NuHAG Numerical Harmonic Analysis Group and TUM

The time-frequency approach to distribution theory and Banach Gelfand triples

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Introductory Statement

This talk will provide a kind of *introductory overview* **over the family of function spaces which can be described using time-frequency methods.**

The relevant family of spaces is the family of so-called **modulation spaces** which are characterized by the properties of their short-time Fourier transforms (STFT). Drawn from a reservoir of tempered or ultra-distributions one chooses those, whose STFT belongs to a certain translation invariant Banach function space (resp. *solid BF-space*, see [2]).

Modulation spaces in turn are special cases of the principle of so-called **coorbit spaces** [fegr89], which allows to exchange the STFT by the continuous wavelet or shearlet transform.

Aspects of the theory of function spaces

In my recent paper on "Choosing Function Spaces in Harmonic Analysis" ([3]) I have addressed the question, of the *information content of a function space* in a given situation.
The key questions are something like

- Which properties of a function f are described by the membership of f in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$?
- What are general principles to create new function spaces?
- In which sense is a new result stronger than existing ones? (maybe only marginally stronger from a logical point of view, but much harder to use!?);

Aspects of the choice of function spaces

Even in mathematics it is in principle a good guide-line for the choice of topics or the design of proofs and lines of arguments to behave similar to real life. Let us think of **furniture**. We usually have many **choices**:

- Visit one of the big furniture houses and pick from the pre-fabricated production lines;
- Combine pieces yourself or ask experts to do it;
- Ask a joiner/carpenter to fabricate some nice piece of furniture that fits perfect;
- Design your own furniture, then produced by the joiner!

If course most people choice the first versions because it is much easier and does not require specific skill, budget and/or time to go through the process.

The easiness of use of function spaces

There are two more aspects concerning function spaces:

- While young couples usually spend a lot of time looking through the options offered by the different furniture houses to find the optimal fit for their needs young scientists are all too often willing to use the function spaces suggested by their predecessors; so in principle the possibility of choosing is not on the table;
- Some companies (especially a Swedish one) do not only offer flexibility in picking the right piece, but also is focussing for years on the easiness of "do-it-yourself", i.e. the implementation using a simple set of tools (a hammer and a screw-driver);

So we need better tools and user-friendly descriptions of function spaces to help people to take educated choices!

Classical Function Spaces

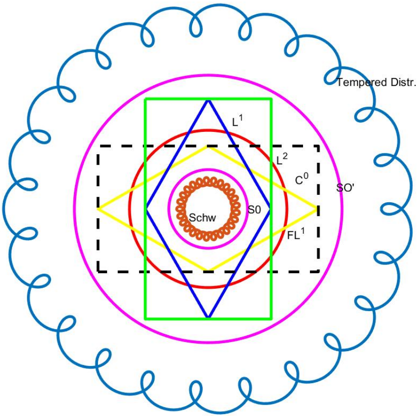
Classical function spaces start from the idea, that smoothness (of integere degree) can be expressed in terms of differentiability. Using Fourier methods the whole scale of **Sobolev spaces** was created, which correspond on the Fourier transform side to weighted $\boldsymbol{L}^2$-spaces.
The idea of Lipschitz continuity, expressed using $\boldsymbol{L}^p$-norms via higher order differences led to the theory of **Besov spaces**.
Fourier methods (Paley-Littlewood decompositions) led to the modern view on these classical function spaces, developed systematically by the leaders of interpolation theory (J. Peetre, H. Triebel) and of course E. Stein. Nowadays we have the families of **Besov-Triebel-Lizorkin spaces**.

Function Spaces and Wavelet Theory

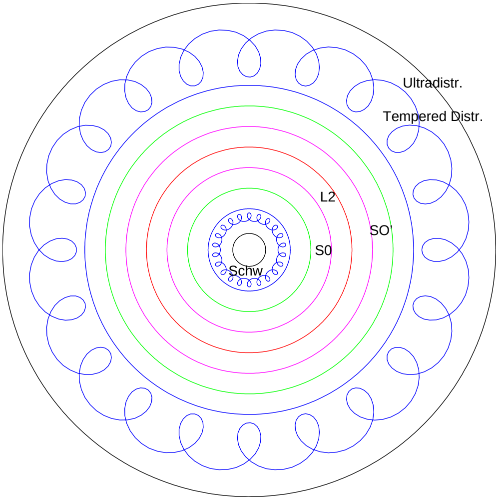
The Fourier approach paved the way to the modern view on these spaces using the *continuous wavelet transform* (CWT). The work of Frazier-Jawerth on atomic decompositions for the classical function spaces showed that for *certain atoms* one could characterize all these function spaces via atomic decompositions with coefficients from a solid BK-space over a suitable, countable index set. So in this way one could view families of function spaces as **retracts** of families of sequence spaces. In fact, these atoms for a frame for $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ and a *Banach frame* form the other spaces. Orthogonal wavelet give unconditional bases for the spaces.

Alternatively one can characterize them by the speed of convergence of the solution of the *heat equation* as the time goes to zero.

Summarizing the landscape of spaces used



Ultradistributions and the Fourier Transform



Classical spaces and the Banach Gelfand Triple

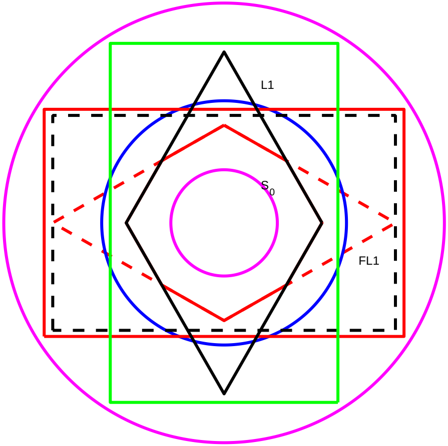


Figure: SOclassSOP.eps

The zoo of function spaces used in Fourier analysis

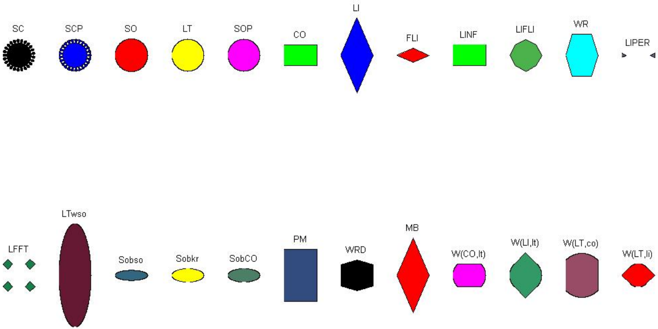


Figure: The collection of all function spaces

Domain of the Fourier inversion theorem

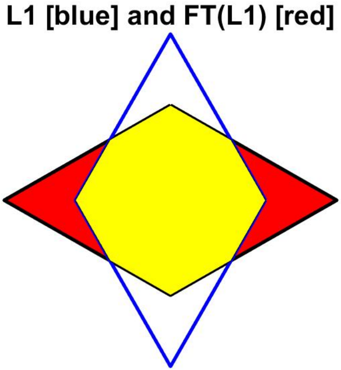


Figure: $\boldsymbol{L}^1 \cap \mathcal{F}\boldsymbol{L}^1$

The Wiener algebra (of absolutely R-integrable fcts)

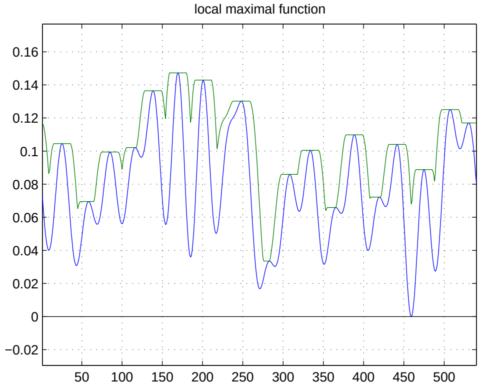


Figure: Integrability of the local maximal function

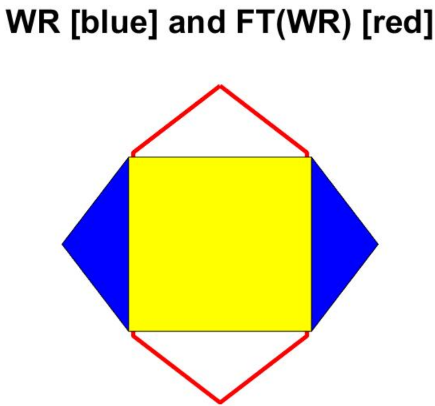
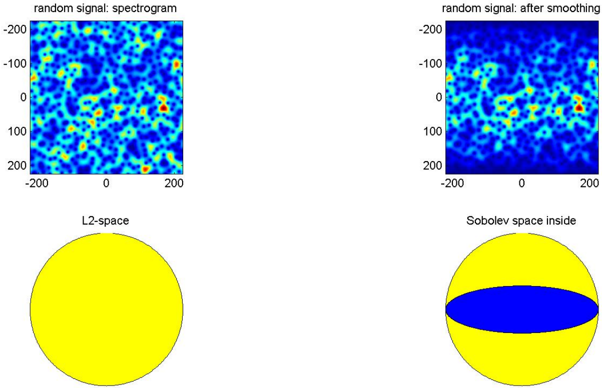


Figure: $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d) \cap \mathcal{F}(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d))$

Spectrogram of functions in Sobolev Spaces



Sobolev Embedding and $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

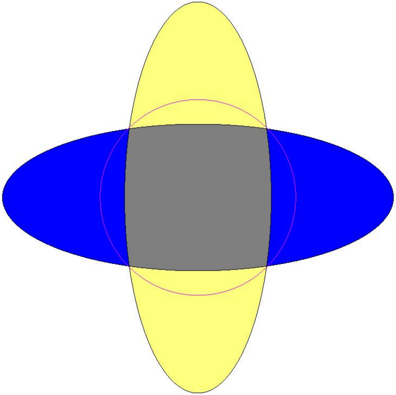


Figure: blue = Sobolev space, yellow = weighted L^2

Sobolev Embedding and $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ II

We will denote (for now) by $\boldsymbol{L}_s^2$ the weighted $\boldsymbol{L}^2$-space with weight $v_s(t) = (1+|t|^2)^{s/2}$, for $s \in \mathbb{R}$. Then the Sobolev space $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s})$ is defined as the Fourier inverse image of $\boldsymbol{L}_s^2(\mathbb{R}^d)$ (with natural norm).

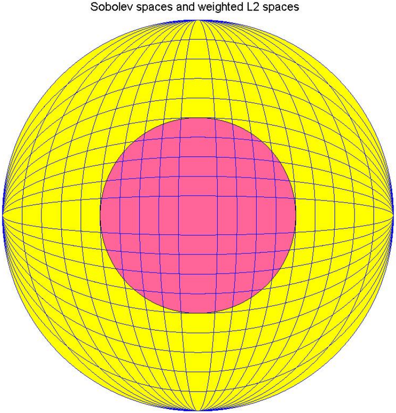
Theorem

For $s > d$ one has

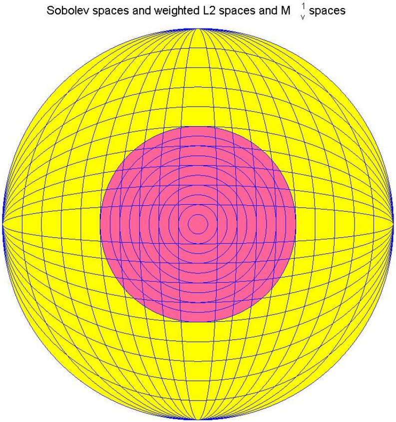
$$\mathcal{H}_s(\mathbb{R}^d) \cap \boldsymbol{L}_s^2 \subset \boldsymbol{S}_0(\mathbb{R}^d),$$

with continuous embedding with respect to the natural norms.

Sobolev and weighted L^2-spaces



Sobolev and weighted L^2-spaces



Wiener's algebra and $S_0(\mathbb{R}^d)$

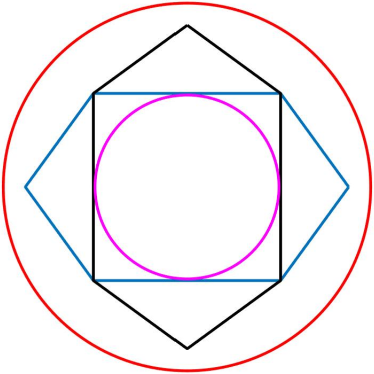
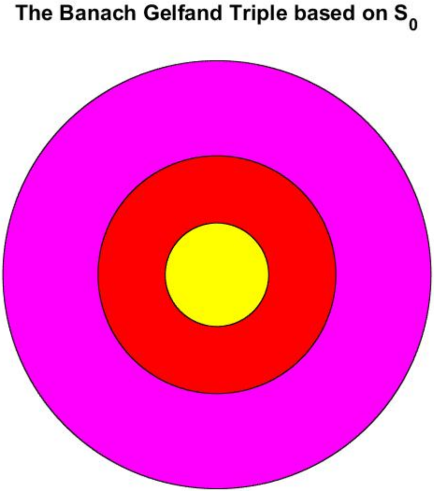


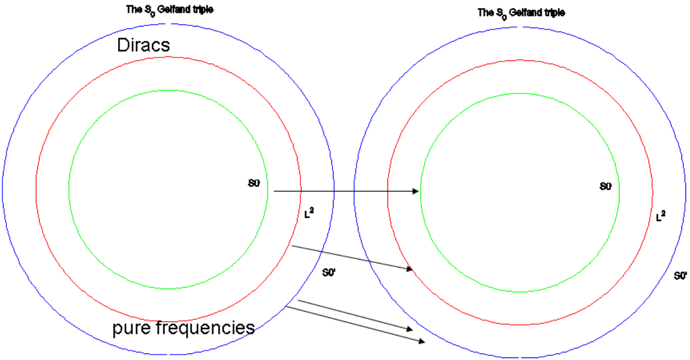
Figure: It was shown by V. Losert that the inclusion of $\boldsymbol{S}_0$ into $\boldsymbol{W} \cap \mathcal{F}(\boldsymbol{W})$ is a proper one.

The Banach Gelfand Triple

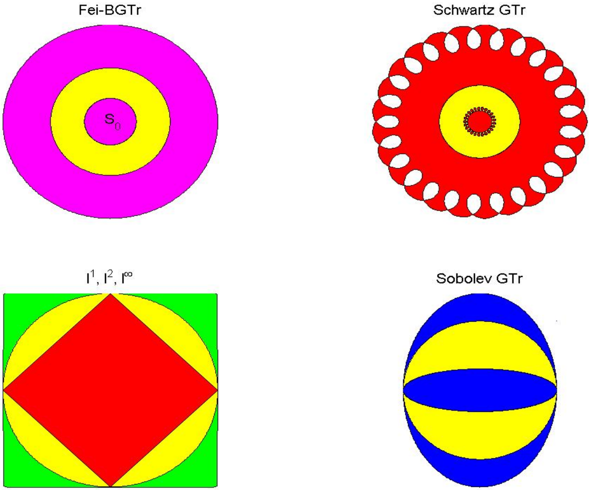


Banach Gelfand Triple (auto)morphism

Gelfand triple mapping



Various Gelfand Triples



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then automatically w^*-w^*--continuous!

The idea behind modulation spaces

Similar to the convergence properties of the heat equation, resp. the decay rate of the CWT, which should be viewed as a function over the affine group, the so-called $ax + b$-group, one can now look at the behavior of the STFT resp. the spectrogram (the absoulte value squared of the STFT) as a function over the so-called time-frequency plane, also called *phase space*.

By the Riemann-Lebesgue Lemma the STFT of any locally integrable function tends to zero as frequency goes to infinity. (Local) Differentiability even provides decay rates.

Micro-local analysis suggests to analyse directional smoothness of a function of several variables at a point by first localizing it and then verifying *directional decay* of the FT.

The STFT provides a localized view-point without zoom, by computing the Fourier transform of local patches, resp. of the signal content within a moving window.

Local Fourier Analysis of a Zebra

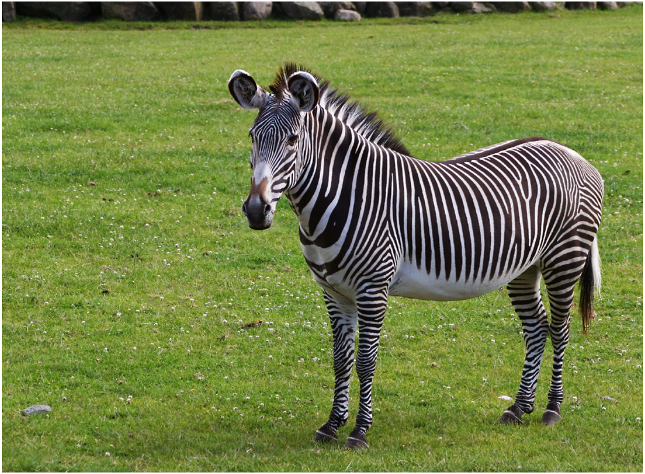
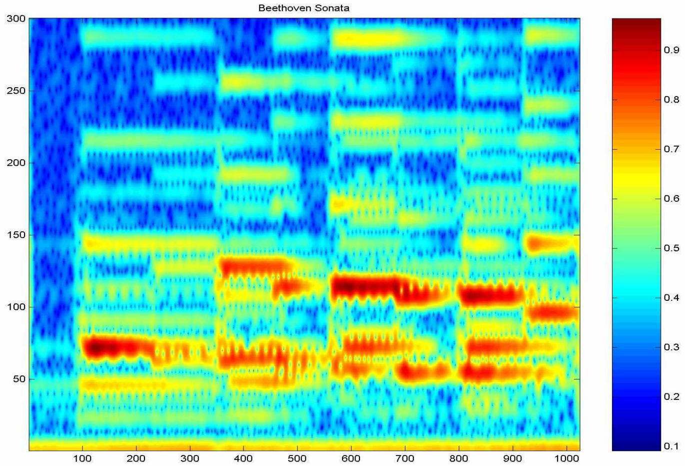


Figure: Demo of a local view on the Zebra

A Typical Musical STFT



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

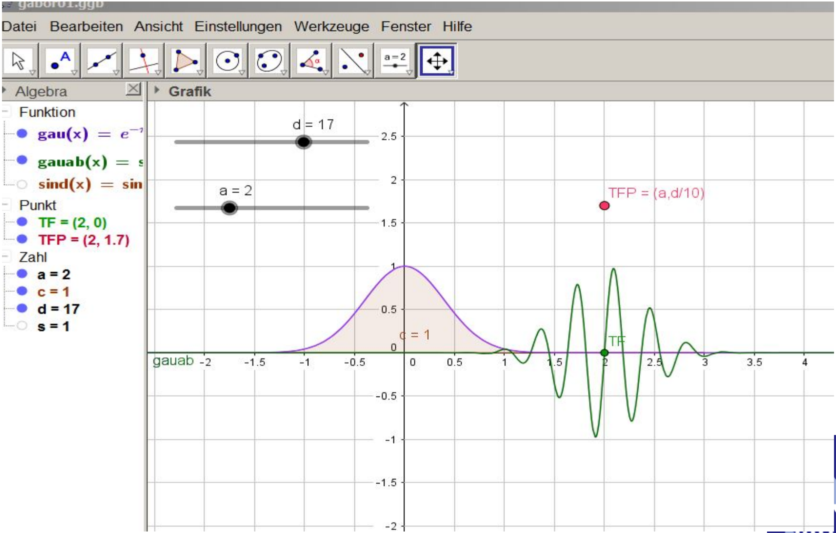
Behavior under Fourier transform

$$(T_x f)\widehat{} = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^* Th \rangle_{\mathcal{H}_1}, \; \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (1)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (2)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \qquad (3)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \qquad (4)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \qquad (5)$$

with $\chi_s(t) = exp(2\pi i \langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (5) (Fourier inversion) and the new formula formula (4). While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see for example [7]).

A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

General MODULATION SPACES

Given any translation invariant and solid BF-space (i.e. a translation invariant Banach function space) $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ with control on the norm of the TF-shift operators by some polynomial (of two variables, such as $v_s(\lambda) = (1+|\lambda|^2)^{s/2}$) one can define the corresponding modulation space $\boldsymbol{M}(\boldsymbol{Y})$ as follows (here g_0 denotes the Fourier invariant Gauss function):

$$\boldsymbol{M}(\boldsymbol{Y}) := \{\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d) \,|\, V_{g_0}(\sigma) \in Y\}.$$

Of course, as expected $\|V_{g_0}(\sigma)\|_{\boldsymbol{Y}}$ is the natural norm for this space, which is continuously embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$.

This construction is well compatible with *duality* and *interpolation*!

The classical modulation spaces

Obviously one has $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) = \mathbf{M}(\mathbf{L}^1)$.

Moyal implies $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right) = \mathbf{M}(\mathbf{L}^2)$, and

furthermoe $\mathbf{S}_0'(\mathbb{R}^d) = \mathbf{M}(\mathbf{L}^\infty)$.

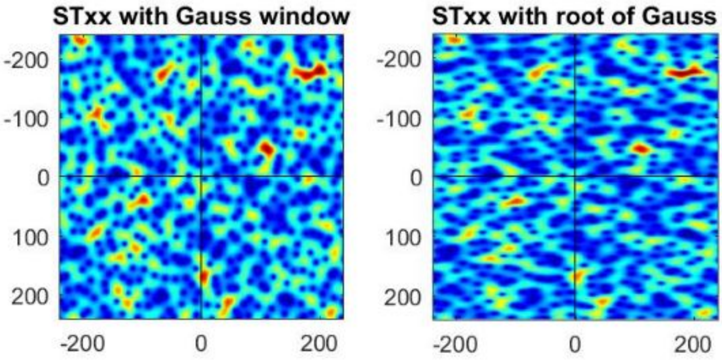
We also have $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s}) = \mathbf{M}(\mathbf{L}^2_{1\otimes v_s})$ and

$\mathbf{L}^2_{v_s}(\mathbb{R}^d) = \mathbf{M}(\mathbf{L}^2_{v_s\otimes 1})$.

The by now classical modulation spaces $\left(\mathbf{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}^s_{p,q}}\right)$ are modeled in analogy with the corresponding Besov spaces $(\mathbf{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\mathbf{B}^s_{p,q}})$ and are obtained as $\mathbf{M}(\mathbf{Y})$, where $\mathbf{Y}$ is a mixed norm, $\mathbf{L}^p - \mathbf{L}^q$ with weight v_s in the frequency.

Shubin classes $\mathbf{Q}_s(\mathbb{R}^d)$ equal $\mathbf{M}(\mathbf{L}^2_{v_s}) = \mathcal{H}_s(\mathbb{R}^d) \cap \mathbf{L}^2_{v_s}$.

Spectrogram with two different windows



Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).
There are many other independent characterization of this space, spread out in the literature since 1980, e.g. atomic decompositions using ℓ^1-coefficients, or as $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{M}^0_{1,1}(\mathbb{R}^d)$.

Basic properties of $M^\infty(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$

It is probably no surprise to learn that the dual space of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, i.e. $S_0'(\mathbb{R}^d)$ is the largest Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
As an amalgam space one has
$S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)' = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded STFT $V_g(\sigma)$.

Norm convergence in $S_0'(\mathbb{R}^d)$ equals uniform convergence of the STFT. Certain atomic characterizations of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ imply that w^*-convergence is equivalent to *locally uniform convergence* of the STFT. – HIFI recordings!

Analogies to the Schwartz space

For most applications (!!except for PDEs, see Hörmander Theory) $\boldsymbol{S}_0(G)$ is a more simple space than the Schwartz-Bruhat space, also defined over general LCA (locally compact Abelian groups).

Fourier invariance: $\mathcal{F}(\boldsymbol{S}_0(G)) = \boldsymbol{S}_0(\widehat{G})$ for LCA groups.

One can regularize distributions from $\boldsymbol{S}_0'(\mathbb{R}^d)$ using *Wiener amalgam* convolution and pointwise multiplier results:

$$\boldsymbol{S}_0 \cdot (\boldsymbol{S}_0' * \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0, \quad \boldsymbol{S}_0 * (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) \subseteq \boldsymbol{S}_0 \tag{6}$$

Although it is NOT a *nuclear Frechet space* there is a kernel theorem, which extends the usual kernel theorem for Hilbert Schmidt operators (which are exactly the operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ with *integral kernels* in $\boldsymbol{L}^2(\mathbb{R}^{2d})$!)

Important messages concerning $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$

The space (of test functions) has all the good properties one knows from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, but it is a Banach space.
It is Fourier invariant (even under the fractional Fourier transform), it is contained in all the $\boldsymbol{L}^p$-spaces, for $1 \leq p \leq \infty$ (and contains $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ as a dense subspace).
All the classical summability kernels (used for the Fourier inversion theorem) are in this class (this is why they are useful), and also Poisson's formula is valid in the strict sense

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n), \quad \forall f \in \boldsymbol{S}_0(\mathbb{R}^d). \tag{7}$$

Important messages concerning $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$

Being the *largest* Banach space of tempered distributions which is isometric under TF-shifts, it contains all the L^p-spaces for $1 \leq p \leq \infty$. It is Fourier invariant via

$$\hat{\sigma}(f) := \sigma(\widehat{f}), f \in S_0(\mathbb{R}^d).$$

It also contains any Haar measure of a subgroup, in particular $⊔⊔_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, for an arbitrary discrete subgroup $\Lambda \lhd \mathbb{R}^d$ (often called Dirac Comb). Moreover $\mathcal{F}(⊔⊔_\Lambda) = C_\Lambda \cdot ⊔⊔_{\Lambda^\perp}$.

Since pointwise multiplication goes to convolution this implies that "sampling on the time side" $(f \mapsto f ⊔⊔_\Lambda)$ corresponds to periodization of $\widehat{f}$ on the Fourier transform side $(\widehat{f} \mapsto ⊔⊔_{\Lambda^\perp} * \widehat{f})$. Similarly the *Fourier slice theorem* (tomography) is proved.

It allows to define spec(f), for $f \in L^\infty$, as supp$(\widehat{f})$.

TF-homogeneous Banach Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \; f \in \boldsymbol{B}. \tag{8}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous on often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$).

TF-homogeneous Banach Spaces II

We can summarize:

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra
$(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$.
There is also a biggest in this family, namely $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

The family of all TF-homogeneous Banach spaces (slightly more general ones) is *currently studied by the author* under the name of *Fourier Standard Spaces*, see

`www.nuhag.eu/talks`

They share many interesting properties with $\boldsymbol{L}^p$-spaces.

Further properties of $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ II

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ if and only its spectrogram for one non-zero window/atom $g \in S_0(\mathbb{R}^d)$ is bounded (and then for all such windows, due to the atomic characterizations of $S_0(\mathbb{R}^d)$).

Norm convergence in $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ corresponds to *uniform convergence* of the spectrograms.

The w^*-convergence in $S_0'(\mathbb{R}^d)$ corresponds to *uniform convergence over compact subsets of the TF plane*. A good HiFi recording covers the range of $0 - 20kHz$ for the duration of a song!

Elements of $S_0'(\mathbb{R}^d)$ have a *support* and a *Fourier transform*. Spectral synthesis implies that a distribution supported by a subgroup "comes from the subgroup" (adjoint of restriction).

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{9}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Banach Gelfand Triples, etc.

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{U}) = \boldsymbol{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as a first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The BGT (S_0, L^2, S_0') and Wilson Bases

Among the many different orthonormal bases the wavelet bases turn out to be exactly the ones which are well suited to characterize the distributions by their membership in the classical Besov-Triebel-Lizorkin spaces.
For the analogue situation (using the modulation operator instead of the dilation, resp. the Heisenberg group instead of the "ax+b"-group) on finds that *local Fourier bases* resp. the so-called Wilson-bases are the right tool. They are formed from tight Gabor frames of redundancy 2 by a particular way of combining complex exponential functions (using Euler's formula) to cos and sin functions in order to build a Wilson ONB for $\boldsymbol{L}^2(\mathbb{R}^d)$.
In this way another BGT-isomorphism between $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ and $(\ell^1, \ell^2, \ell^\infty)$ is given, for each concrete Wilson basis.

Guide to the literature

Most of our relevant papers at NuHAG are found at

`www.nuhag.eu/bibtex`

A good survey about the state of the art in Gabor analysis around 2000 is given in Charly Gröchenig's book [6], or [5].
The purely algebraic part of Gabor analysis is described in the paper [4]. The linear algebra aspects (overcomplete systems, etc.) is fully described in [1].

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

We will call $(\pi(\lambda)g)_{\lambda\in\Lambda}$ a Gabor family with Gabor atom g.

Theorem

Given $g \in \mathbf{S}_0(\mathbb{R}^d)$. Then there exists $\gamma > 0$ such that any γ-dense lattice Λ (i.e. with $\cup_{\lambda\in\Lambda} B_\gamma(\lambda) = \mathbb{R}^d$) the Gabor family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame. Hence there exists a linear mapping (the unique MNLSQ solution) $f \mapsto (c_\lambda) = \langle f, \tilde{g}_\lambda\rangle, \lambda\in\Lambda$, for a uniquely determined function $\tilde{g} \in \mathbf{S}_0(\mathbb{R}^d)$, thus

$$f = \sum_{\lambda\in\Lambda} \langle f, \tilde{g}_\lambda\rangle g_\lambda, \quad \forall f \in \mathbf{L}^2(\mathbb{R}^d).$$

In other words, the minimal norm representation of any $f \in \mathbf{L}^2(\mathbb{R}^d)$ can be obtained by just sampling the STFT with respect to the *dual window* $\tilde{g}$.

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

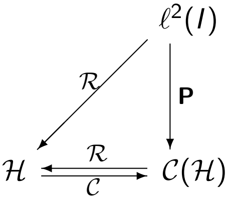
The dual Gabor atom $\tilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ provides not only the minimal norm coefficients, but also $\ell^1(\Lambda)$-coefficients for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and is well defined on $\boldsymbol{S}_0$, $\sigma \mapsto \sigma(\tilde{g}_\lambda)$ and defines representation coefficients in $\ell^\infty(\Lambda)$.
So in fact $f \mapsto (\langle f, \tilde{g}_\lambda \rangle)$ defines a Banach Gelfand triple morphism from the triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ to $(\ell^1, \ell^2, \ell^\infty)$. The (left) inverse mapping is the synthesis mapping

$$(c_\lambda) \mapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda,$$

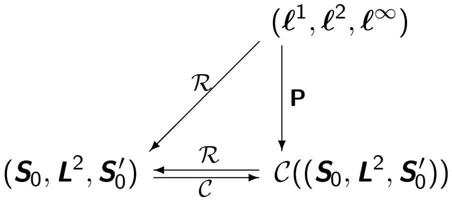
with norm convergence for $c \in \ell^1$ or ℓ^2, and still w^*-sense in $\boldsymbol{S}_0'(\mathbb{R}^d)$ for $c \in \ell^\infty(\Lambda)$.

Frames described by a diagram

Similar to the situation for matrices of maximal rank (with row and column space, null-space of $\boldsymbol{A}$ and $\boldsymbol{A}'$) we have:
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Gelfand triples (S_0, L^2, S_0'):



$S_0(\mathbb{R}^d)$ and Generalized Stochastic Processes

The joint generalization of stochastic processes (mapping from points to a Hilbert space $\mathcal{H}$ of probability measures) and distribution theory (linear mappings from a space of test functions to the complex numbers $\mathcal{H} = \mathbb{C}$) is of course the concept of *Generalized Stochastic Processes*, viewed as linear operators from $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ to a Hilbert Space $(\mathcal{H}, \|\cdot\|_{\mathcal{H}})$.
Such a theory has been developed together with my PhD student Wolfgang Hörmann a while ago.
Key points are the existence of a *Fourier transform* of a process (the *spectral process*), a spectral representation, the existence of an autocorrelation distribution in $\boldsymbol{S}_0'(G \times G)$. The autocorrelation of the spectral process is the 2D-Fourier transform (in the $\boldsymbol{S}_0'$-sense) of the autocorrelation of the process.

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem.
Naively the operator has a representation as an integral operator: $f \mapsto Tf$, with

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \quad x, y \in \mathbb{R}^d.$$

But clearly no multiplication operator can be represented in this way (not even identity), for any locally integrable function $K(x,y)$.
But we can reformulate the connection distributionally, as

$$\langle Tf, g \rangle = \langle K, f \otimes g \rangle,$$

and still call K (the uniquely determined) distributional kernel (on $\mathbb{R}^{2d}$) corresponding to T (and vice versa).

The Kernel Theorem in the S_0-setting

We will use $\boldsymbol{B} = \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ and observe and $\boldsymbol{B}'$ coincides with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$ (correctly: the linear operators which are w^* to norm continuous!), using the scalar product of Hilbert Schmidt operators: $\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*)$, $T, S \in \mathcal{HS}$.

Theorem

There is a natural BGT-isomorphism between $(\boldsymbol{B}, \mathcal{H}, \boldsymbol{B}')$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$.
This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

Custom-made modulation spaces

Coming back to my initial comparison with self-designed furniture on has to mention that there IS THE POSSIBILITY to create modulation spaces satisfying certain requirements, e.g. which allow to give a description of the idea of variable band-width in a Gabor context.

The idea is to create a reproducing kernel Hilbert space, which behaves locally like a Sobolov space (at least for high frequencies), while it is more or less like $\boldsymbol{L}^2(\mathbb{R}^d)$ up the a certain limit (the local maximal frequency or local bandwidth).

Such an approach, using specifically designed weights W on on phase space can be used in order to define $\boldsymbol{M}(\mathbb{L}^2_W)$. They are found in the work of Roza Aceska (PhD thesis Vienna and follow-up, joining us from Novi Sad in 2006).

Real World Gabor Multipliers

Gabor coefficients trhesholding for an image

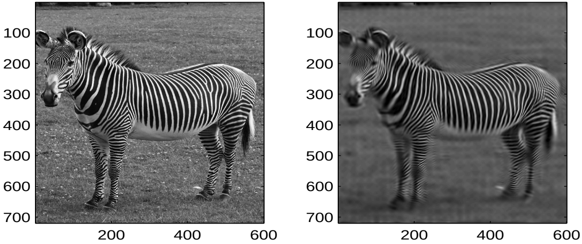


Figure: The reconstruction on the right is using only the large .... Gabor coefficients (separable case).

Further information, reading material

The NuHAG webpage offers a large amount of further information, including talks and MATLAB code:

www.nuhag.eu
www.nuhag.eu/bibtex (all papers)
www.nuhag.eu/talks (all talks)
www.nuhag.eu/matlab (MATLAB code)
www.nuhag.eu/skripten (lecture notes)

Enjoy the material!!

Thanks for your attention!

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