furthermore we have that Parseval's formula

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle \tag{3}$$

is valid for $(f, g) \in S_0(\mathbb{R}^d) \times S_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(S_0, L^2, S_0')(\mathbb{R}^d)$.

The properties of Fourier transform can be expressed by a Gelfand bracket

$$\langle f, g\rangle_{(S_0, \boldsymbol{L}^2, S_0')} = \langle \hat{f}, \hat{g}\rangle_{(S_0, \boldsymbol{L}^2, S_0')} \tag{4}$$

which combines the functional brackets of Banach spaces and of the inner-product for the Hilbert space.

One can characterize the Fourier transform as the *uniquely determined* unitary Gelfand triple automorphism of $(S_0, \boldsymbol{L}^2, S_0')$ which maps pure frequencies into the corresponding Dirac measures [2]

[2] as one would expect in the case of a finite Abelian group.

The Kernel Theorem

Theorem 4. *If K is a bounded operator from $\mathcal{S}_0(\mathbb{R}^d)$ to $\mathcal{S}_0{}'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \mathcal{S}_0{}'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \mathcal{S}_0(\mathbb{R}^d)$, where $g \otimes f(x, y) = g(x)f(y)$.*

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Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbf{R}^d} k(x, y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in S_0{}'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} k(x, y)f(y)dyg(x)dx = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} k(x, y)g(x)f(y)dxdy.$$

This result is the "outer shall of the Gelfand triple isomorphism, which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The complete picture can again be expressed by a unitary Gelfand triple isomorphism:

Theorem 5. *The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with* $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ *as scalar product on* $\mathcal{HS}$ *and the usual Hilbert space structure on* $\boldsymbol{L}^2(\mathbb{R}^{2d})$ *on the kernels.*
Moreover, such an operator has a kernel in $S_0(\mathbf{R}^{2d})$ *if and only if the corresponding operator* K *maps* $\mathcal{S}_0'(\mathbb{R}^d)$ *into* $\mathcal{S}_0(\mathbb{R}^d)$*, but not only in a bounded way, but also continuously from* w^*-*topology into the norm topology of* $\mathcal{S}_0(\mathbb{R}^d)$*.*

Remark: Note that for "regularizing" kernels in $S_0(\mathbf{R}^{2d})$ the usual identification (recall that the entry of a matrix $a_{n,k}$ is the coordinate number n of the image of the $n-$th unit vector under that action of the matrix $A = (a_{n,k})$:

$$k(x,y) = K(\delta_y)(x) = \delta_x(K(\delta_y).$$

Since $\delta_y \in \mathcal{S}_0'(\mathbb{R}^d)$ and thus $K(\delta_y) \in \mathcal{S}_0(\mathbb{R}^d)$ the pointwise evaluation makes sense.

With this understanding our claim is that the kernel theorem provides a (unitary) isomorphism between the Gelfand triple (of kernels) $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0')(\mathbf{R}^{2d})$ into the Gelfand triple of operator spaces

$$\left(\mathcal{L}(\mathcal{S}_0', \mathcal{S}_0), \, \mathcal{HS}\,, \, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0')\right).$$

The Kohn Nirenberg Symbol and Spreading Function

The Kohn-Nirenberg symbol $\sigma(T)$ of an operator T (respectively its *symplectic* Fourier transform, the *spreading distribution* $\eta(T)$ of T) can be obtained from the kernel using some automorphism and a partial Fourier transform, which again provide unitary Gelfand isomorphisms. The symplectic Fourier transform is another unitary Gelfand Triple automorphism of $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$.

Theorem 6. *The correspondence between an operator* T *with kernel* K *from the Banach Gelfand triple* $\big(\mathcal{L}(\mathcal{S}_0{}', \mathcal{S}_0), \mathcal{HS}\,, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0{}')\big)$ *and the corresponding spreading distribution* $\eta(T) = \eta(K)$ *in* $S_0'(\mathbf{R}^{2d})$ *is the uniquely defined Gelfand triple isomorphism between* $\big(\mathcal{L}(\mathcal{S}_0{}', \mathcal{S}_0), \mathcal{HS}\,, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0{}')\big)$ *and* $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ *which maps the time-frequency shift operators* $M_y \circ T_x$ *onto the Dirac measure* $\delta_{(x,y)}$.

Kohn-Nirenberg and Spreading Symbols of Operators

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$

- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$

- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$

- *partial Fourier transform in the first variable:* $\mathcal{F}_1$

- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

Kohn-Nirenberg correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator with *symbol* σ

$$K_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega} d\omega$$

is the *pseudodifferential operator* with Kohn-Nirenberg symbol σ.

$$\begin{aligned} K_\sigma f(x) &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \sigma(x,\omega) e^{-2\pi i (y-x)\cdot\omega} d\omega \Big) f(y) dy \\ &= \int_{\mathbb{R}^d} k(x,y) f(y) dy. \end{aligned}$$

2. Formulas for the (integral) kernel k: $k = \mathcal{T}_a \mathcal{F}_2 \sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_2 \sigma(\eta, y-x) = \mathcal{F}_1^{-1} \widehat{\sigma}(x, y-x) \\ &= \widehat{\sigma}(\eta, y-x) e^{2\pi i \eta\cdot x} d\eta. \end{aligned}$$

3. The *spreading representation* of the same operator arises from the identity
$$K_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta, u) M_\eta T_{-u} f(x) du d\eta.$$
$\widehat{\sigma}$ is called the spreading function of the operator K_σ.

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the so-called *Rihaczek distribution* is defined by

$$R(f,g)(x,\omega) = e^{-2\pi i x \cdot \omega} \overline{\widehat{f}(\omega)} g(x).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, R(f,g) \rangle = \langle K_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

Weyl correspondence

1. Let σ be a tempered distribution on $\mathbb{R}^d$ then the operator

$$L_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\xi, u) e^{-\pi i \xi \cdot u} f(x) du d\xi$$

is called the *pseudodifferential operator* with *symbol* σ. The map $\sigma \mapsto L_\sigma$ is called the *Weyl transform* and σ the Weyl symbol of the operator L_σ.

$$\begin{aligned} L_\sigma f(x) &= \iint_{\mathbb{R}^{2d}} \widehat{\sigma} e^{-\pi i u \cdot \xi} T_{-u} M_\xi f(x) du d\xi \\ &= \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \widehat{\sigma}(\xi, y - x) e^{-2\pi i \xi \frac{x+y}{2}} \right) f(y) dy. \end{aligned}$$

2. Formulas for the kernel k from the KN-symbol: $k = \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma$

$$\begin{aligned} k(x,y) &= \mathcal{F}_1^{-1}\widehat{\sigma}\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{F}_2^{-1}\sigma\Big(\frac{x+y}{2}, y-x\Big) \\ &= \mathcal{T}_s^{-1}\mathcal{F}_2^{-1}\sigma. \end{aligned}$$

3. $\langle L_\sigma f, g\rangle = \langle k, g \otimes \overline{f}\rangle$. (Weyl operator vs. kernel)

If $f, g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, then the *cross Wigner distribution* of f, g is defined by

$$W(f,g)(x,y) = \int_{\mathbb{R}^d} f(x+t/2)\overline{g}(x-t/2)e^{-2\pi i\omega\cdot t}dt = \mathcal{F}_2\mathcal{T}_s(f\otimes\overline{g})(x,\omega).$$

and belongs to $\mathcal{S}(\mathbb{R}^{2d})$. Consequently, for any $\sigma \in \mathcal{S}'(\mathbb{R}^d)$

$$\langle \sigma, W(f,g) \rangle = \langle L_\sigma f, g \rangle$$

is well-defined and describes a uniquely defined operator L_σ from the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ into the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$.

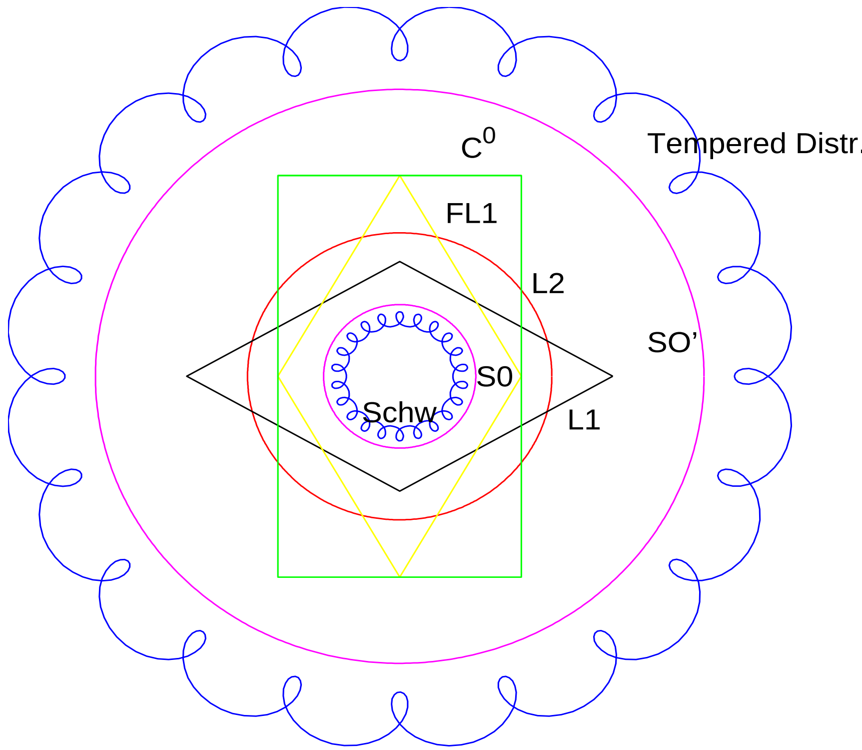
$$(\mathcal{U}\sigma)(\xi, u) = \mathcal{F}^{-1}(e^{\pi i u \cdot \xi} \widehat{\sigma}(\xi, u)).$$

$$K_{\mathcal{U}\sigma} = L_\sigma$$

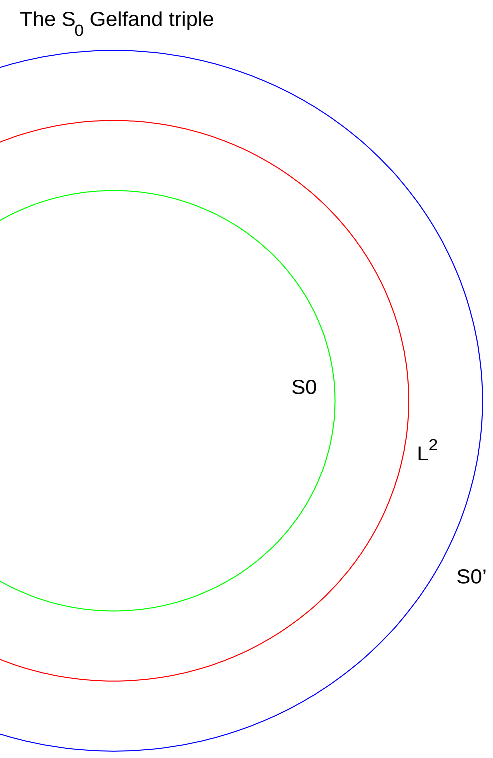
describes the connection between the Weyl symbol and the operator kernel.

In all these considerations the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ can be correctly replaced by $\mathcal{S}_0(\mathbb{R}^d)$ and the tempered distributions by $S_0'(\mathbf{R}^d)$.

Schwartz space, S_0, L^2, S_0', tempered distributions



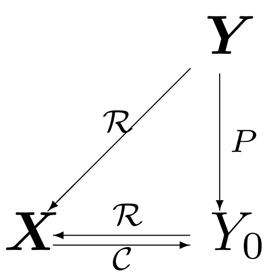
The Gelfand Triple $(\mathcal{S}_0, L^2, \mathcal{S}_0{}')$



Fourier transform is a prototype of a unitary Gelfand triple isomorphism.

Gelfand Triples, Modulations Spaces, . . . continued

We have seen so far, that the notions of "generating sets of vectors in $\mathbb{R}^d$" ($>$ frames, Banach frames) resp. that of a linear independent set ($>$ Riesz bases, Riesz projection bases) are characterized by a "triangular diagram" [3] The combination of both properties (i.e. the case $\boldsymbol{Y}_0 = \boldsymbol{Y}$ exactly describes *exact frames* resp. unconditional bases. Such a diagram - characterizing a retract - also makes sense for Banach Gelfand triples (even families), leading to Banach frames and Riesz projection bases.



[3] representing the fact that the range of a 5×3-matrix A in $\mathbb{R}^5$, i.e. the column space of A, can be identified with $\mathbb{R}^3$ if A has maximal rank, and sits within $\mathbb{R}5$ as a complemented subspace. Moreover the so-called pseudo-inverse (denoted by PINV in MATAB) describing the minimal norm least square solution of $A * x = b$ defines a left inverse $\mathcal{R}$ to $\mathcal{C} : x \mapsto A * x$, completing the diagram.

Examples provided so far

(1) The standard Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$.
(2) The family of orthonormal Wilson bases (obtained from Gabor families by suitable pairwise linear-combinations of terms with the same absolute frequency) extends the natural unitary identification of $\boldsymbol{L}^2(\mathbb{R}^d)$ with $\ell^(I)$ to a unitary Banach Gelfand Triple isomorphism between $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$ and $(\ell^1, \ell^2, \ell^\infty)(I)$.
(3) The Fourier transform is a prototype of a unitary GT-automorphism for $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$.
(4) There is an important Gelfand triple of Operator spaces, namely $\big(\mathcal{L}(\mathcal{S}_0{}', \mathcal{S}_0),\, \mathcal{HS}\,,\, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0{}')\big)$, which is characterized by its mapping property, but due to suitable unitary Gelfand triple isomorphisms to $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^{2d})$ (kernel theorem), or $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ (using the spreading $\eta(T)$ or the Kohn-Nirenberg $\sigma(T)$ relation, which are connected between the symplectic Fourier transform over $\mathbf{R}^d \times \widehat{\mathbf{R}}^d$).

The following lemma is more or less a reformulation of the definition of $\mathcal{S}_0(\mathbb{R}^d)$ (or rather one of the many equivalent characterizations of this space).

Lemma 2. *For any* $g \in \mathcal{S}_0(\mathbb{R}^d)$ *the short-time Fourier transform* $f \mapsto V_g f$ *establishes a retract from* $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$ *into* $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$, *with left inverse*

$$V_g^* : F \mapsto \int_{\mathbf{R}^d \times \widehat{\mathbf{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda$$

One can however show that $V_g f \in S_0(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ if and only if $f, g \in \mathcal{S}_0(\mathbb{R}^d)$ and therefore one can also formulate the following claim:

Lemma 3. *For any* $g \in \mathcal{S}_0(\mathbb{R}^d)$ *the short-time Fourier transform* $f \mapsto V_g f$ *establishes a retract from* $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$ *into* $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$, *with left inverse:*

$$V_g^* : F \mapsto \int_{\mathbf{R}^d \times \widehat{\mathbf{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda.$$

It is even more interesting that for any TF-lattice Λ the restriction of $V_g f$ to Λ, i.e.

Theorem 7. *For $g \in \mathcal{S}_0(\mathbb{R}^d)$ the mapping $C : f \mapsto (V_g f(\lambda))_{\lambda \in \Lambda}$ is a bounded Gelfand triple morphism into $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$.*
The mapping C defined above is a retract if and only if the family $(\pi(\lambda)g)_{\lambda \in \Lambda}$ is a (Gabor) frame. Indeed, the dual Gabor atom $\widetilde{g}$ is automatically in $\mathcal{S}_0(\mathbb{R}^d)$ if the frame operator is invertible. Consequently the mapping $\mathcal{R} : \mathbf{c} \mapsto \sum_\Lambda c_\lambda \pi(\lambda)\widetilde{g}$ is a Gelfand triple morphism from

$$(\ell^1, \ell^2, \ell^\infty) \quad \textit{into} \quad (\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')(\mathbf{R}^d).$$

In the background the following important result by Gröchenig and Leinert has to be formulated:

Theorem 8. *Assume that for* $g \in S_0(\mathbb{R}^d)$ *the Gabor frame operator*

$$S : f \mapsto \sum_{\lambda \in \Lambda} \langle f, \pi(\lambda) g \rangle \, \pi(\lambda) g$$

is invertible as an operator on $\boldsymbol{L}^2(\mathbb{R}^d)$*, then it is also invertible on* $S_0(\mathbb{R}^d)$ *and in fact on* $S_0{}'(\mathbb{R}^d)$*.*
In other words: Invertibility at the level of the Hilbert space automatically implies that S *is (resp. extends to) an isomorphism of the Gelfand triple automorphism for* $(S_0, \boldsymbol{L}^2, S_0{}')(\mathbb{R}^d)$*.*

Gabor Riesz Projection Bases

In some other cases, e.g. for applications in applications such as mobile communication, one would like to recover coefficents from linear combinations of the form $\sum_{\Lambda}^{\circ} c_{\lambda^\circ} g_{\lambda^\circ}$, in other words, one needs Gabor Riesz bases.

Theorem 9. *For $g \in S_0(\mathbb{R}^d)$ the mapping $\mathcal{C} : \mathbf{c} \mapsto \sum_{\Lambda^\circ} c_{\lambda^\circ} g_{\lambda^\circ}$ is a bounded Gelfand triple morphism $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$ into $(S_0, \boldsymbol{L}^2, S_0{}')$.*
The mapping $\mathcal{C}$ defined above is a retract if and only if the family $(\pi(\lambda^\circ)g)_{\lambda^\circ \in \Lambda^\circ}$ is a (Gabor) Riesz basis. Indeed, the generator of the biorthogonal Gabor atom $\widetilde{g}$ is automatically in $S_0(\mathbb{R}^d)$ and the Gram operator is invertible on $(\ell^1, \ell^2, \ell^\infty)(\Lambda^\circ)$. Consequently the mapping $\mathcal{R} : f \mapsto V_{\widetilde{g}} f(\lambda^\circ)$ is a Gelfand triple morphism from

$$(S_0, \boldsymbol{L}^2, S_0{}')(\mathbf{R}^d) \quad \textit{into} \quad (\ell^1, \ell^2, \ell^\infty).$$

A result of this type is of course the basis for many results about Gabor multipliers, arising by multiplying the Gabor coefficients with some sequence, to be called upper symbol. So every bounded sequence $m \in \ell^\infty(\mathbb{Z}^{2d})$ defines a bounded linear operator. Even more is true:

Theorem 10. *For a fixed pair of $\mathcal{S}_0(\mathbb{R}^d)$-functions g, γ, the mapping from the upper symbol $m \in (\ell^1, \ell^2, \ell^\infty)$ to the Gabor multiplier*

$$f \mapsto GM_m(f) := \sum_{\lambda \in \Lambda} V_\gamma f(\lambda) m_\lambda g_\lambda$$

is a GT-morphism into $\big(\mathcal{L}(\mathcal{S}_0{}', \mathcal{S}_0), \mathcal{HS}\,, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0{}')\big)$.

There is an alternative description of Gabor multipliers GM_m is to express it as a sum of rank-one operators $P_\lambda : f \mapsto \langle f, g_\lambda \rangle g_\lambda$

Riesz Projection bases for Spline-type spaces

Think of a translation invariant (say wavelet) closed subspace V with a Riesz (or even orthonormal) basis of the form $(T_\lambda \varphi)_{\lambda \in \Lambda}$. If φ is of some mild quality, namely $\varphi \in \boldsymbol{W}(\boldsymbol{L}^2, \ell^1)$ then we have $\varphi * \varphi^* \in \boldsymbol{W}(\mathcal{F}L^1, \ell^1) = \mathcal{S}_0(\mathbb{R}^d)$, hence the sampled autocorrelation function is in ℓ^1.

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The orthonormal projection from the Hilbert space $f \mapsto P_V$ is obtained by the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi.$$

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But this mapping is not only well defined on $\boldsymbol{L}^2$, but also on a $\boldsymbol{L}^p$, for the full range of $1 \leq p \leq \infty$ and - again due to the properties of Wiener amalgams brings us for $f \in (\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ coefficients which are in $(\ell^1, \ell^2, \ell^\infty)$, which in turn implies that the function $\sum_{\lambda\in\Lambda}(\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi$ is a well defined element of $\boldsymbol{W}(\boldsymbol{C}^0, (\ell^1, \ell^2, \ell^\infty))$, hence in particular $(\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)(\mathbb{R}^d)$.

The Gabor Multipliers and Spline Type Spaces

The question, whether a Gabor multiplier GM_m is uniquely determined can be recast into a question about the property of the family of rank 1 operators. Let us assume for simplicity that $\gamma = g$ (and perhaps that we have tight Gabor atoms, so that the constant multiplier $m(\lambda) \equiv 1$ gives the Id-operator.

Theorem 11. *The family of rank-1 operators* P_λ, $f \mapsto \langle f, g_\lambda \rangle g_\lambda$ *is a Riesz basis with* $\mathcal{HS}$ *if and only if the circulant matrix generated from the "vector"* $|V_g g(\lambda)|^2$ *is invertible.*

If this is the case, the family $(P_\lambda)_{\lambda \in \Lambda}$ *is in fact a Riesz projection basis for* $\big(\mathcal{L}({\mathcal{S}_0}', \mathcal{S}_0), \mathcal{HS}\,, \mathcal{L}(\mathcal{S}_0, {\mathcal{S}_0}')\big)$*, i.e. the mapping* $Q: T \mapsto$ *best-approx-to* T *in the* $\mathcal{HS}$ *sense extends to a retract from* $\big(\mathcal{L}({\mathcal{S}_0}', \mathcal{S}_0), \mathcal{HS}\,, \mathcal{L}(\mathcal{S}_0, {\mathcal{S}_0}')\big)$ *into the spaces of Gabor multipliers with multiplier symbols in* $(\ell^1, \ell^2, \ell^\infty)(\Lambda)$ *(which is surjective).*

Time-Frequency Concentration via Modulation Spaces

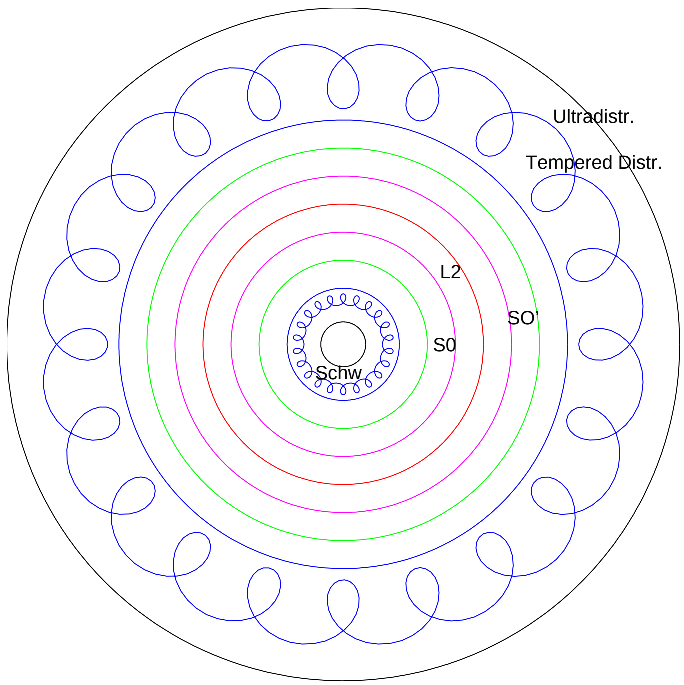
Recall that the Gauss function is given by $g_0(t) = e^{-\pi t^2}$, $z = (x, \xi) \in \mathbb{R}^d$. The Short-Time Fourier Transform (STFT) with Gaussian window is therefore

$$\begin{aligned} V_{g_0} f(z) &= \int_{\mathbb{R}^d} f(t)\, g_0(t-x)\, e^{2\pi i t \cdot \xi}\, dt \\ &= \langle f, M_\xi T_x g_0 \rangle \end{aligned}$$

Modulation space M_v^p ("good pulses") consists of all functions such that

$$\|f\|_{M_v^1} := \Big(\int_{\mathbb{R}^d} |V_{g_0} f(z)|^p\, v^p(z)\, dz\Big)^{1/p} < \infty$$

Typical examples for such weight functions v on phase space are either weights depending on frequency only, such as $v(x, \xi) = (1 + |x|)^s$ (leading to the "classical" modulation spaces), or more interesting (because they lead to Fourier invariant spaces) weights which are radial: $v_s := (1+|x|^2+|\xi|^2)^{s/2}$. The intersection of all spaces $M^p_{v_s}$ is just the Schwartz space $\mathcal{S}(\mathbb{R}^d)$. $M^2_s = Q_s =$ (Shubin Class).



Frames and Riesz Bases

A family is a Riesz basis in a Hilbert space if it behave like a finite linear independent sequence in a Hilbert space: The set of all linear combinations (infinite linear combinations with square summable coefficients) is a closed subspace (whose orthogonal complement is the null-space of the adjoint mapping). It always has a biorthogonal Riesz basis (obtained via the inverse Gram matrix).

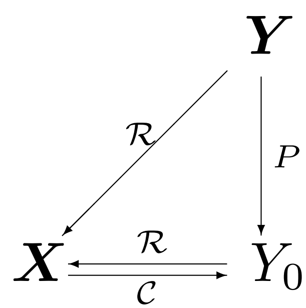
A family is a frame in a Hilbert spaces, if it behaves like a generating set (in the sense that the set of all linear combinations with square summable coefficient equals the whole Hilbert space. There is always a dual frame (obtained by inverse frame operator).

There is also a natural "duality between these problems, either in the abstract Hilbert space setting, or better understandable in the context of matrices. A rectangular $m \times n$ matrix A of $maximal\ rank$ is either linear independent $m \geq n$ or a generating set $n \geq m$, and therefore its transpose (conjugate) is of the "other type". In this contact the frame matrix is just the Gramian to A'.

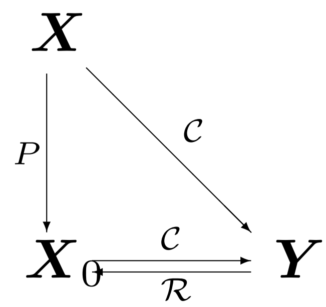
Frames and Riesz Bases: Commutative Diagrams

Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range Y_0 of $\mathcal{C}$ in $\boldsymbol{Y}$:



Riesz Basis case: E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$:



X
P
C
X0
C
R
Y

Unconditional Banach Frames

A suggestion for making the bring the well established notion of Banach frames closer to the setting we are used from the Hilbert space and ℓ^2-setting:

Definition 7. *A mapping* $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ *defines an unconditional (or solid) Banach frame for* $\mathbf{B}$ *w.r.t. the sequence space* $\mathbf{Y}$ *if*

1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, *with* $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,

2. $(\mathbf{Y}, \|\|_{\mathbf{Y}})$ *is a solid Banach space of sequences over* I, *with* $\mathbf{c} \mapsto c_i$ *being continuous from* $\mathbf{Y}$ *to* $\mathbb{C}$ *and solid, i.e. satisfying* $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$ *(hence, w.l.o.g.,* $\mathbf{e}_i \in \mathbf{Y}$*),*

3. *finite sequences are dense in* $\mathbf{Y}$ *(at least* W^**).*

Corollary 4. *By setting* $h_i := \mathcal{R}e_i$ *we have* $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i\in I} c_i h_i$ *unconditional in* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$*, hence* $f = \sum_{i\in I} T(f)_i h_i$ *as unconditional series.*

Corollary 4. *By setting* $h_i := \mathcal{R}e_i$ *we have* $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ *unconditional in* $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, *hence* $f = \sum_{i \in I} T(f)_i h_i$ *as unconditional series.*

We may talk about Gelfand frames (or Banach frames for Gelfand triples) resp. Gelfand Riesz bases (as opposed to a Riesz projection basis for a given pair of Banach spaces).

Fundamental facts in Gabor Analysis

Here is a list of "fundamental questions" in Gabor analysis:

1. Given a "sufficiently nice" atom, can we show, that it provides us with good Gabor frames (regular or irregular)?

2. Given the fact that the Gabor frame operator for a regular Gabor family $g_\lambda = \pi(\lambda)g$ (with λ from a lattice Λ in phase space) commutes with TF-shifts from Λ, how can we derive Janssen's representation from this fact (and what can be said about the convergence of Janssen's representation)?

3. An important ingredient for obtaining the Janssen is the fact that the periodization of an operator along a TF-lattice Λ corresponds exactly to sampling in the spreading domain. How can that be described in mathematically correct terms?

4. Recalling that the dual frame for a regular Gabor frame is a Gabor frame with $\tilde{g} = S^{-1}g$ as generator. How can one ensure good properties of $\tilde{g}$, given a good quality of g.

5. In which way do we have robustness of the choice of dual Gabor atoms, in the sense that "similar lattices" induce similar dual Gabor atoms?

Sufficient Dense Lattices Λ provide Gabor Frames

It is just a simple reformulation that a (regular or irregular) Gabor family $(g_i) = g_{\lambda_i}$ is a (Gabor) frame if and only if the Gabor frame operator

$$S : f \mapsto \sum_{i \in I} \langle f, g_i \rangle g_i$$

is a bounded and positive definite (hence invertible) operator on the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$. The usual way to understand S is to view it as a composition of the coefficient operator $T : f \mapsto (\langle f, g_i \rangle)_{i \in I}$ and its adjoint T^*, given by $(c_i) \mapsto \sum_i c_i g_i$. Obviously T can be reinterpreted as the mapping $f \mapsto (V_g f(\lambda_i))_{i \in I}$.
Since it is well known that for any two functions $f, g \in \boldsymbol{L}^2(\mathbb{R}^d)$ one has $V_g f \in \boldsymbol{C}^b(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^{2d})$ one would not expect that there is a problem with sampling. At least for the regular case, i.e. if $\{\lambda_i, i \in I\} = \Lambda$ is a lattice.

However, examples by F-Janssen show that one may have a situation, where the family $(g_\lambda)_{\lambda \in \Lambda}$ is a Bessel family (equivalent to the boundedness of T) for any rational lattice Λ, whereas it is not a Bessel family for a "dense set" of irrational lattices.

Sufficient Dense Lattices Λ provide Gabor Frames, ctd.

It is now maybe not so surprising the hear, that windows in $\mathcal{S}_0(\mathbb{R}^d)$ are the good ones:
Not only can one claim that the coefficient mapping is bounded for any $lattice$ Λ, but even more generally for arbitrary reatively separated point sets $(\lambda_i)_{i\in I} \subset \mathbf{R}^d \times \widehat{\mathbf{R}}^d$
This may not come as a surprise as soon as one has understood that $V_g f$ is not only in $\boldsymbol{C}^b(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^{2d})$, but that

$$g \in \mathcal{S}_0(\mathbb{R}^d) \Rightarrow V_g f \in \boldsymbol{W}(\mathbb{C}^0, \ell^2)(\mathbf{R}^d \times \widehat{\mathbf{R}}^d) \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d)$$

This in term can be justified by the (non-trivial) fact that $g \in \mathcal{S}_0(\mathbb{R}^d)$ if and only if $V_g g$ (not just $V_{g_o} g \in \boldsymbol{L}^1(\mathbb{R}^{2d})$) if and only if $V_g g \in \boldsymbol{W}(\mathbb{C}^0, \ell^1)(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$, combined with a $twisted$ convolution relation between short time Fourier transforms:

$$V_g f = V_g f *_{tw} V_g g.$$

The fact that $V_g f \in \boldsymbol{W}(\mathbb{C}^0, \ell^2)$ (even in $\boldsymbol{W}(\mathcal{F}L^1, \ell^2)$!) in connection with the reproducing formula is also the basis for coorbit theory which shows that *any sufficiently dense* family $(\lambda_i)_{i \in I} \subset \mathbf{R}^d \times \widehat{\mathbf{R}}^d$ generates a Gabor frame (g_i).

Automatic Quality of Irregular Gabor Families

It is one of the standard results of classical coorbit theory (Fei/Gro) that - again given sufficient density of the (possibly irregular) family $(\lambda_i)_{i\in I}$ - the Gabor frames (g_i) arising are not just frames for the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, but also allow to characterize various coorbit spaces. In the present situation coorbit spaces characterized by the behaviour of STFTs of their elements are just the modulation spaces.

One concrete and compact way to express the essential feature in the context of Banach Gelfand triples is the following result, which can be seen as examples of Banach frame features of coorbit theory:

Theorem 12. *For the (irregular or regular) Gabor described above, with Gabor atom $g \in \mathcal{S}_0(\mathbb{R}^d)$ and (λ_i) sufficiently dense one has: the mapping $T : f \mapsto (V_g f(\lambda_i))_{i\in I}$ describes a retract from the Gelfand triple $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$ into the Gelfand triple $(\ell^1, \ell^2, \ell^\infty)$.*

In particular one can recognize the membership of f in one of the space

$(S_0, \boldsymbol{L}^2, {S_0}')$ *by checking the membership of the family* $(V_g f(\lambda_i))_{i \in I}$ *in one of the spaces from* $(\ell^1, \ell^2, \ell^\infty)$*, and resynthesis from the sampling values can be achieved in a stable way at each level.*

Robustness and Jitter Stability of such Gabor Frames

Once one has found that a family $(g_i) = (\pi(\lambda_i)g)_{i \in I}$ is a Gabor frame, what the *quality of that Gabor frame is*. How robust is it against minor perturbation of the underlying family (λ_i)?
It is natural to ask whether it is true that not only Gabor families which are obtained from an exact lattice are frames, but also those obtained by a (sufficiently small) uniform jitter error.

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That such a theorem cannot hold for general L^2-frames is easily seen by considering the complete orthonormal system on $\mathbb{R}$ obtained from $g = \mathbf{1}_{[0,1)}$ and the critical lattice $\Lambda = \mathbb{Z} \times \mathbb{Z}$.

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In contrast, one has for $g \in S_0(\mathbb{R}^d)$: Assume that $(\pi(\lambda_i)g$ is a Gabor frame, then there exists $\delta > 0$ such that for any family (λ_i') with $|\lambda_i' - \lambda_i| < \delta$ for all $i \in I$ one has: $\pi(\lambda_i')g$ is a Gabor frame as well.

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Even more is true: not only are those frames off all those families of points

"sufficiently close" are all of similar quality, they are also preservering the property of characterizing the Gefand Triple $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0{}')$ by having the $canonical\ coefficients$ in $(\ell^1, \ell^2, \ell^\infty)$.

Gabor Atom, the Tight, and the Dual Atom

For the *regular case*, i.e. for the case that the discrete point set generating the regular Gabor family $(g_\lambda)_{\lambda \in \Lambda}$ from one or several Gabor atoms the Gabor frame operator has the crucial addition property of commuting with the TF-shifts used in building it. Although the mapping $\lambda \mapsto \pi(\lambda)$ is not a unitary group representation, but still a projective representation (with some phase factors arising in the composition law), one arrives via resummation at the crucial fact that

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$$S \circ \pi(\lambda) = \pi(\lambda) \circ S \quad \forall \lambda \in \Lambda$$

representations using the dual Gabor atom $\widetilde{g}$:

representations using the dual Gabor atom $\widetilde{g}$:

$$f = \sum_{\lambda \in \Lambda} V_{\widetilde{g}} f(\lambda) \cdot \pi(\lambda) g = \sum_{\lambda \in \Lambda} V_g f(\lambda) \cdot \pi(\lambda) \widetilde{g}.$$

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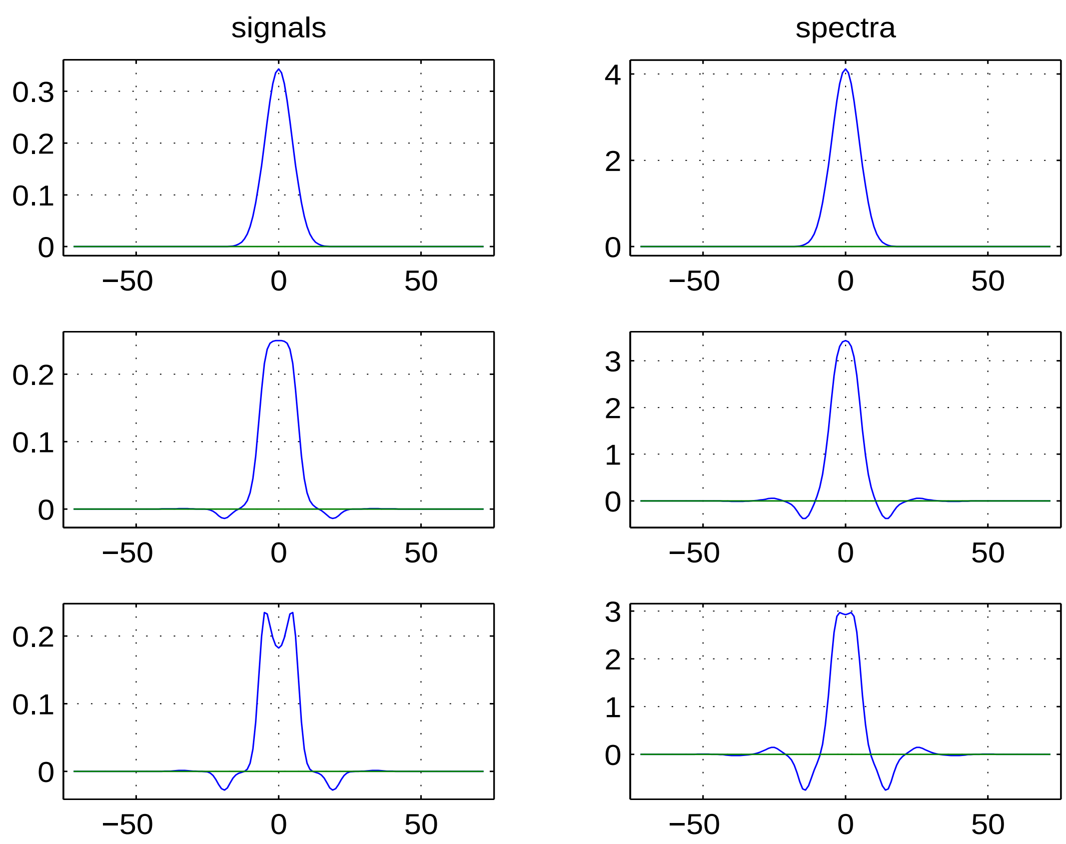


Janssen's Representation of the Gabor Frame Operator

Definition 8. *For a given TF-lattice* Λ *the adjoint TF-lattice* Λ° *is defined as follows:*

$$\Lambda^\circ := \{\lambda^\circ \,|\, \pi(\lambda) \circ \pi(\lambda^\circ) = \pi(\lambda^\circ) \circ \pi(\lambda), \quad \forall \lambda \in \Lambda\}$$

The so-called Janssen representation of the Gabor frame operator describes the (Gabor) frame operator assocatied with a Gabor family $(g_\lambda) := (\pi(\lambda)g)_{\lambda\in\Lambda}$ as a superposition of TF-shifts from Λ°, more precisely, there is a constant $C_\Lambda > 0$ such that

$$S = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ)\pi(\lambda^\circ)$$

An important consequence of this representation is the fact that $S = A \cdot Id$ if and only if (comparison of coefficients) the so-called Wexler-condition is satisfied, i.e. $V_g g(\lambda^\circ) = 0$ for all $\lambda^\circ \neq 0, \lambda^\circ \in \Lambda^\circ$. In other words, for normalized $g \in L^2$ one has: the family (g_λ) forms a tight Gabor frame if and only if (g_{λ°) is an orthonormal system.

Of course it is of interest to study the form of the convergence of the Janssen presentation above, depending on the quality of the window. For example, $g \in \mathcal{S}_0(\mathbb{R}^d)$ implies that $V_g g \in S_0(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ and consequently the Janssen representation of absolutely convergent in the sense of operators (on $\mathcal{S}_0(\mathbb{R}^d), \boldsymbol{L}^2(\mathbb{R}^d), S_0'(\mathbf{R}^d)$ of $\boldsymbol{L}^p(\mathbb{R}^d)$ for any $p \geq 1$).
If $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ one can not even claim that the coefficients $V_g g \in \ell^2(\Lambda^\circ)$, but one has norm convergence at least in the sense of $S_0'(\mathbf{R}^d)$ of

$$C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ)\pi(\lambda^\circ) f$$

for each $f \in \mathcal{S}_0(\mathbb{R}^d)$.

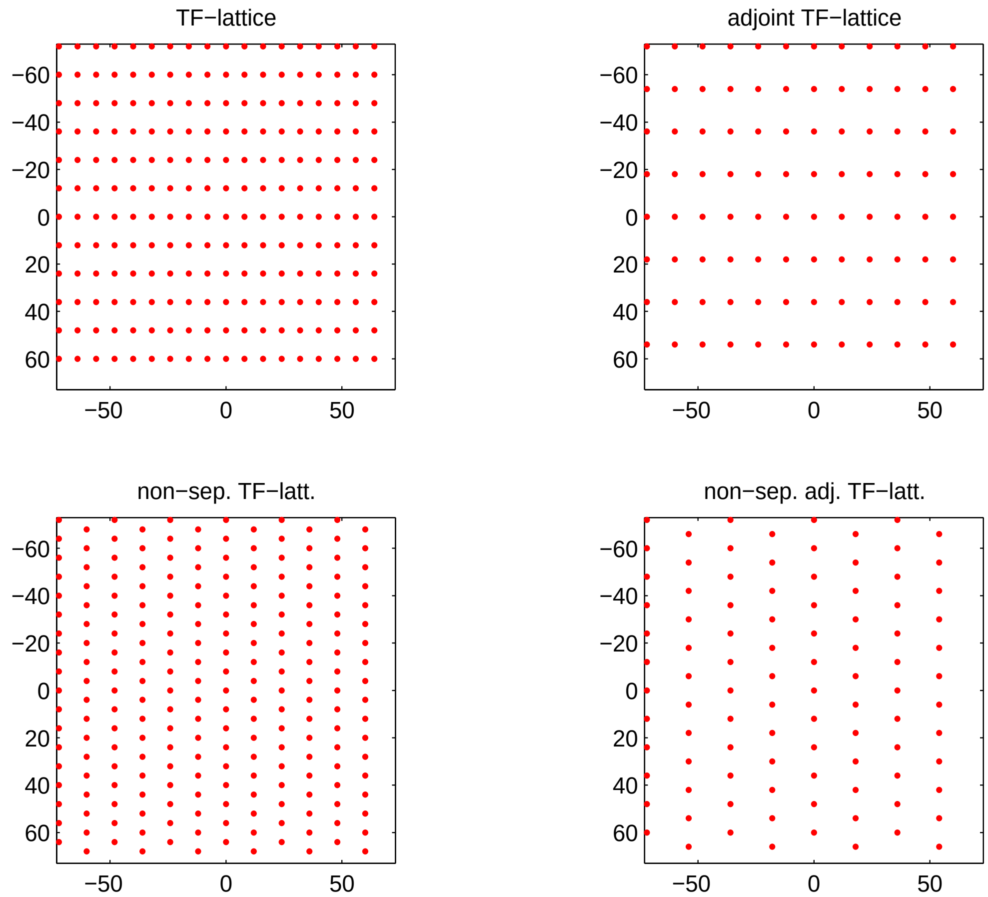
Note that in the case of a separable group $\Lambda = \Lambda_1 \times \Lambda_2$, with Λ_1 and Λ_2 being lattices in the time- and frequency domain respectively it is easy to derive Walnut's representation from the Janssen's representation, describing

S as a superposition of the form

$$Sf = \sum_{\lambda_1 \in \Lambda_1} G_{\lambda_1} T_{\lambda_1} f,$$

where the pointwise multipliers G_{λ_1} are Λ_1-periodic functions.
Concerning the convergence of this series it is not hard to show that $g \in \boldsymbol{W}(\boldsymbol{C}^0, \ell^1)(\mathbf{R}^d)$ or just $g \in \boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)(\mathbf{R}^d)$ implies that $\sum_{\lambda_1 \in \Lambda_1} \|G_{\lambda_1}\|_\infty < \infty$, and consequently that S is bounded on any $\boldsymbol{L}^p, 1 \leq p \leq \infty$.

Adjoint Lattices, Janssen's Representation



Local properties of STFTs with $\mathcal{S}_0(\mathbb{R}^d)$- windows

Corollary 5. *Let* $g \in \mathcal{S}_0(\mathbb{R}^d)$. *Then* $|V_g f|^2 \in S_0(\mathbf{R}^{2d}) \subset \boldsymbol{W}(\boldsymbol{C}^0, \ell^1)$ *for* $f \in L^2(\mathbb{R}^d)$.

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This will be an important fact in the background of the following study: Consider the rank-one operators $P_\lambda : f \mapsto \langle f, g_\lambda \rangle g_\lambda$, for $\lambda \in \Lambda$. For g normalized in $\boldsymbol{L}^2$ these are the projections on the 1D-space generated by g_λ, and for $g \in \mathcal{S}_0(\mathbb{R}^d)$ they are "good quality operators in $\mathcal{L}(\mathcal{S}_0{}'(\mathbb{R}^d), \mathcal{S}_0(\mathbb{R}^d))$.

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We them as elements of $\mathcal{HS}$ and want to find out, whether they are a Riesz basis in $\mathcal{HS}$ by checking for the invertibility of their Gram matrix.

$$\langle P_\lambda, P_{\lambda'} \rangle_{\mathcal{HS}} = |\langle g_\lambda, g_{\lambda'} \rangle|^2_{\boldsymbol{L}^2} = |V_g g(\lambda - \lambda')|^2$$

this in turn is a circulant matrix, and its invertibility is equivalent to the fact that the $\Lambda^\perp$ periodic version of $\mathcal{F}_\Lambda(|V_g g|^2)$ is free of zeros (note that we can apply Wiener's inversion theorem because $\mathcal{F}_\Lambda(|V_g g|^2) \in \mathcal{S}_0(\mathbb{R}^d)$, hence its periodization has an absolutely convergent Fourier series (as well as its inverse with respect to convolution).

Gabor Multipliers: Overview of Questions

$$G_m(f) = \sum_{\lambda \in \Lambda} m_\lambda V_g f(\lambda) g_\lambda = \sum_{\lambda \in \Lambda} m_\lambda P_\lambda(f)$$

where we assume for simplicity that $g \in \mathcal{S}_0(\mathbb{R}^d)$ generates a tight Gabor frame, or equivalently, we assume that $m \equiv 1$ gives us the identity operator.

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- Gabor multiplier results can be obtained from the mapping properties of the $\mathcal{C} : f \mapsto V_g f|_\Lambda$ and the synthesis mapping $\mathcal{R} : c \mapsto \sum c_\lambda g_\lambda$.
- in addition one may ask in which sense the quality of the Gabor multipliers depends on the ingredients;
- what can be said about the linear mapping from sequences (m_λ) to operators G_m (injectivity, etc.);

- best approximation by Gabor multipliers;
- questions of stability (condition numbers);

Gabor Multipliers: Basic Facts

Theorem 13. *The mapping GM from the "upper symbol"* (m_λ) *to the Gabor multiplier* G_m *(for arbitrary* $g \in \boldsymbol{L}^2(\mathbb{R}^d)$*) is a Gelfand triple isomorphism from the Gelfand triple* $(\ell^1, \ell^2, \ell^\infty)$ *to the Gelfand triple of operator spaces on* $\boldsymbol{L}^2(\mathbb{R}^d)$ *consisting of* $(\mathcal{S}_1, \mathcal{HS}, \mathcal{B}(\boldsymbol{L}^2))$.

For $g \in \mathcal{S}_0(\mathbb{R}^d)$ we have something stronger:

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For $g \in \mathcal{S}_0(\mathbb{R}^d)$ we have something stronger:

Theorem 14. *Assume* $g \in \mathcal{S}_0(\mathbb{R}^d)$*. Then the mapping GM from the "upper symbol"* (m_λ) *to the Gabor multiplier* G_m *is a bounded linear Gelfand triple mapping from the Gelfand triple* $(\ell^1, \ell^2, \ell^\infty)$ *to the Gelfand triple of operator spaces* $\big(\mathcal{L}(\mathcal{S}_0{}', \mathcal{S}_0), \mathcal{HS}, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0{}')\big)$*.*

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Gabor Multipliers: Basic Facts II

Theorem 15. *Assume that* $g \in \mathcal{S}_0(\mathbb{R}^d)$ *and that* (P_λ) *is a Riesz basis in* $\mathcal{HS}$*, then the mapping* GM *defines a Gelfand Riesz basis for* $(\ell^1, \ell^2, \ell^\infty)$ *into* $(\mathcal{S}_0, \boldsymbol{L}^2, \mathcal{S}_0')$.
In particuar, the orthogonal projection $T \mapsto P(T)$*, mapping a given Hilbert Schmidt operator to its coefficients of the best approximation by Gabor multipliers with respect to the given Gabor frame generated from* (g, Λ) *is extending to a Gelfand triple mapping from* $\big(\mathcal{L}(\mathcal{S}_0', \mathcal{S}_0),\, \mathcal{HS}\,,\, \mathcal{L}(\mathcal{S}_0, \mathcal{S}_0')\big)$ *to* $(\ell^1, \ell^2, \ell^\infty)$.

Janssen's Representation of the (regular) Gabor Frame Operator II

$$S = \sum_{\lambda \in \Lambda} g_\lambda \otimes g_\lambda^*$$

can be written as

$$S = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g g(\lambda^\circ) \pi(\lambda^\circ)$$

Sometimes one finds in the literature the so-called condition (A) (due to Tolimieri-Orr, which simply means that the coefficient sequence $(V_g g(\lambda^\circ))$ belongs to $\ell^1(\Lambda^\circ)$

Oviously this is true for the case that $g \in \mathcal{S}_0(\mathbb{R}^d)$, because then $V_g g \in S_0(\mathbf{R}^d \times \widehat{\mathbf{R}}^d)$ and consequently sampling to any discrete subgroup gives absolutely summable coefficients.

Stability of Gabor Frames with respect to Dilation

Recent results (Trans. Amer. Math. Math. Soc.), obtained together with N. Kaiblinger.

For a subspace $X \subseteq \boldsymbol{L}^2(\mathbb{R}^d)$ define the set

$$F_g = \Big\{(g, L) \in X \times GL(\mathbb{R}^{2d}) \text{ which generate a Gabor frame } \{\pi(Lk)g\}_{k \in \mathbb{Z}^{2d}}\Big\}. \tag{5}$$

The set F_L^2 need not be open (even for good ONBs!). But we have:

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Theorem 16. (i) *The set $F_{\mathcal{S}_0(\mathbb{R}^d)}$ is open in $\mathcal{S}_0(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.*
(ii) *$(g, L) \mapsto \widetilde{g}$ is continuous mapping from $F_{\mathcal{S}_0(\mathbb{R}^d)}$ into $\mathcal{S}_0(\mathbb{R}^d)$.*

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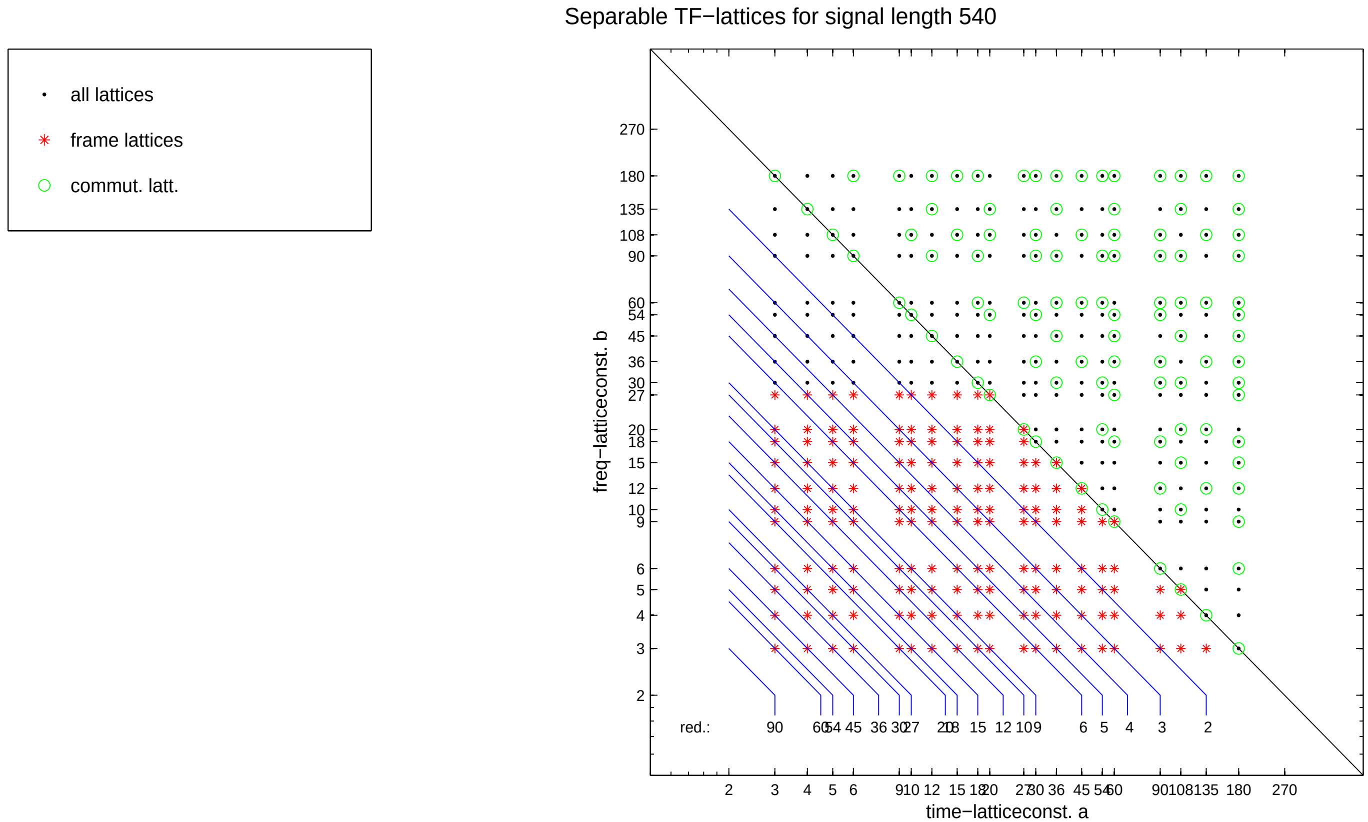
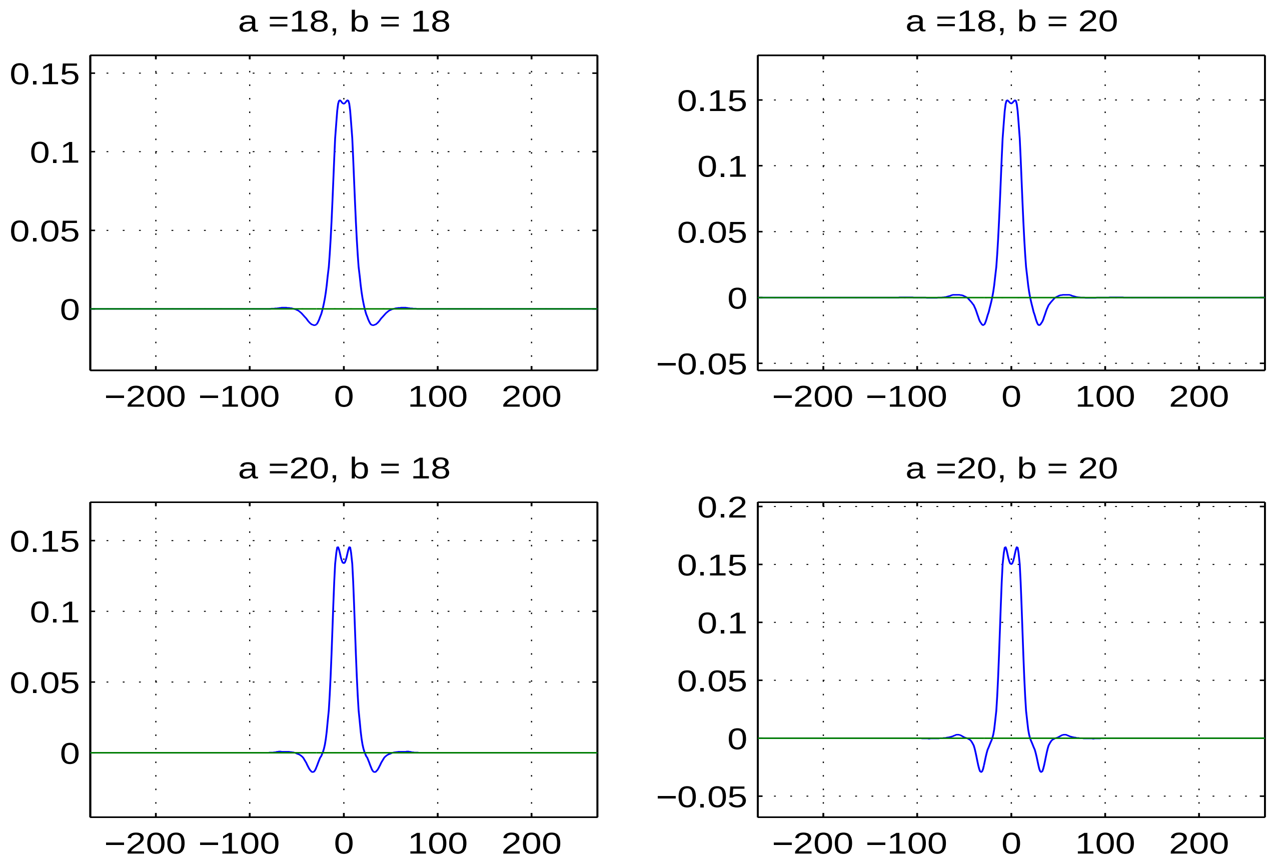
There is an analogous result for the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

Corollary 6. (i) *The set* $F_{\mathcal{S}}$ *is open in* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d) \times GL(\mathbb{R}^{2d})$.
(ii) *The mapping* $(g, L) \mapsto \widetilde{g}$ *is continuous from* $F_{\mathcal{S}}$ *into* $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$.

A Discrete Version: Each Point "is" a Lattice, $n = 540$



On the continuous dependence of dual atoms on the TF-lattice



Why is it so relevant to know it for the $S_0(\mathbb{R}^d)$ norm?

Isn't the description above self-referential? Wouldn't it be reasonable to look out for the same results for "more standard" function spaces? (assuming that we are only interested in the $\boldsymbol{L}^2$*-setting*

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NO!

- Simply because the continuous dependence is not valid in the ordinary $\boldsymbol{L}^2$-setting!

- Even if it was true for some other norm it would not imply, that the overall system, i.e. the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} V_h(f) g_\lambda$$

would *not* be close to the Identity operator in the *operator norm on* $\boldsymbol{L}^2(\mathbb{R}^d)$, for all functions h which arise as dual windows for a pair (g', Λ'), with g' close to g and Λ' close to Λ (in the sense of having very similar generator!).

Other ongoing activities

Together with N. Kaiblinger a paper on quasi-interpolation is on the way. It is shown that piecewise linear interpolation resp. quasi-interpolation (using for example cubic splines), i.e. operators of the form

$$Q_h f = \sum_{k \in \mathbb{Z}^d} f(hk) T_{hk} D_h \psi$$

are norm convergent to $f \in S_0(\mathbb{R}^d)$ in the S_0-norm.
This is an important step for his work on the approximation of "continuous Gabor problems" by finite ones (handled computationally using MATLAB, for example).

In his recent PhD thesis (Nov. 05) F. Luef has established very interesting connections between the existing body of Gabor analysis and early work by

A. Connes and M. Rieffel. The results will be published in several papers. Among others he could show that the use of $\mathcal{S}_0(\mathbb{R}^d)$ or $\mathcal{S}_0(G)$, for G a LCA group, allows to verify many of the results obtaine by Rieffel using the (complicated) Schwartz-Bruhat space, in an easier way.

Wiener Amalgams and Gabor Analysis:

- Recall: Banach Gelfand triples, unconditional Banach frames,
- suggest: Gelfand frames and Gelfand Riesz bases
- kernel theorem, various symbols
- use of amalgam spaces for the basic questions of Gabor analysis
- sufficiently dense lattices Λ generate good Gabor frames
- robustness: perturbation of either the atom or the lattice is continuous in the $\mathcal{S}_0$-setting
- and much more . . . , e.g. $\alpha-$modulation spaces;