Irregular Sampling and Wiener Amalgam Spaces

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A WORD OF ORIENTATION!

Central questions in **Harmonic Analysis** are connected with properties of a variety of *Banach spaces of (generalized) functions*, bounded operators between them, but also Banach algebras, e.g. $(\boldsymbol{L}^1(G), \|\cdot\|_1)$ with respect to convolution, or intertwining operators. In many cases one has by now rather good knowledge concerning unconditional bases for such spaces, or at least Banach frames or atomic decompositions.

Having a sufficiently broad basis in this field allows to ask (and answer!) more interesting questions (sometimes with less effort) compared to a mindset where "classical spaces are given and sacrosanct" (e.g. $\boldsymbol{L}^p$-spaces only), see [4, 5].

The Landscape of Function Space Theory

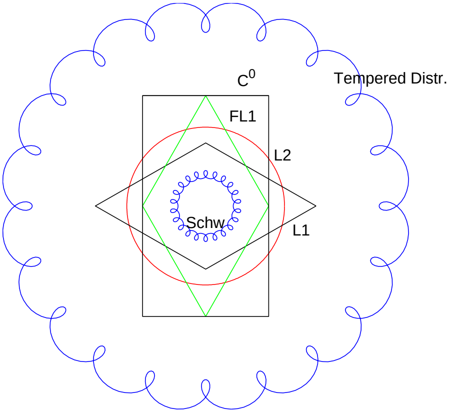
The territory of "**function spaces**" is vast, and even the term itself is subject to quite different interpretations. We would like to understand it in the spirit of Hans Triebel's "Theory of Function Spaces", which means Banach spaces of functions or (usually tempered) distributions (maybe ultra-distributions). Many of these function spaces have been introduced to allow a clean description of certain operators.

Function spaces are prototypical objects in *functional analysis* and many general principles have been first developed in the context of function spaces, while on the other hand the abstract principles of linear functional analysis can be quite nicely illustrated by applying them to (new and old) function spaces.

A listing of examples would be another talk.

The Landscape of Function Space Theory, II

Over the years I have developed a *"symbolic language"* for the different function spaces which should help to better understand the relative inclusion relations.



Choice of Spaces and Criteria

In my article [4] entitled "Choosing function spaces..." I argue, that - similar to real life - those function spaces which serve a purpose, which can be shown to be useful in different situations, the ones which are easy to use and/or help to derive strong results will gain popularity, should be taught and studied more properly, then those who are just "fancy" or which "can be constructed", because at the end the possible gain of using a very complicated function space to derive a statement which in practice is almost impossible to be applied is very modest.

Of course it is a *long way from the suggestion to discuss criteria of usefulness* comparable to what is in real-life a *consumer report*, but I am convinced that such an approach is important for a healthy development of the community. It will help us to keep contact with any kind of applications, and reduces the risk of abstract and finally complicated but close to useless theorems.

Goals of my presentation

Since I have made extensive use of Wiener amalgam spaces I can present a long list of arguments why they represent a relevant construction scheme for function spaces. In summary I would like to discuss:

1. Motivate the use of Wiener amalgam spaces ;
2. Define and characterize Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$;
3. Indicate where and why they are useful;
4. Specifically discuss $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$;
5. Indicate their role for Gabor analysis;

The classical "Wiener-type spaces"

The fact, that there are no inclusions between any two of the spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p \leq \infty$ has global (in one direction) and local (in the other direction) reasons.
$\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$
The Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ allow to get rid of these restrictions, because they behave locally like $\boldsymbol{L}^p$ while globally their behaviour is that of ℓ^q. The family of these spaces is (more or less) closed under duality, under complex interpolation, but also pointwise multiplications and convolutions respect the local and the global component independently!
See [3]

Recalling the concept of Wiener Amalgam Spaces

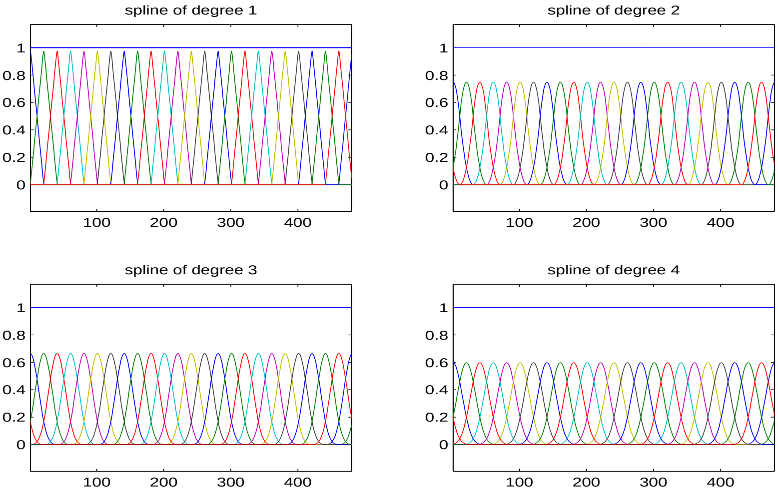
Wiener amalgam spaces are a generally useful family of spaces with a wide range of applications in analysis. The main motivation for the introduction of these spaces came from the observations that the non-inclusion results between spaes $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for different values of p are either of *local* or of *global* nature. Hence it makes sense to separate these to properties using BUPUs.

Definition

A bounded family $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathbb{R}^d$ is called a regular **Uniform Partition of Unity** if $\psi_n = T_{\alpha n}\psi_0, n \in \mathbb{Z}^d$, $0 \leq \psi_0 \leq 1$, for some ψ_0 with compact support, and

$$\sum_{n\in\mathbb{Z}^d} \psi_n(x) = \sum_{n\in\mathbb{Z}^d} \psi(x - \alpha n) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Illustration of the B-splines providing BUPUs



Recalling the concept of Wiener Amalgam Spaces II

Note that one can **define the Wiener amalgam space $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$** by the condition that the sequence $\|f\psi_n\|_{\boldsymbol{B}}$ belongs to $\ell^q(\mathbb{Z}^d)$ and its norm is one of the (many equivalent) norms on this space.

Different BUPUs define the same space and equivalent norms. Moreover, for $1 \leq q \leq \infty$ one has Banach spaces, with natural inclusion, duality and interpolation properties.
Many known function spaces are also Wiener amalgam spaces:

- $\boldsymbol{L}^p(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{L}^p, \ell^p)$, same for weighted spaces;
- $\mathcal{H}_s(\mathbb{R}^d)$ (the Sobolev space) satisfies the so-called ℓ^2-puzzle condition (P. Tchamitchian): $\mathcal{H}_s(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{H}_s, \ell^2)$, and consequently for $s > d/2$ (Sobolev embedding) the pointwise multipliers (V. Mazya) equal $\boldsymbol{W}(\mathcal{H}_s, \ell^\infty)$.

Minimality of Wiener's algebra

The Wiener amalgam spaces are essentially a generalization of the original family $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$, with local component $\boldsymbol{L}^p$ and global q-summability of the sequence of local $\boldsymbol{L}^p$ norms.
In contrast to the "scale" of spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right), 1 \leq p \leq \infty$ which do *not allow for any non-trivial inclusion relations* we have nice (and strict) inclusion relations for $p_1 \geq p_2$ and $q_1 \leq q_2$:

$$\boldsymbol{W}(\boldsymbol{L}^{p_1}, \ell^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \ell^{q_2}).$$

Hence $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)$ is the smallest among them, and $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)$ is the largest among them. The closure of the space of test functions, or also of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)$ is just *Wiener's algebra* $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$, which was one of Hans Reiter's list *Segal algebras*. It can also be characterized as the smallest of all *solid* Segal algebras.

Introducing Modulation Spaces

Having the possibility to define Wiener amalgam spaces with $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ (the Fourier image of $\boldsymbol{L}^p(\mathbb{R}^d)$ in the sense of distributions) as a local component allowed to introduce **modulation spaces** in analogy to *Besov spaces*, replacing more or less the dyadic decompositions on the Fourier transform side by uniform ones.

Formally one can define the (unweighted) modulation spaces as

$$\boldsymbol{M}^{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)\right). \tag{1}$$

or more generally the now classical modulation spaces

$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q_{v_s})\right). \tag{2}$$

Fourier invariant modulation spaces

It is an interesting variant of the classical Hausdorff-Young theorem to observe that one has

Theorem

- *For* $1 \leq r \leq p \leq \infty$ *one has*

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{F}^p, \ell^r)) \subseteq \boldsymbol{W}(\boldsymbol{F}^r, \ell^p);$$

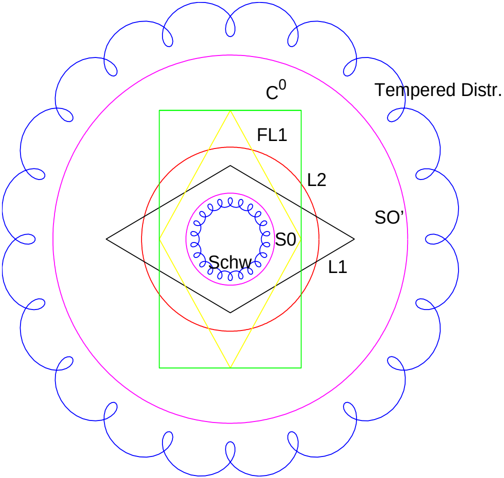
- *and as a consequence for* $1 \leq p, q \leq 2$*:*

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)) \subseteq \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'}).$$

The Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$

Within the family of Banach spaces of (tempered) distributions of the form $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$ we have natural inclusions. The smallest in this family is the space $M_0^{1,1}(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$, which is a *Segal algebra* and the smallest non-trivial Banach space isometrically invariant under time-frequency shifts.
It is Fourier invariant, as well as all the spaces $\boldsymbol{M}^p := \boldsymbol{M}^{p,p}$, with $1 \leq q \leq \infty$. This last mentioned space $\boldsymbol{M}^\infty(\mathbb{R}^d)$ coincides with $\boldsymbol{S}_0'(\mathbb{R}^d)$, the dual of $\boldsymbol{S}_0(\mathbb{R}^d)$, and is the largest TF-invariant Banach space.
In the middle we have the space $\boldsymbol{M}^2 := \boldsymbol{M}^{2,2} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2)$.
Together the triple of space $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ forms a so-called **Banach Gelfand Triple** which is highly useful for many applications (especially TF-analysis).

My favorite Function Space plot



Usefulness of $S_0(\mathbb{R}^d)$ in Fourier Analysis

Most consequences result form the following inclusion relations:

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d);$$

$$\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subseteq \boldsymbol{S}_0(\mathbb{R}^d);$$

$$(\boldsymbol{S}_0'(\mathbb{R}^d) * \boldsymbol{S}_0(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d);$$

$$(\boldsymbol{S}_0'(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d);$$

1. $\boldsymbol{S}_0(\mathbb{R}^d)$ is a valid domain of Poisson's formula;
2. all the classical Fourier summability kernels are in $\boldsymbol{S}_0(\mathbb{R}^d)$;
3. modelling of stationary stochastic processes;
4. the elements $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ are the natural building blocks for Gabor expansions;

Irregular sampling often takes place over point sets which have, at a rough scale, a kind of uniform density. The natural terminology for this setting turns out to be given by the following definition:

Definition

An indexed family $(y_j)_{j \in J}$ in a metric space is uniformly separated, if there is a positive $\delta > 0$ such that $d(y_j, y_{j'}) \geq \delta$ for all pairs $j \neq j'$. A family of points $(x_i)_{i \in I}$ in $\mathbb{R}^d$ is called relatively separated if it is the finite union of uniformly separated sets.

Sampling and Wiener amalgam spaces, II

The typical function spaces which allow reconstruction from sufficiently dense samples can be well described by means of Wiener amalgam spaces, typically of the form $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$.

The classical *Shannon Sampling Theorem* tells us that the reconstruction of a band-limited function $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ from sufficiently dense (regular) samples can be realized by a series expansion, involving shifted copies of a template function g (which satisfies $\hat{g}(s) \equiv 1$ on Ω, the spectral support of f.

$$f = \sum_{\lambda \in \Lambda} f(\lambda) T_\lambda g. \qquad (3)$$

Sampling and Wiener amalgam spaces, III

A crucial argument in the proof results about irregular sampling, i.e. about the "complete reconstruction" of smooth (e.g. band-limited) functions or functions in spline-type spaces (e.g. cubic spline functions in $\boldsymbol{L}^p$) involves a suitable characterization sampling sets.

1. On the space of band-limited functions (for Ω compact)
$$B_\Omega^p := \{f \in \boldsymbol{L}^p(\mathbb{R}^d) \mid \operatorname{supp}(\hat{f}) \subseteq \Omega\}$$
the $\boldsymbol{L}^p$-norm and the $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$-norm are equivalent;
2. Sampling such a function at a lattice $\Lambda = \boldsymbol{A}(\mathbb{Z}^d)$ results in a measure in $\boldsymbol{W}(\boldsymbol{M}, \ell^p)$, because $\sum_{\lambda\in\Lambda} \delta_\lambda \in \boldsymbol{W}(\boldsymbol{M}, \ell^\infty)$, hence
$$\sum_{\lambda\in\Lambda} f(\lambda)\delta_\lambda = f \cdot \sum_{\lambda\in\Lambda} \delta_\lambda \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^p) \cdot \boldsymbol{W}(\boldsymbol{M}, \ell^\infty) \subseteq \boldsymbol{W}(\boldsymbol{M}, \ell^p).$$
3. The Shannon-type series expansion (3) is convergent in $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$, in particular in $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ and uniformly.

Sampling and Wiener amalgam spaces, IV

Theorem

Let $X = (x_i)_{i \in I}$ *be relatively separated set,* $\mu_X = \sum \delta_{x_i}$*:*

1. $\mathbf{c} = (c_i) \in \ell^p(I)$ *iff* $\sum_{i \in I} c_i \delta_{x_i} \in \boldsymbol{W}(\mathbf{M}, \ell^p)$ *and* $\|\sum_{i \in I} c_i \delta_{x_i}\|_{\boldsymbol{W}(\mathbf{M}, \ell^p)} \leq \|\mathbf{c}\|_{\ell^p} \cdot \|\mu\|_{\boldsymbol{W}(\mathbf{M}, \ell^p)}$.

Sampling and Wiener amalgam spaces, IV

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2. *For* $g \in \boldsymbol{W}(\mathbf{C}^0, \ell^1)$ *and* $\mathbf{c} = (c_i) \in \ell^p(I)$, $\sum_{i \in I} c_i T_{x_i} g =$

$$\left(\sum_{i \in I} c_i \delta_{x_i}\right) * g \in \boldsymbol{W}(\mathbf{M}, \ell^p) * \boldsymbol{W}(\mathbf{C}, \ell^1) \subseteq \boldsymbol{W}(\mathbf{C}, \ell^p) .$$

Sampling and Wiener amalgam spaces, IV

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$$\left(\sum_{i \in I} c_i \delta_{x_i} \right) * g \in \boldsymbol{W}(\mathbf{M}, \ell^p) * \boldsymbol{W}(\mathbf{C}, \ell^1) \subseteq \boldsymbol{W}(\mathbf{C}, \ell^p) .$$
3. $h \in \boldsymbol{W}(\mathbf{C}, \ell^p)$ *implies*
$$h \cdot \mu = \sum_{i \in I} f(x_i) \delta_{x_i} \in \boldsymbol{W}(\mathbf{C}, \ell^p) \cdot \boldsymbol{W}(\mathbf{M}, \ell^\infty) \subseteq \boldsymbol{W}(M, \ell^p) .$$

Sampling and Wiener amalgam spaces, IV

The important advantage of methods which rely on Wiener amalgam space methods compared to the classical approach is the fact that estimates do *not rely anymore on variants of Poisson's formula* but on convolution relations for Wiener amalgam spaces going back to [3].
The general setup of Wiener amalgam spaces provides the advantage that the constants involved do *not depend on the choice of the concrete local or global components*. They are rather independent of such choices and thus corresponding results are *valid for families of spaces*. This is important in the following sense, which I will describe by a story related to irregular sampling.

Sampling results for families of spaces

It is clear that sampling results which are only valid for the Hilbert space $\boldsymbol{B}_\Omega^2$ are much less relevant for applications than similar results which are valid for a range for spaces, say $\boldsymbol{B}_\Omega^p$.

Theorem

Given a compact subset $\Omega \subset \widehat{\mathbb{R}^d}$ *there exists some* $\delta > 0$ *such that for every* $\delta -$ *dense family* $(x_i)_{i \in I}$ *in* $\mathbb{R}^d$ *the following is true: There exists an iterative algorithm, mapping functions from* $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$ *into itself for any* $p \in [1, \infty)$*, with the property, that the only input required is information about the set* Ω *(in fact its diameter) and the sampling values of* $(f(x_i))_{i \in I}$*, which is convergent at a geometric rate (depending on the density of the sampling set), uniformly over the full range of parameters p, providing convergence to f in the* $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$*-norm, for any* Ω*-bandlimited function in* $\boldsymbol{L}^p(\mathbb{R}^d)$*.*

Crucial Ingredients

The essential ingredients are the following facts

1. Every band-limited function $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ (for any $p \in [1, \infty]$) satisfies $f = f * g$ if $g \in \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)$ with $\hat{g}(\omega) = 1$ for $\omega \in \Omega$;
2. Given a $\delta-$dense family $(x_i)_{i \in I}$ one can form a partition of unity $\Phi = (\phi_i)_{i \in I}$ with $\mathrm{supp}(\phi_i)$ in $B_{\delta'}(x_i), i \in I$.
3. $\mathrm{Sp}_\Phi f := \sum_{i \in I} f(x_i)\phi_i$ is close to f in $\boldsymbol{W}(\boldsymbol{C}_0, \ell^p)$-sense:

$$|(\mathrm{Sp}_\Phi f - f)(x)| \leq osc_\delta f(x), \; x \in \mathbb{R}^d,$$

with $osc_\delta f(x) = \max_{|u| \leq \delta} |f(x) - f(x+u)|$.

4. Hence $\|f - \mathrm{Sp}_\Phi f * g\|_p \leq \varepsilon > 0$.

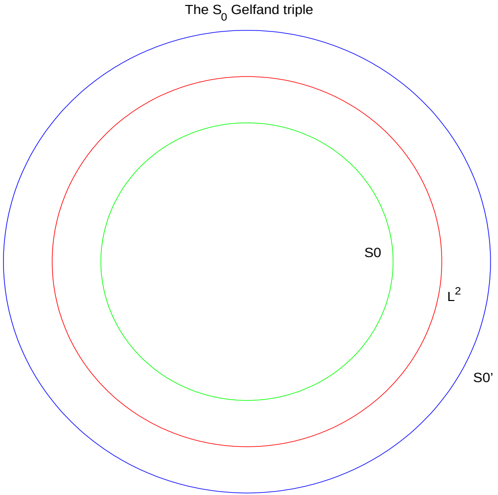
The Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$

The Segal algebra $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has been introduced as the Wiener amalgam space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$. It has many interesting properties, for example the invariance under the Fourier transform. Its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is of course $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty)$ (space of translation bounded quasi-measures, which are locally pseudo-measures).

Together with the Hilbert space $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^2, \ell^2) = \boldsymbol{W}(\boldsymbol{L}^2, \ell^2)$ they form the so-called *Banach Gelfand triple*, which allows to describe many operators (e.g. the Fourier transform, the kernel-theorem for operators, the mapping between the kernel and the spreading function of an operator, etc.) in a much better way than traditional function spaces (cf. [1]).

The Banach Gelfand Triple (S_0, L^2, S_0')

The "rigged Hilbert space" $\boldsymbol{L}^2(G)$



The Banach Gelfand Triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$

This triple shares many properties with the Schwartz Gelfand triple $(\mathcal{S}, \mathbf{L}^2, \mathcal{S}')(\mathbb{R}^d)$, altough in fact it is easier to use and also well defined over general LCA groups.
For example, one has - similar to the setting of $\mathcal{S}(\mathbb{R}^d)$ - the following way of regularizing (tempered) distributions, now elements of $\mathbf{S}_0'(\mathbb{R}^d)$ by convolution combined with pointwise multiplication of test functions.

$$\mathbf{S}_0 * (\mathbf{S}_0 \cdot \mathbf{S}_0') \subset \mathbf{S}_0, \quad \text{and} \quad \mathbf{S}_0 \cdot (\mathbf{S}_0 * \mathbf{S}_0') \subset \mathbf{S}_0. \tag{4}$$

These relations can be verified as follows:
$\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) \cdot (\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) * \mathbf{W}(\mathcal{F}\mathbf{L}^\infty, \ell^\infty)) \subseteq$
$\mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1) \cdot \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^\infty) \subseteq \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)$.

References and further resources

At the end we provide a few links to "not so easy to find literature" on the subject.

Basic source

`www.nuhag.eu/bibtex`

A good introduction is provided by [8] and the classical paper [3]. The continuous description of Wiener amalgam spaces (at that time Wiener-type spaces) is given in [2].

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