NuHAG Numerical Harmonic Analysis Group

A Fresh Look at HARMONIC ANALYSIS

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The DFT matrix of size 17, using unit roots of order 17

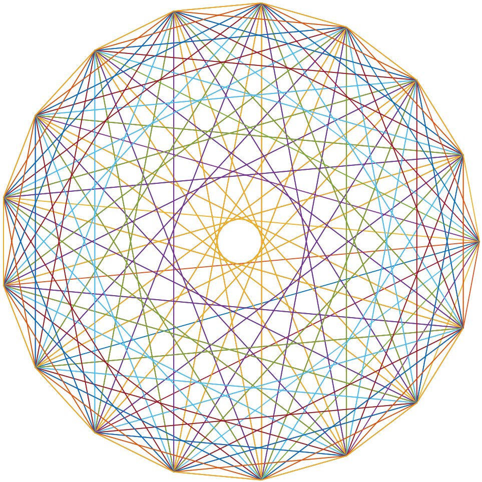


Figure: Obtained by the MATLAB command: `plot(fft(eye(17)))`

Goals of the Course

1. overview over aspects of time-frequency analysis
2. the role function spaces, in particular $S_0(\mathbb{R}^d)$, $S_0'(\mathbb{R}^d)$
3. Classical Fourier analysis and distribution theory
4. The Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$
5. various typical Applications (TF-Analysis, classical)
6. Fourier transform, sampling, linear systems
7. the idea of *Conceptual Harmonic Analysis*
8. Gabor analysis, and the kernel theorem
9. numerical realization and simulation
10. survey of aspects of modern Harmonic Analysis

Historical Background

- Fourier Analysis has a 200 year history (J.B.Fourier, 1822)
- Since H. Lebesgue it can be seen as an integral transform with domain $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$; inversion using summability methods
- by *Plancherel's Theorem* $\mathcal{F}$ is unitary on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$
- it converts convolution into pointwise multiplication

Most of the relevant background information has been given in the preperatory courses at the beginning of this course, making use of the theory of Lebesgue integrals, using several times the Dominated Convergence Principle.

The objects of Fourier Analysis

- first of all the $\mathcal{F}$ is defined for **functions**
- both non-periodic (decaying) and periodic ones
- but it can also be defined for discrete functions resp. linear combinations of Dirac measures
- finally for **distributions**, i.e. *generalized functions*
- we will discuss the connection between these viewpoints
- as well as mutual approximation of one by the other

The analogy to the number system

A good model of how we want to get a holistic picture of *Fourier Analysis* is the *number system*. Let us recall that we have the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$ and the *complex numbers* $\mathbb{C}$. They are quite *different* in appearance but still have a quite common structure, also embedded on into the other, via the natural chain

$$\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}. \tag{1}$$

It is more correct to say, that the conversion of a rational number p/q into an infinite decimal expression allows to identify a (dense) subset of the real numbers with the rational numbers (exactly the periodic ones).

The trick with the complex numbers

Instead of just "adding" the **mysterious** *complex unit* **i** or **j** by the definition `i = sqrt(-1)` (which works well in MATLAB!) mathematicians have a clear-cut way of defining the complex numbers as pairs of real numbers $z = (a, b), a, b \in \mathbb{R}$ (the more traditional way is to simply write $z = a + i * b$.
Clearly a real number $r \in \mathbb{R}$ corresponds in to the complex number $(r, 0)$ resp. $r = r + 0 * i$ to fulfill the formal requirements.
The embedding is not only "natural", all the operations, i.e. addition and multiplication, as well as taking the group inverse, i.e. the negative $-a$ or the reciprocal $1/a$ (for $a \neq 0$) can be taken in either context before or after applying the embedding.
We have no problem to accept the validity of these formulas:

$$\sqrt{2}^2 = 2; \; e^{2\pi i} = 1, e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}; s = \pi^2, \; \pi * 1/s = 1/\pi,$$

but how would we actually "compute" all these numbers?

Connection to MATLAB I

During the course I will often refer to certain MATLAB experiments which are supposed to support presented sometimes in a relatively abstract manner, and on the other hand in order to *encourage* numerical experiments in order to provide a better understanding to those who are willing to carry out a bit of programming work.

We will not need any specific toolbox, but there is a large collection of MATLAB M-files provided by the NuHAG web-site for this interested in downloading these tools.

For those who do not want to get involved in the programming using MATLAB (or any other similar language) still GEOGEBRA might be an option to consider (free download). Also occasionally I will make use if this program for demonstration purposes.

Finally, otherwise, I hope that all of you *enjoy* looking at the output (figure, plots) obtained via MATLAB.

Connection to MATLAB II

Important message relevant for the spirit of the course:

Since the *natural setting for Fourier or time-frequency analysis (including Gabor Analysis)* is the world of LCA (locally compact Abelian) groups $\mathcal{G}$ the special case of $\mathcal{G} = \mathbb{Z}_N$ (the cyclic group of order N, resp. the group of unit roots of order N within $\mathbb{C}$) is within the scope of the general theory.

So most of the experiments will compute numbers or functions on such a group which are "the analogue" of something we might be interested in the continuous setting, so over $\mathbb{R}$, providing only a numerical method providing!

Operator Notation I

Since many of the participants of this course might have an engineering background, some others a more mathematical education, let us explain some possible difference between the way how this course describes things and the usual way in those communities.

We talk about *functions* f, g, h on some group $\mathcal{G}$. Typically engineers and physicists would write $f(t), g(\mathbf{x}), h(\omega)$ etc., indicating that this is a (perhaps continuous) function of "time" ($t \in \mathcal{G} = \mathbb{R}$) or "location" $\mathbf{x} \in \mathcal{G} = \mathbb{R}^2$, or frequency $\omega \in \mathbb{R}$, with continuous variables. Alternatively the discrete setting, e.g. $\mathcal{G} = \mathbb{Z}_N$ or $\mathcal{G} = \mathbb{Z}$ covers the case of "discrete variables" ($N \in \mathbb{N}$ or $N = \infty$, the non-compact case).

Ex: Check group properties of $\mathbb{Z}_N$.

The details of such an identification will be given *over and over* again during this course!

Operator Notation II

We will consider a large variety of operators, i.e. linear mappings. Typically these operators (whenever they are abstract objects) will be defined on vector spaces of signals. We will usually clearly define the domain the target space of such operators, even if there are many possible choices. So a *linear operator* $T : \mathbf{V}_1 \to \mathbf{V}_2$ satisfies (by definition of linearity)

$$T(\lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{w}) = \lambda_1 T(\mathbf{v}_1) + \lambda_2 T(\mathbf{w}) \tag{2}$$

where $\mathbf{v}, \mathbf{w}$ are general elements in $\mathbf{V}_1$ and λ_1, λ_2 belong to the underlying field (e.g. $\mathbb{R}$ or $\mathbb{C}$). This is (by a simple induction argument) equivalent to the *preservation of linear combinations*.

$$T\left(\sum_{k=1}^{K} \lambda_k \mathbf{v}_k\right) = \sum_{k=1}^{K} \lambda_k T(\mathbf{v}_k). \tag{3}$$

Operator Notation III

Given any operator T the input is a function with *name* f, g, h and not a function $f(t), g(\mathbf{x})$ etc.. and of course, any operator mapping functions to functions can be described as $f \mapsto Tf$ and we may define it pointwise by e.g.

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y)f(y)dy, \quad x, y \in \mathbb{R}^d$$

for the case of an integral operator. Still Tf or $T(f)$ is the result of applying the operator T to the function f.

In this way we avoid confusion with terms such as Tf(t), which for US will always mean (Tf)(t) in contrast to other traditions which read these symbols as T[f(t)]. But the output may have the same argument t or another variable name.

Concrete operators I

The most important operators, first mostly for functions on $\mathbb{R}$ or $\mathbb{R}^d$ are *translation, modulation and dilation operators.*

The *translation operator* moves the graph of a function from one position to another position. It preserves the values, but moves them to other positions.
The *modulation operators* multiply a given function by some *pure frequency*. As we will see it corresponds in each case to a translation on the Fourier transform side, therefore it is also often called a *frequency shift* operator.
The *dilation operators* are special cases of linear transformations of the argument of the function, here by *rescaling*.
Later one we will consider the *Fourier transform, convolution operators* and so on.

Concrete Operators II

We will first make use of the following operators. Note that some of them can be defined for general LCA groups (translation, modulation) while others are specifically tuned for $\mathbb{R}^d$ (dilation etc.). For simplicity let us first work on $\mathbb{R}^d$ (e.g. $d = 1$).

$$T_x f(z) = f(z - x), \quad x, z \in \mathbb{R}^d \tag{4}$$

$$M_s f(z) = e^{2\pi i s \cdot z} f(z), \quad x, s \in \mathbb{R}^d. \tag{5}$$

Here $s \cdot z = \langle x, s \rangle_{\mathbb{R}^d} = \sum_{k=1}^d x_k s_k$ is the scalar product in $\mathbb{R}^d$.

$$D_\rho f(z) = f(\rho z), \quad \rho \neq 0, z \in \mathbb{R}^d \tag{6}$$

denotes the (value preserving) *dilation operator*.

Concrete Operators III

It is a good first exercise to verify that each of these families of operators, namely

$$(T_x)_{x \in \mathbb{R}^d}, \quad (M_s)_{s \in \mathbb{R}^d}, \quad (D_\rho)_{rho \neq 0}$$

for *commutative groups* of operators, with

$$Id = T_0 = M_0 = D_1.$$

and

Concrete Operators IV

Let us look at these three families of operators, given by

$$(T_x)_{x\in\mathbb{R}^d}, \quad (M_s)_{s\in\mathbb{R}^d}, \quad (D_\rho)_{\rho\neq 0}$$

forming **commutative groups** of operators, with

$$Id = T_0 = M_0 = D_1, \quad T_x^{-1} = T_{-x}, \quad M_s^{-1} = M_{-s}, \quad D_\rho^{-1} = D_{1/\rho}.$$

The commutativity follows from these composition rules:

$$T_{x_1} \circ T_{x_2} = T_{x_1+x_2};\ M_{s_1} \circ M_{s_2} = M_{s_1+s_2};\ D_{\rho 1} \circ D_{\rho 2} = D_{\rho 1 \cdot \rho 2}.$$

One can also restrict attention to $\rho > 0$.

Concrete Operators V

The reason for putting a negative value into the definition of the shift-operator is of course the fact that T_4 should describe a shift by 4 units in the *positive* direction.
Let us recall, that for a continuous function f the **support** is defined as $\mathrm{supp}(f) := \{x \mid f(x) \neq 0\}^-$, the closure of the set of "interesting points" for f.

M_s does not change the support, but the effect of D_ρ is

$$\mathrm{supp}(D_\rho f) = 1/\rho \cdot \mathrm{supp}(f).$$

This is one of the reasons of introducing another dilation operator named St_ρ which we call the *area preserving* dilation operator

$$\mathrm{St}_\rho f(x) = \rho^{-d} f(x/\rho), \quad x \in \mathbb{R}^d, \rho > 0 \tag{7}$$

$$\text{with} \quad \int_{\mathbb{R}^d} \mathrm{St}_\rho f(y) dy = \int_{\mathbb{R}^d} f(x) dx. \tag{8}$$

Goals for the course: Inexact Formulas

Among the most important goals of this course is a better understanding of expressions as they are commonly used in engineering books or physics articles. All to often **mathematicians** consider expression and explanations in such a context at least as *"morally incorrect"* and at least questionable. On the other hand **applied scientist** tend to either ignore the inaccuracies which are part of such explanations (e.g. at the level of well-deformedness) and either try to use *wordy explanations*, claiming quite often that the pedantic view-point taken by mathematicians is not so relevant for applications and that the use of these "inaccurate formulas" has *made practical life so much easier* than going through *"all the technical details"* (!as mathematicians would require to do!) that it is just a matter of convenience to use such things.

A better interpretation of inaccurate formulas

At best, applied scientist claim that mathematicians would know how to fix the claims which - when taken literally - might be considered as mathematically incorrect. So they *refer to the pedantic community of mathematicians* who would know how to turn the *intuitive and plausible* statements made in engineering books into mathematically correct (but cumbersome and complicated) claims.
It is the **purpose of this course to encourage the participants** to not give up on either an intuitive approach to the subject or alternatively have correct mathematical statements, but rather to **combine both view by means of a "fresh look" on Harmonic Analysis**, with some aspects of *functional analysis* (theory of function spaces, approximation, dual spaces).

Examples of "incorrect" statements

Sifting property of the Delta Dirac

$$\psi(x) = \int_{-\infty}^{\infty} \delta(x-y)\psi(y)dy$$

or the integration of the pure frequencies adding up to a Dirac:

$$\int_{-\infty}^{\infty} e^{2\pi isx} ds = \delta(x)$$

One can use a combination of both statements in order to derive a "highly formal" version of the Fourier inversion theorem.

Turning inaccurate formula into correct statements

In the setting of *tempered distributions* one can rewrite the first equation as

$$\psi = \psi * \delta$$

resp.

$$\mathcal{F}^{-1}(\mathbf{1}) = \delta,$$

or equivalently giving a "meaning" to the formula (see WIKIPEDIA)

$$\int_{-\infty}^{\infty} 1 \cdot e^{2\pi i x \xi} d\xi = \delta(x). \quad (9)$$

Strange formulas in WIKIPEDIA (2018)

WIKIPEDIA contains (p.4 on the **Dirac Delta function**)

$$\int_{-\infty}^{\infty} \delta(\xi - x)\delta(x - \eta)dx = \delta(\xi - \eta). \tag{10}$$

This is pretty confusing (to a mathematician). You have to first multiply one delta-function with another (is this possible?) an then even integrate out, with a result which is not a number but another Dirac function.

For us the "underlying" statement will become

$$\delta_0 * \delta_\eta = \delta_\eta$$

which is just a simple special case of the general rule

$$\delta_x * \delta_y = \delta_{x+y} = \delta_y * \delta_x, \quad x, y \in \mathbb{R}^d;$$

It can be seen as a special case of convolution of two measures.

The first elements of the Fourier Landscape

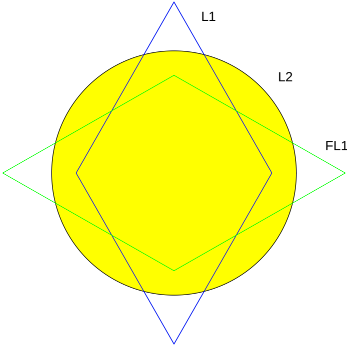


Figure: The Lebesgue spaces $\mathbf{L}^1(\mathbb{R}^d)$ and $\mathbf{L}^2(\mathbb{R}^d)$, as well as the Fourier image of $\mathbf{L}^1(\mathbb{R}^d)$, which we call the Fourier algebra $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$

Adding Schwartz to the Fourier Landscape

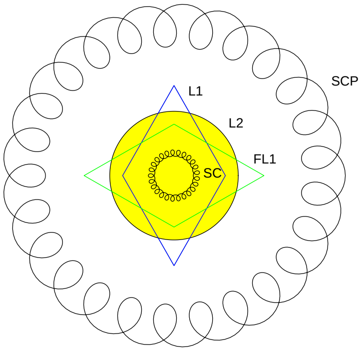


Figure: Adding the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ inside all the spaces $\boldsymbol{L}^p(\mathbb{R}^d)$, with $1 \leq p \leq \infty$ as well as the dual space, the space $\mathcal{S}'(\mathbb{R}^d)$ of *tempered distributions*.

The Riemann-Lebesgue Lemma

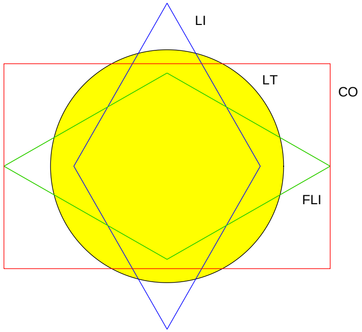


Figure: Observe: There are $\boldsymbol{L}^1(\mathbb{R}^d)$-functions which are not in $\boldsymbol{L}^2(\mathbb{R}^d)$ and vice versa, but $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d) \subset \boldsymbol{L}^2(\mathbb{R}^d)$! Obviously $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is a proper subset of $\boldsymbol{C}_0(\mathbb{R}^d)$, and so on ...

Adding the Wiener Algebra $W(C_0, \ell^1)(\mathbb{R}^d)$

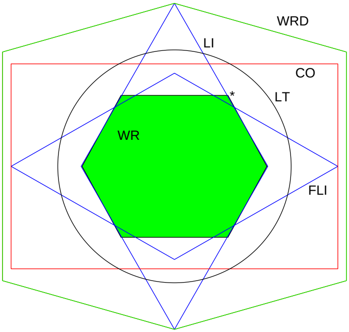


Figure: Wiener's algebra WR $:= \boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ is contained in $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$, while its dual space WRD contains all the spaces $\boldsymbol{L}^p(\mathbb{R}^d)$. It is NOT contained in the Fourier algebra! (see *)

Long-term goal: Adding *SORd* and $S_0'(\mathbb{R}^d)$

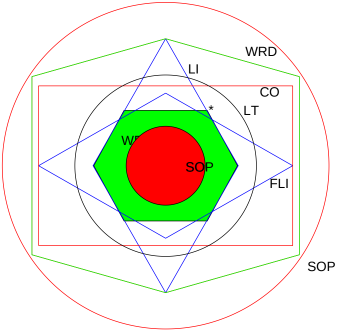


Figure: The classical function spaces, adding Wiener's algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ and is dual, but also $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$.

$\boldsymbol{L}^1(\mathbb{R}^d)$ and the Fourier Algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$

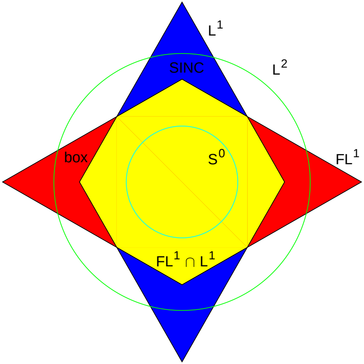


Figure: $\boldsymbol{L}^1(\mathbb{R}^d), \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ and their intersection: The domain of the Fourier inversion theorem is the yellow domain, strictly inside of $\boldsymbol{L}^2(\mathbb{R}^d)$.

Fourier transform for $W(L^1, \ell^2)(\mathbb{R}^d)$

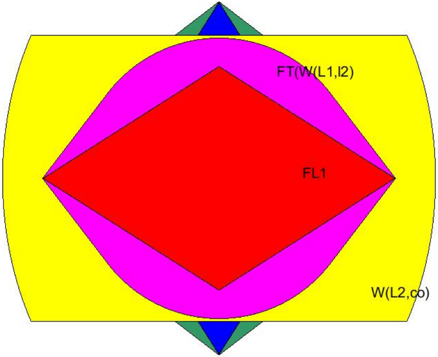


Figure: $\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^1, \ell^2))(\mathbb{R}^d) \subset \boldsymbol{W}(\boldsymbol{L}^2, \boldsymbol{c}_0)(\mathbb{R}^d)$

The philosophy behind these pictograms

Using these *pictograms* should encourage to speculate about properties of these spaces and their mutual relationships, such as

1. (proper) containment, including intersections;
2. Fourier invariance (rotation by 90 degrees!)
3. invariance under *fractional Fourier transforms* corresponding to arbitrary rotations.
   This property is only valid for $\boldsymbol{L}^2(\mathbb{R}^d)$, $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$ (and of course $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$)!)

Banach algebras considered so far

In the introductory part we have seen that the usual approach to Fourier analysis, making use of the Lebesgue space $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ of Lebesgue-integrable functions allows as to turn this Banach space into a Banach algebra with respect to convolution and define the Fourier transform.
As a combination of the *Riemann-Lebesgue Lemma* and the *Convolution Theorem* we can describe the situation as follows:

Theorem

The Fourier transform defines an injective Banach algebra homomorphism from $(\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1)$ *into* $(\boldsymbol{C}_0(\mathbb{R}^d), \cdot, \|\cdot\|_\infty)$, *which also is compatible with respect to the involutions* $f \mapsto f^*$, *with* $f^*(t) = \overline{f(-x)}$, *which corresponds to the involution* $h \mapsto \overline{h}$ *on the Fourier transform side.*

Families of automorphisms of these Banach algebras

Theorem

1. *The family* $(\mathrm{St}_\rho)_{\rho\neq 0}$ *is a commutative group of isometric automorphisms of* $\big(\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1\big)$.
2. *The family* $(\mathrm{D}_\rho)_{\rho\neq 0}$ *is a commutative group of isometric automorphisms of* $\big(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \cdot\,, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\big)$, *but also of* $\big(\boldsymbol{C}_0(\mathbb{R}^d), \cdot, \|\cdot\|_\infty\big)$.
3. *The family* $(M_s)_{s\in\mathbb{R}^d}$ *is a commutative group of isometric automorphisms of* $\big(\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1\big)$.

There are also other (non-commutative) groups of automorphism compatible with convolution, namely automorphism of the underlying group $\mathbb{R}^d$, i.e. linear mappings $\mathbf{x} \to \boldsymbol{A} * \mathbf{x}$ for some non-singular $d \times d$-matrix $\boldsymbol{A}$, especially rotations via

$$D_{\boldsymbol{A}}(f)(z) = f(\boldsymbol{A}^{-1} * \mathbf{x})$$

Bounded approximate units via dilation/scaling

Theorem

1. *For every function* $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ *with* $\int_{\mathbb{R}^d} g(x)dx = 1$ *the family* $(\mathrm{St}_\rho g)_{\rho \to 0}$ *defines a bounded approximate unit for the Banach algebra* $\left(\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1\right)$*;*
2. *For any function* $h \in \boldsymbol{C}_0(\mathbb{R}^d)$ *with* $h(0) = 1$ *the family* $(\mathrm{D}_\rho h)_{\rho \to 0}$ *forms a BAI for* $\left(\boldsymbol{C}_0(\mathbb{R}^d), \cdot, \|\cdot\|_\infty\right)$*;*
3. *For any function* $h \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ *with* $h(0) = 1$ *the family* $(\mathrm{D}_\rho h)_{\rho \to 0}$ *forms a BAI for* $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \cdot, \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$*;*

For $g = g_0$, the **Gauss function** given by $g_0(t) = e^{-\pi|t|^2}$ also $h = \widehat{g_0} = g_0$ (Fourier invariance!) provides a *Dirac sequence* by compression and a pointwise approximate unit for $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ and $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ by dilation.

But NOW LET us TAKE A COMPLETE RESTART

We have seen that with a good background on Lebesgue integration (and we have really used all the strong results, including the Lebesgue dominated convergence theorem and Fubini's theorem) we can define $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, the Fourier transform and convolution, up to the convolution theorem and the Riemann-Lebesgue Lemma and Plancherel theorem and the inversion theorem. We have established $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a commutative Banach algebra with bounded approximate identities. All this is based on the observation that the Fourier transform, defined as an *INTEGRAL TRANSFORM* through

$$\hat{f}(x) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i <s,t>} dt$$

has a natural domain (namely $\boldsymbol{L}^1(\mathbb{R}^d)$), etc...

But NOW LET us TAKE A COMPLETE RESTART II

One of the main arguments for the usefulness of the Fourier transform is the fact that it *converts the complicated convolution into simple pointwise multiplication*. But why should we be interested in CONVOLUTION?

If one follows the (excellent) Stanford course by Brad Osgood (who also strives for a distributional view-point, trying not to stress the audience to much with details on the Schwartz space), then convolution could be introduced by the questions: Assume you multiply two Fourier transforms: is this the Fourier transform of "something", and if so, what is it. And of course he comes up with the convolution as we know it.

I am afraid that this is not a convincing approach for non-mathematicians!

Where does convolution appear in nature?

We have studied a few simple cases in the practical part: Knowing how to multiply numbers (e.g. by looking at 111111^2) we get a first idea what convolution is.
But already kids learn how to multiply out polynomials and compute the coefficients of a product polynomial, by forming (in a concrete way) the so-called *Cauchy product*
It is possible (and in fact not difficult) to relate this multiplication of polynomials to probability in the following way: addition of *independent random variables*: we will illustrate the sume of two dices, each associated with the polynomial

$$p(x) = (x + x^2 + x^3 + x^4 + x^5 + x^6)/6$$

easily with the coefficients of $p(x)^2$!!
(Verbal explanation, illustrated by some MATLAB experiments).
Similar case: the **Binomial Theorem** (*Pascal's triangle*)!

Translations invariant systems

Translation-invariant linear systems play a great role. Courses on the subject appear in most electrical engineering curricula.

Definition

The Banach space of all "translation invariant linear systems" (TLIS) on $C_0(\mathbb{R}^d)$ is denoted by[a]

$$\mathcal{H}_{\mathbb{R}^d}(C_0(\mathbb{R}^d)) = \{T \in \mathcal{L}(C_0(\mathbb{R}^d)) \;\; T \circ T_z = T_z \circ T, \forall z \in \mathbb{R}^d\} \quad (11)$$

[a] The letter $\mathcal{H}$ in the definition refers to *homomorphism* [between normed spaces], while the subscript G in the symbol refers to "commuting with the action of the underlying group $G = \mathbb{R}^d$ realized by the so-called regular representation, i.e. via ordinary translations.

Translations invariant systems as a Banach algebra

Lemma

1. *The space $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is in fact a closed subalgebra of $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ (with operatornorm), hence endowed with the operator norm it is*
2. *$\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ is even closed with respect to the strong operator topology, i.e.if you have a sequence of operator $(T_n)_{n\geq 1}$ in $\mathcal{L}(\boldsymbol{C}_0(\mathbb{R}^d))$ with the property that*

$$lim_{n\to\infty}\|T_nf - T_0f\|_\infty = 0, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),$$

then the limiting operator also belongs to $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$.

3. *Clearly $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ contains all the translation operators $T_x, x \in \mathbb{R}^d$, and their closed linear span forms a commutative subalgebra of $(\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d)), |||\cdot|||)$.*

Convolution operators as Moving Averages

The outline of our study of TILS (translation invariant system) on $\boldsymbol{C}_0(\mathbb{R}^d)$ is roughly the following:

1. Show that every translation invariant system T can be viewed as a moving average, or alternatively as a convolution operator, characterized completely by some linear functional, i.e.some $\mu \in \boldsymbol{M}(\mathbb{R}^d)$;
2. Then show how thanks to discretization operators, which are based on the existence of arbitrary fine partitions of unity, measures can be approximated by discrete measures;
3. Then show that the convolution operators based on these discrete measures, we call them $D_\Psi \mu$, are approximating the convolution operators $f \mapsto C_\mu f = \mu * f$ in the strong operator sense, i.e.they converge uniformly for any given $f \in \boldsymbol{C}_0(\mathbb{R}^d)$.

Recalling the choice of BUPUs

A very simple and broad BUPU. Shift parameter $a = 64$, a divisor of the signal length $n = 480$, with four extra convolutions with a box function of width 16. So total support size $a + 4 * 16 = 128$.

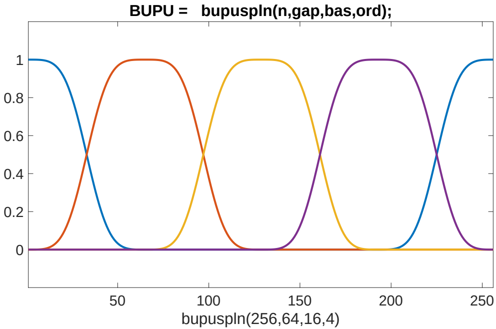


Figure: bupuspline00.eps

Background information, Spline-quasi-interpolation

We have discussed BUPUs, which are bounded uniform partitions $\Psi = \Psi = (\psi_i)_{i\in I}$ of unity, where for now boundedness refers to boundedness in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, or practically speaking we assume $0 \leq \psi_i(x) \leq 1$ for all $i \in I$.
On $\mathbb{R}^d$ the *size of a BUPU* can simply be determined as [1]

$$|\Psi| = \inf\{\gamma \mid \mathrm{supp}(\psi_i) \subset B_\gamma(x_i)\},$$

which by assumption is finite.
We then defined the two operators *SpPsi* on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and its transpose operator on D_Ψ on $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.

$$\mathrm{Sp}_\Psi(f) = \sum_{i\in I} f(x_i)\psi_i, \quad \mathrm{D}_\Psi(\mu) = \sum_{i\in I} \mu(\psi_i)\delta_{x_i}.$$

[1] Also taking a little bit the family $(x_i)_{i\in I}$ into account.

Interpretation in a classical sense

For various special choices these operators are acutally quite simple to understand. Let us restrict our attention to the case of BUPUs of triangular shape (B-spline of order 2 or degree 1).
We can take the standard triangular system (convolution square of the box-function) and its shift along $\mathbb{Z}$ and then compress this system by the D_ρ-operator, for $\rho \to \infty$, say $\rho = 2^n$.
Then the resulting operator Sp_Ψ produces out a piecewise linear interpolation of f from the samples of the form for $\alpha = 2^{-n}$.
On the other hand, just for the sake of illustration, assume you take the spline-BUPU of order one (shifted) box functions, the think of $x_i = \xi_i$ as in Riemann sums. Then $D_\Psi(f)$ can be interpreted as Riemannian sum (even irregular Riemannian sums, by using $\lambda(\mathbf{1}_{[a_i,b_i)}) = a_i - a_i$, with λ = Lebesgue measure).

w^*-convergence of $D_\Psi \mu$ to μ

We all have learned that Riemannian sums form a *Cauchy-net*, i.e. for $f \in \boldsymbol{C}([a,b])$ we know that they are convergent to $\int_a^b f(x)dx$, the so-called Riemann integral. They corresonding Cauchy-condition is of the following form: Given $f \in \boldsymbol{C}([a,b])$ and $\varepsilon > 0$ we can find some $\delta = \delta(f, \varepsilon)$ such that for all Riemannian sums which are at least as fine as δ (maximal length of intervals occuring) two Riemannian sums will not differ more than that given $\varepsilon > 0$. By completeness of $\mathbb{R}$ there is a limit: $\int_a^b f(x)dx$!
In our setting we claim

$$\text{For any } f \in \boldsymbol{C}_0(\mathbb{R}^d) \text{ we have } \lim_{|\Psi| \to 0} D_\Psi \mu(f) = \mu(f). \tag{12}$$

PROOF: $D_\Psi \mu(f) = \mu(\mathrm{Sp}_\Psi(f)) \to \mu(f)$ for $|\Psi| \to 0$.

Consequences for convolution approximation

Using this last observation it is clear that we have for every $x \in \mathbb{R}^d$, by replacing f by $T_x f^{\checkmark}$ and starting to write $\mu * f$ for the application of the convolution operator C_μ:

$$D_\Psi \mu * f(x) \to \mu * f(x), \quad \forall x \in \mathbb{R}^d.$$

But in fact the speed of convergence depends only on the expression $\| \operatorname{osc}_\delta(f)\|_\infty$ resp. here on the quantity

$$\| \operatorname{osc}_\delta(T_x f^{\checkmark})\|_\infty = \| \operatorname{osc}_\delta f\|_\infty.$$

This implies finally the required convergence in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \| \cdot \|_\infty\right)$:

$$\lim_{|\Psi|\to 0} D_\Psi \mu * f = \mu * f, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d). \tag{13}$$

Properties of $D_\Psi \mu$

Although it is clear that Sp_Ψ is not normexpanding, since obviously

$$\|\mathrm{Sp}_\Psi(f)\|_\infty \leq \|f\|_\infty, \quad \forall f \in \boldsymbol{C}_0(\mathbb{R}^d),$$

we could derive this using the (anyway useful) estimate

$$|\mu(\psi_i)| \leq \|\mu\psi\|_{\boldsymbol{M}} \,. \tag{14}$$

Proof: We just define $\psi_i^* = \sum_{j:\psi_j\cdot\psi_i\neq 0} \psi_j$ and find that $\|psi_i^* = \sum_{j\in F} \psi_j$ for some finite set, hence $\|\psi_i^*\|_\infty \leq 1$. Hence

$$|\mu(\psi_i)| = |\mu(\psi_i^* \cdot \psi_i)| \leq \|\mu\psi_i\|_{\boldsymbol{M}},$$

and in particular

$$\sum_{i\in I} |\mu(\psi_i)| \leq \|\mu\|_{\boldsymbol{M}}.$$

illustration of the functions ψ_i^*

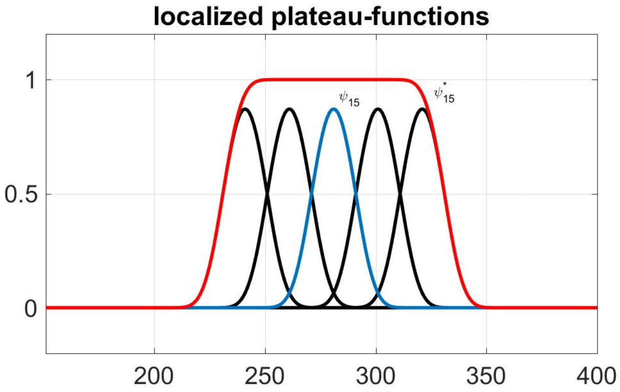


Figure: locplautpsistar.jpg

Tightness in $C_0(\mathbb{R}^d)$

The idea of *uniform concentration* of functions resp. of measures (most of the "mass" of a finite measure on $\mathbb{R}^d$ is concentrated on a bounded subdomain) will play a role in our consideration. We will use the word *tight* for this concept. Since unbounded but well-concentrated sets will not be of any relevance for our considerations we make boundedness a part of the definition.

Definition

A bounded subset M in $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ is called *tight* (or uniformly tight) if for $\varepsilon > 0$ there exists some compactly supported function $p \in \boldsymbol{C}_c(\mathbb{R}^d)$ (we think of plateau-functions) such that

$$\|f - p \cdot f\|_\infty \leq \varepsilon \quad \forall f \in M.$$

A short remark on tightness

Remark: It is a good exercise to show that any bounded approximate unit in $(\mathbf{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ qualifies in the same way to characterize tightness. The usual description could be considered as a variant of this description, with $p \in \mathbf{C}_c(\mathbb{R}^d)$ being replaced by the net of indicator functions of compact sets $K \subset \mathbb{R}^d$, ordered by size.

Tightness in $M(\mathbb{R}^d)$

In a very similar way tightness of a set S in $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is valid if

$$\|\mu - p \cdot \mu\|_{\boldsymbol{M}} \leq \varepsilon \quad \forall \mu \in S.$$

It is, for example, not difficult to show:

Lemma

For any tight set S of measures, also the set

$$\{D_\Psi \mu \,|\, |\Psi| \leq 1, \mu \in S\}$$

is also a tight subset of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.

Tightness and convolution

Often results valid about e.g. the w^*-convergence of $D_\Psi \mu$ to μ are also valid for w^*- convergent and tight nets (potentially arising in a different way than discretization), e.g.

Lemma

Assume a (bounded and) tight net $(\mu_\alpha)_{\alpha inI}$ is w^*-convergent to some $\mu_0 \in \boldsymbol{M}(\mathbb{R}^d)$. Then we also have

$$\lim_\alpha \|\mu_\alpha * f - \mu_0 * f\|_\infty = 0.$$

Introducing convolution on $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$

Having identified now on the one hand $\mathcal{H}_{\mathbb{R}^d}(\boldsymbol{C}_0(\mathbb{R}^d))$ with $\boldsymbol{M}(\mathbb{R}^d)$ (isometrically) and also realized that - as the strong closure of a commutative algebra of discrete convolution operator - we can transfer the commutitive multiplicative structure onto $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. In other words we check that the convolution can be defined reflecting the composition laws of the corresponding operators T, thus turning the Banach space $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ into a Banach algebra!

Clearly we get associativity for free (in the same we get associativity of matrix multiplication for free as soon as we have verified that matrix multiplication just corresponds to the composition of the corresponding linear mappings). We also can prove (using natural arguments) that

$$\lim_{|\Psi|\to 0} \mathrm{D}_\Psi \mu_1 * \mathrm{D}_\Psi \mu_2 * f \to \mu_1 * \mu_2 * f.$$

Consistency considerations

Within the "convolution" that we obtain by transfer of structure we can now check what the concrete action of a given measure on function is, resp. on measures.
Important starting point:

$$\delta_x * f = T_x f, \quad f \in \boldsymbol{C_0}(\mathbb{R}^d), x \in \mathbb{R}^d.$$

Just look at

$$\delta_x * f(z) = \delta_x(T_z f^{\checkmark}) = \delta_x([T_{-z} f]^{\checkmark})) = [T_{-z} f](-x)$$

$$= f(-x - (-z)) = f(z - x) = T_x f(z)$$

Since $T_x \circ T_y = T_y \circ T_x$ we have

$$\delta_x * \delta_y = \delta_{x+y}, \quad x, y \in \mathbb{R}^d.$$

Comparing our approach with $L^1(\mathbb{R}^d)$-theory

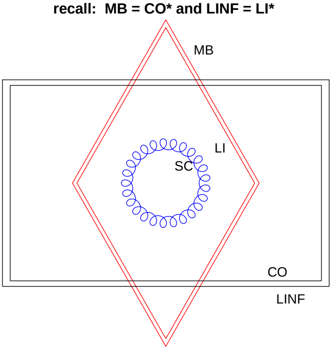


Figure: LILINFCOMB.eps

Comparing the situation

So far we have $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ and its dual $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.
We have also seen that w^*-convergence of measures (elements) of the dual space is relevant, because the discrete measures form a proper, closed subspace of $\boldsymbol{M}_d(\mathbb{R}^d)$.
There are different ways of characterizing $\boldsymbol{L}^1(\mathbb{R}^d)$ within $\boldsymbol{M}(\mathbb{R}^d)$, mostly (measure theoretic) as the "absolutely continuous" measures, alternatively via $\|T_x\mu - \mu\|_{\boldsymbol{M}(\mathbb{R}^d)} \to 0$ for $x \to 0$.
This viewpoint will help us to understand $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a closed ideal within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$.
We will have of course a dual of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$.
The embedding $k \to \mu_k$ resp. the realization of $\boldsymbol{C}_b(\mathbb{R}^d)$ as a part of the dual space of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ requires the Haar measure on $\mathbb{R}^d$ (i.e. the Riemann integral, not more!).

Illustration of the D_ψ operator

Given a probability density and a relatively course BUPU we have this situation in a discrete situation. The density was created from a random lowpass signal, be raising the real part and then normalizing the sum of these non-negative values to 1.

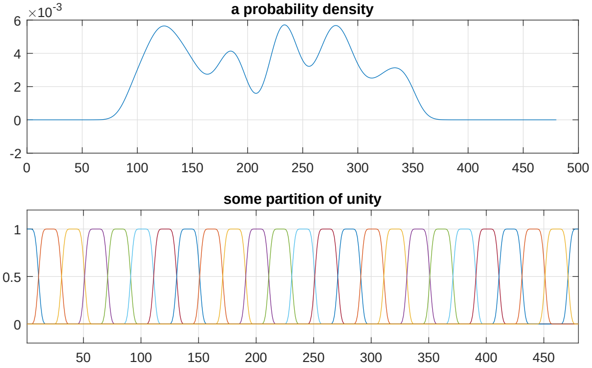


Figure: probBUP1.eps

The corresponding distribution functions

The corresponding distribution functions then look like this. The jumps (Dirac measures) arise here at regular sampling positions, coordinates 1 : 20 : 480 , so the BUPU has 24 entries.

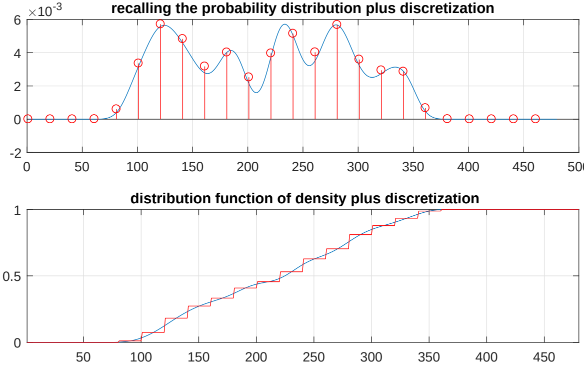


Figure: probBUP2.eps

Concerning the inequivalence of sup-norm and $\mathcal{F}L^1$-norm

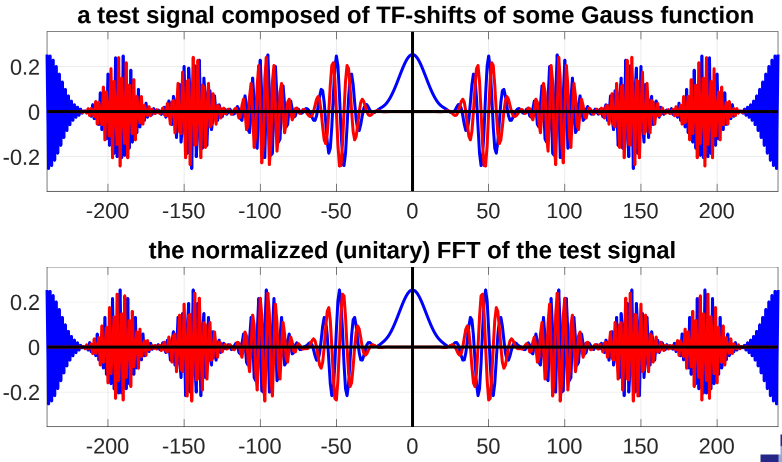


Figure: inequavCOFLI.eps

L^1 norm grows with the number of terms, $\|\hat{f}\|$ is stable

'Poor approximation by simple quasi-interpolator'

The next picture shows some smooth (complex-valued) and band-limited signal (max. frequency 15, hence 31 random Fourier coefficients and $Sp_{\Psi}(f)$ using the above BUPU.

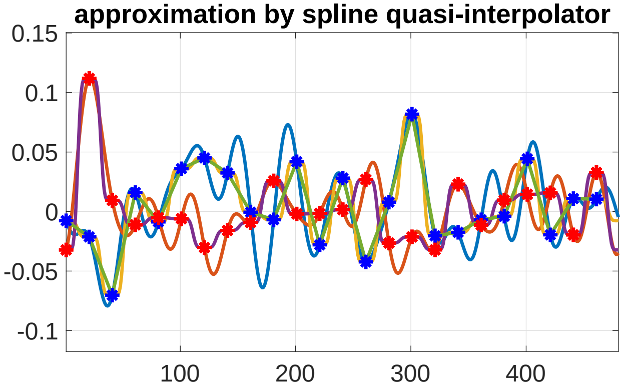


Figure: quasiapprox3.eps

The same signals, with their Fourier spectra

The left hand sides show real part of the signals, with $\mathrm{Sp}_{\Psi}(f)$ at the lower level the right hand sides their (normalized) Fourier transforms.

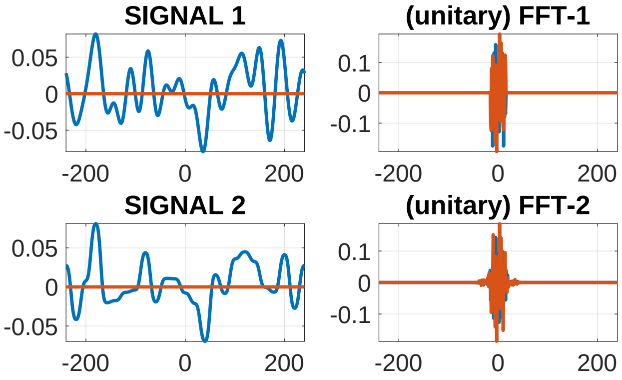


Figure: quasiapprox3b.eps

The Gaborator: Free online spectrogramm

The Gaborator is a software which came out in November 2017 and was presented at an AUDIO Conference then (see Youtube).

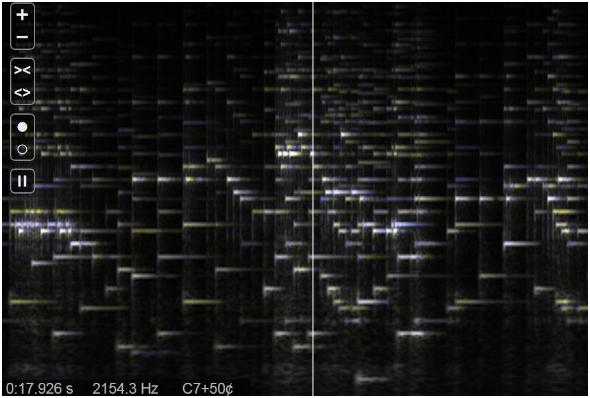


Figure: Observe the information about frequency and musical description of score in the left lower corner!

The Gaborator

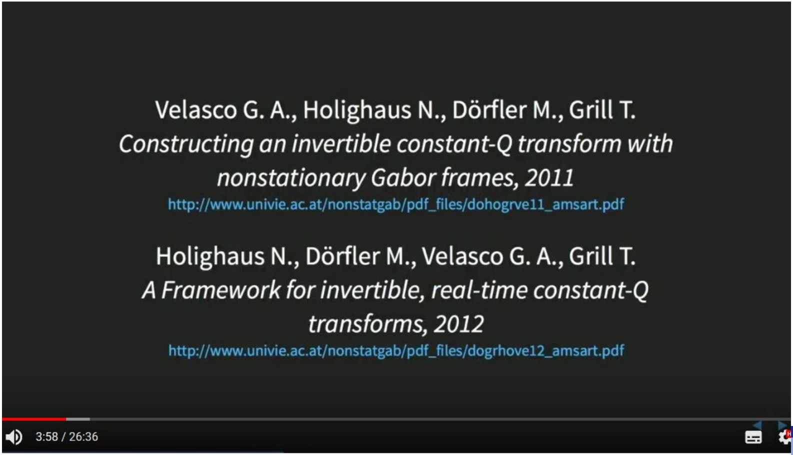


Figure: Gaborator: bibliographic hint

Vandermonde matrices and Lagrange interpolation

The **columns** of the inverse Vandermonde matrix describe just the coefficients of the Lagrange interpolating polynomial.

```
>> inv(vander(0:3))
-0.1667 0.5000 -0.5000 0.1667
1.0000 -2.5000 +2.0000 -0.5000
-1.8333 3.0000 -1.5000 0.3333
+1.0000 0.0000 +0.0000 0.0000
>> cof2 = conv(conv([1,0],[1,-2]),[1,-3])
>> cof2 = 1 -5 6 0
>> lagr2 = ans/polyval(cof2,1)
=  0.5000 -2.5000 3.0000 0.0000
```

It is also easy to illustrate it via GEOGEBRA, as a product of terms of the form $(x - x_k)$.

Some GEOGEBRA FILES prepared for the course

See (as usual) `www.nuhag.eu/chennai18` for the corresponding files (file extension is ".ggb").

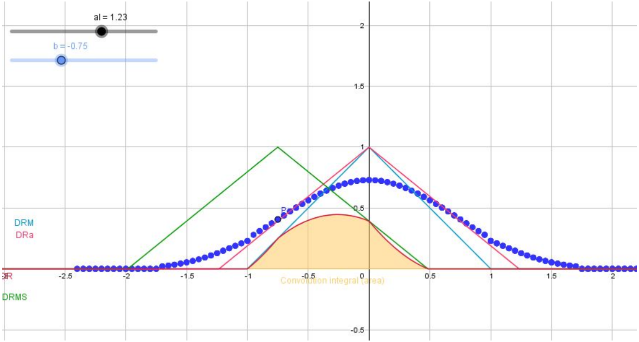


Figure: The convolution product of the standard triangular function with a dilated triangular function, computed pointwise via integration.

Some GEOGEBRA FILES: Product of dilated SINCs

The same situation on the Fourier transform side

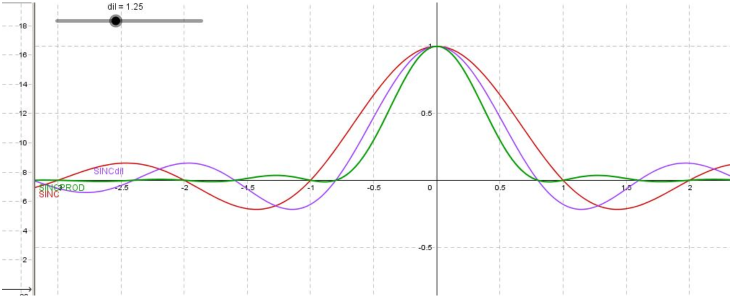


Figure: The convolution product of the standard triangular function with a dilated triangular function, computed pointwise via integration.

Banach modules already observed

In the introductory part concerning Banach spaces and Banach algebras we have already seen a couple of Banach modules. For example the family of ℓ^p-spaces of $\boldsymbol{L}^p$-spaces.

Banach modules already observed

In the introductory part concerning Banach spaces and Banach algebras we have already seen a couple of Banach modules.
For example the family of ℓ^p-spaces of $\boldsymbol{L}^p$-spaces. Cleary they are Banach modules over the Banach algebras $\left(\ell^\infty, \|\cdot\|_\infty\right)$ resp. or $\left(\boldsymbol{L}^\infty, \|\cdot\|_\infty\right)$ with respect to pointwise multiplication, which are in fact commutative C^*-algebras with unit.
In fact these properties are *equivalent* to the property of *solidity*.
For the sequence spaces $\left(\ell^p, \|\cdot\|_p\right)$, $1 \leq p \leq \infty$ this means.

$$\mathbf{a} = (a_i)_{i \in I} \in \ell^p(I), |b_i| \leq |a_i| \ \forall i \in I \Rightarrow \mathbf{b} \in \ell^p \text{ and } \|\mathbf{b}\|_p \leq \|\mathbf{a}\|_p.$$

However we find it more interesting (to understand situations arising in the context of $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ or $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$) to work with Banach algebras having only BAI, hence we view these spaces as Banach modules over $\left(\boldsymbol{c}_0, \|\cdot\|_\infty\right)$ / $\left(\boldsymbol{C}_0, \|\cdot\|_\infty\right)$.

Banach algebras of relevance for Harmonic Analysis

$(\boldsymbol{M}_b(G), *, \|\cdot\|_{\boldsymbol{M}_b}), (\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1), (\boldsymbol{C}_0(\mathbb{R}^d), \cdot, \|\cdot\|_\infty), (\boldsymbol{c}_0, \|\cdot\|_\infty)$

and of course various operator algebras, including pointwise multiplier algebras of Fourier multipliers, i.e.algebras of operators commuting with translations.

1. $(\ell^p, \|\cdot\|_p)$ is an Banach module over $(\boldsymbol{c}_0, \cdot, \|\cdot\|_\infty)$ with respect to "coordinatewise multiplication" for $1 \leq p \leq \infty$; essential for $p < \infty$ (density of finite sequences);
2. $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ is an essential Banach module over $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ with respect to pointwise multiplication;
3. $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$ is a Banach module with respect to $(\boldsymbol{M}_b(G), *, \|\cdot\|_{\boldsymbol{M}_b})$ (thanks the a suitable definition of convolution within $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$);
4. $(\boldsymbol{L}^1(\mathbb{R}^d), *, \|\cdot\|_1)$ is a closed ideal within $(\boldsymbol{M}_b(G), *, \|\cdot\|_{\boldsymbol{M}_b})$;

Ordinary functions viewed as bounded measures

It is not surprising that we can identify many (integrable, in fact) functions as bounded measures (depending on our concept of what an integral is), but certainly for $k \in \boldsymbol{C}_c(\mathbb{R}^d)$, using Riemannian integrals.

$$\mu = \mu_k, \quad \text{resp.} \quad \mu(f) = \int_{\mathbb{R}^d} f(x)k(x)dx \tag{15}$$

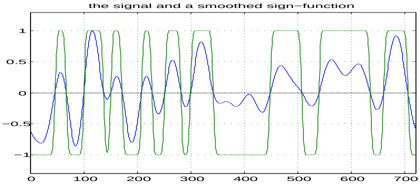
Then we have the following crucial embedding:

Lemma

The mapping $k \to \mu_k$ described above defines an isometric embedding from $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ into $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$. Hence we may identify the closure of $\boldsymbol{M}_{C_c} = \{\mu_k \mid k \in \boldsymbol{C}_c(\mathbb{R}^d)\}$ with the completion of the normed space $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$.

Visual justification for the embedding, real case

Green: continuous variant of signum function for $k \in \boldsymbol{C}_c(\mathbb{R})$.



Completion by closure within $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$

Since we can now identify the test-functions, endowed with the $\boldsymbol{L}^1$-norm (computed via Riemann integrals!) we can just close this space (actually, its isometric copy) with $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ in order to obtain one version of its completion, which we call $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$.
In the classical literature one characterizes this space via Lebesgue integrability, resp. identifies this space using measure theoretic methods as the subspace of all bounded regular Borel measures which are *absolutely continuous with respect to the Lebesgue measure* on $\mathbb{R}^d$

Alternative characterizations of $(L^1(\mathbb{R}^d), \|\cdot\|_1)$

There are some other characterizations of the closed subspace generated by $C_c(\mathbb{R}^d)$ via the *natural embedding* into $M(\mathbb{R}^d)$, which uses the following idea, comparable to the definition of $C_{ub}(\mathbb{R}^d) \subset C_b(\mathbb{R}^d)$:

Definition

$$M_{cs} := \{\mu \in M(\mathbb{R}^d) \,|\, \|T_x\mu - \mu\| \to 0 \quad \text{for} \quad x \to 0.\} \tag{16}$$

In the same way one shows easily:

Lemma

$M_{cs}(\mathbb{R}^d)$ is a closed ideal within $(M_b(\mathbb{R}^d), \|\cdot\|_{M_b})$ with respect to convolution. It is also invariant under the usual involutions (e.g. $\mu \to \mu^{\checkmark}$.

How can we show that $\left(L^1(\mathbb{R}^d), \|\cdot\|_1\right)$ equals $M_{cs}(\mathbb{R}^d)$

It is plausible that the closed subspace $M_{cs}(\mathbb{R}^d) \subset (M(\mathbb{R}^d), \|\cdot\|_M)$ can be characterized by appropriate !norm convergence (within $(M(\mathbb{R}^d), \|\cdot\|_M)$) of convolutions with a Gaussian Dirac sequence $(Strhog_0)_{\rho \to 0}$.
Secondly the convolution, now either viewed WITHIN the Banach convolution algebra $(M(\mathbb{R}^d), \|\cdot\|_M)$ or alternatively as an action of μ on some test function in $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ can actually be identified. We skip the discussion of this detail for the moment, mentioning that the adjoint action on $(M(\mathbb{R}^d), \|\cdot\|_M)$, viewed as a dual space to $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, is just internal convolution with $\mu^{\checkmark}$. This is easily verified for $\mu = \delta$, hence for $D_\Psi \mu$ and by taking limits one can have the desired result.

Although closed in $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ the space $\boldsymbol{L}^1(\mathbb{R}^d)$ is w^* dense

Having observed that $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ is a closed ideal, still containing all the compressed Gauss function and other Dirac sequences which behave well as convolution operators on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ (see Exercises, using uniform continuity) we can claim that $\boldsymbol{L}^1(\mathbb{R}^d)$ is a w^*-dense subspace of $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$. Even more concrete.

Lemma

*Let $(e_\alpha)_{\alpha\in I}$ be any bounded approximate identity in $\boldsymbol{L}^1(\mathbb{R}^d)$ for its action on $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, e.g.$(\mathrm{St}_\rho g)_{\rho\to 0}$ for some $g \in \boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{C}_0(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(t)dt = 1$.
Then $e_\alpha * \mu$ is w^*-convergent to μ, for any $\mu \in \boldsymbol{M}(\mathbb{R}^d)$.*

Approximation of measures by $L^1(\mathbb{R}^d)$-functions via smoothing

Proof.

To see that w^*-convergence happens we only have to observe that for any $f \in \boldsymbol{C}_0(\mathbb{R}^d)$ we have for $g = g^\checkmark$ (for convencience)

$$[\mathrm{St}_\rho g * \mu](f) = \mu(\mathrm{St}_\rho g * f) \to \mu(f) \text{ in } \mathbb{C}, \text{ for } \rho \to 0, \tag{17}$$

because $\|Strhog * f - f\|_\infty \to 0$ and μ is a continuous linear functional. On the other hand we can represent $\mathrm{St}_\rho g * \mu$ by the continuous, bounded (and integrable) function $z \mapsto \mu(T_x[\mathrm{St}_\rho g]^\checkmark)$ (according to above argument, now without compact support..). □

How can we show that $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ equals $\boldsymbol{M}_{cs}(\mathbb{R}^d)$ II

The key argument is then approximately the following one, at least for the dense subspace of compactly supported measures, and for non-negative functions $g \in \boldsymbol{C}_c(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(t)dt = 1$:

$$\mathrm{St}_\rho g * \mu = \mu_{\mathrm{St}_\rho g} * \mu = C_\mu(\mathrm{St}_\rho g) \in \boldsymbol{C}_c(\mathbb{R}^d)! \qquad (18)$$

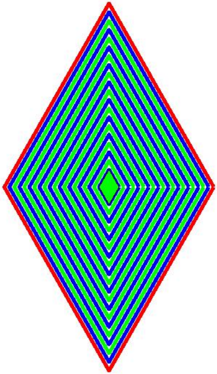
Since these terms tend to μ now in $\boldsymbol{M}(\mathbb{R}^d)$ the limit must be in the closure of $\boldsymbol{C}_c(\mathbb{R}^d) \subset \boldsymbol{M}(\mathbb{R}^d)$ (via natural embedding), and the argument is completed.
From now on we can forget about the definition and use all the properties of $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ that we know from Lebesgue integration theory.

Now we are ready for the Riemann-Lebesgue Lemma

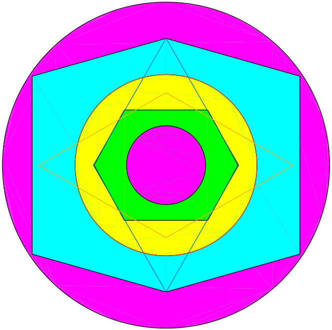
It is not difficult to show that $(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ (now defined differently) is a *continuous* function and that it belongs to $\mathbf{C}_0(\mathbb{R}^d)$. For the first claim (in fact uniform continuity is clear) we may recall that the usual Lebesgue dominated convergence can be replaced by the uniform convergence of characters on compact sets combined wit the "essentially compact" support of elements of $\mathbf{L}^1(\mathbb{R}^d) \subset \mathbf{M}(\mathbb{R}^d)$.
The decay property is proved in the same way as for $\mathbf{L}^1(\mathbb{R}^d)$ defined via Lebesgue integration.
Moreover, the (abstract) *uniqueness* (up to isometric isomorphism) of the completion of a normed space (here $\mathbf{C}_c(\mathbb{R}^d)$ with the $\mathbf{L}^1$-norm) grants that the different approaches give the same space. Obviously we also get that $(\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ **with convolution is a commutative Banach algebra with BAI with involution.**

Both $L^1(\mathbb{R}^d)$ and $M_d(\mathbb{R}^d)$ are w^*-dense in $M(\mathbb{R}^d)$

An attempt to visualize the containment of $L^1(\mathbb{R}^d)$ and $M_d(\mathbb{R}^d)$, the discrete measures. In the norm topology closed they are almost disjoint, but w^* dense in $M(\mathbb{R}^d)$.



a more complete picture



Wiener algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (green) with $\boldsymbol{S}_0$ (small circle) inside, and dual: *translation bounded measures* $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)(\mathbb{R}^d)$ inside $\boldsymbol{S}_0'(\mathbb{R}^d)$ (big circle). Yellow circle indicates $\boldsymbol{L}^2(\mathbb{R}^d)$.

$(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ as universal Banach algebra

The action of $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ as a Banach algebra under convolution on the first space, namely $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$, is the most simple one, but we have seen, there is also an action within the algebra, consistent with the embedding of $(\boldsymbol{C}_c(\mathbb{R}^d), \|\cdot\|_1)$ in to $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$, which restricts to an action on $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, hence it is plausible to expect something for $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p < \infty$, or e.g. Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$. If one does not like to use local Lebesgue integrability on can assume for the definition below simply that the Banach space is continuously embedded into the dual space of Wiener's algebra, which is $\boldsymbol{W}(\boldsymbol{M}, \ell^\infty)$ resp. $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)(\mathbb{R}^d)$, endowed with the vague resp. w^*-convergence.

Homogeneous Banach spaces a la Katznelson

For this purpose let us recall the concept of *homogeneous Banach spaces* (on $\mathbb{R}^d$) as it is found in the book of Y. Katznelson.

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ of locally integrable functions is called a *homogeneous Banach space* if it satisfies:

1. it is isometrically translation invariant, i.e.

$$\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \ \forall f \in \boldsymbol{B};$$

2. Translation is continuous in $\boldsymbol{B}$, i.e.

$$\|T_x f - f\|_{\boldsymbol{B}} \to 0 \text{ for } x \to 0.$$

Homogeneous Banach spaces are $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$-modules

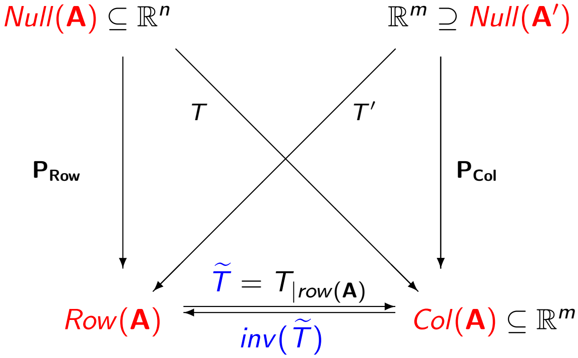
Our next claim (without providing details) is the following one:

Theorem

1. *Any homogeneous Banach space* $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is a Banach module over* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ *with respect to convolution.*
2. *If one restricts the attention to the closed subalgebra* $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$, *then it is even an essential module.*
3. *Conversely, every* $(\boldsymbol{M}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}})$ *module which, viewed as* $(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1)$ *is an essential* $\boldsymbol{L}^1(\mathbb{R}^d)$ *Banach convolution module inside the locally integrable functions, then it is a homogeneous Banach space.*

The proof is based on the action of bounded discrete measures, viewed as Riemannian type sums with values in Banach spaces and the convergence of $D_\Psi \mu * f$ in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

Geometric interpretation of matrix multiplication



$$T = \widetilde{T} \circ P_{Row}, \quad pinv(T) = inv(\widetilde{T}) \circ P_{Col}.$$

Matrices of maximal rank

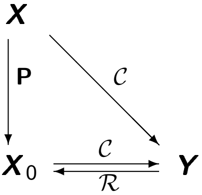
We will be mostly interested (as models for Banach Frames and Riesz projection bases) in the situation of matrices of maximal ranks, i.e. in the situation where $r = \mathrm{rank}(A) = max(m, n)$, where $A = (a_1, \cdots, a_k)$.
Then either the synthesis mapping $x \mapsto A * x = \sum_k x_k a_k$ has trivial kernel (i.e. the column vectors of A are a linear independent set, spanning the column-space of which is of dimension $r = n$), or the analysis mapping $y \mapsto A' * y = (\langle y, a_k \rangle)$ has trivial kernel, hence the column spaces equals the target space (or $r = m$), or the column vectors are a spanning set for $\mathbb{R}^m$.

Moving from $\mathbb{C}^n$ to Hilbert spaces: PROBLEMS

It seems easy to go from the finite dimensional setting to infinite dimensional cases. This is in fact possible, if we have mappings which have closed range!! Then the four spaces (terminology of Gilbert Strang's LINEAR ALGEBRA books!) are still here and we can talk about Moore-Penrose inverse and all this.
WARNING: The usual "generalization" of *linear independence* resp. "generating system", meaning the property of a set to be *total* in the Hilbert space are NOT SUITABLE because they lack stability. The first restricts the attention (wrongly) to finite families, the other talks about approximation by finite linear combinations instead of series representations, convergent in the Hilbert space. THERE IS A BIG difference.
RECOMMENDATION: Use diagrams to understand/describe the case!!!

Riesz basic sequences for Banach spaces

For *Riesz basic sequences* we have the following diagram:



Definition

A sequence (h_k) in a separable Hilbert space $\mathcal{H}$ is a *Riesz basis* for its closed linear span (sometimes also called a Riesz basic sequence) if for two constants $0 < D_1 \leq D_2 < \infty$,

$$D_1 \|c\|_{\ell^2}^2 \leq \Big\| \sum_k c_k h_k \Big\|_{\mathcal{H}}^2 \leq D_2 \|c\|_{\ell^2}^2, \qquad \forall c \in \ell^2 \tag{19}$$

Details on Riesz projection bases are given in the PhD thesis of

What is a generating set in a Hilbert space

We teach in our linear algebra courses that the following properties are equivalent for a set of vectors $(f_i)_{i\in I}$ in $\boldsymbol{V}$:

1. The only vector perpendicular to a set of vectors is $\emptyset$;
2. Every $v \in \boldsymbol{V}$ is a linear combination of these vectors.

An attempt to transfer these ideas to the setting of Hilbert spaces one comes up with several different generalizations:

- a family is *total* if its linear combinations are dense;
- a family is a *frame* if there is a bounded linear mapping from $\mathcal{H}$ into $\ell^2(I)$ $f \mapsto \mathbf{c} = c(f) = (c_i)_{i\in I}$ such that

$$f = \sum_{i\in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{20}$$

The usual definition of frames

There is another, *equivalent* characterization of frames. First, it is an obvious consequence of the characterization given above, that

$$f = \sum_{i \in I} c_i f_i \quad \forall f \in \mathcal{H}. \tag{21}$$

implies that there exists $C, D > 0$ such that

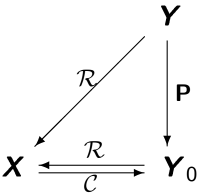
$$C\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq D\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{22}$$

For the converse observe that $Sf := \sum_{i \in I} \langle f, f_i \rangle f_i$ is a strictly positive definite operator and the *dual frame* $(\tilde{f}_i)$ satisfies

$$f = \sum_{i \in I} \langle f, \widetilde{f_i} \rangle f_i = \sum_{i \in I} \langle f, f_i \rangle \widetilde{f_i}$$

The corresponding diagram for frames

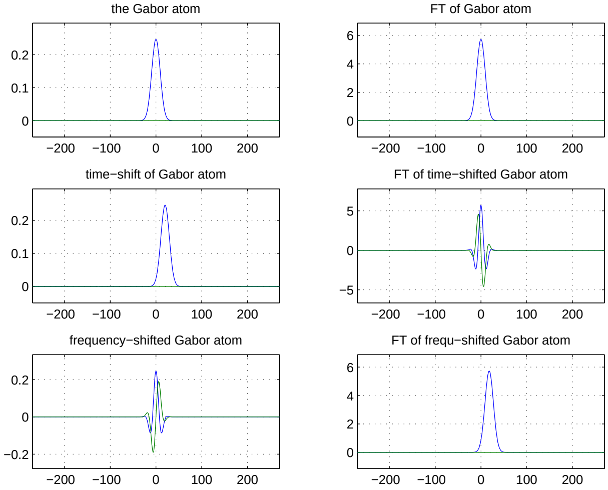
$\mathcal{R}$ being a left inverse of $\mathcal{C}$ implies that $\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



Dennis Gabor's suggestion of 1946

There is one very interesting example (the prototypical problem going back to D. Gabor, 1946): Consider the family of all time-frequency shifted copies of a standard Gauss function $g_0(t) = e^{-\pi|t|^2}$ (which is invariant under the Fourier transform), and shifted along $\mathbb{Z}$ ($T_n f(z) = f(z-n)$) and shifted also in time along $\mathbb{Z}$ (the modulation operator is given by $M_k h(z) = \chi_k(z) \cdot h(z)$, where $\chi_k(z) = e^{2\pi ikz}$).
Although D. Gabor gave some heuristic arguments suggesting to expand every signal from $\boldsymbol{L}^2(\mathbb{R})$ in a unique way into a (double) series of such "Gabor atoms", a deeper mathematical analysis shows that we have the following problems (the basic analysis has been undertaken e.g. by A.J.E.M. Janssen in the early 80s):

TF-shifted Gaussians: Gabor families



Problems with the original suggestion

Even if one allows to replace the time shifts from along $\mathbb{Z}$ by time-shifts along $a\mathbb{Z}$ and accordingly frequency shifts along $b\mathbb{Z}$ one faces the following problems:

1. for $a \cdot b = 1$ (in particular $a = 1 = b$) one finds a *total* subset, which is not a frame nor Riesz-basis for $\boldsymbol{L}^2(\mathbb{R})$, which is redundant in the sense: after removing one element it is still total in $\boldsymbol{L}^2(\mathbb{R})$, while it is not total anymore after removal of more than one such element;
2. for $a \cdot b > 1$ one does not have anymore totalness, but a Riesz basic sequence for its closed linear span ($\subsetneq \boldsymbol{L}^2(\mathbb{R})$);
3. for $a \cdot b < 1$ one finds that the corresponding Gabor family is a *Gabor frame*: it is a redundant family allowing to expand $f \in \boldsymbol{L}^2(\mathbb{R})$ using ℓ^2-coefficients (but one can even remove infinitely many elements!);

The Balian-Low Theorem

In his seminal paper of 1946 D. Gabor chose the integer lattice $a = b = 1$ in $\mathbb{R}^2$ and used the Gaussian in order to define a Gabor system with maximal time-frequency localization. Unfortunately this system is no longer stable though complete/total. The Balian-Low Theorem (early 80th) states that good time-frequency localization and Gabor Riesz bases are not compatible:

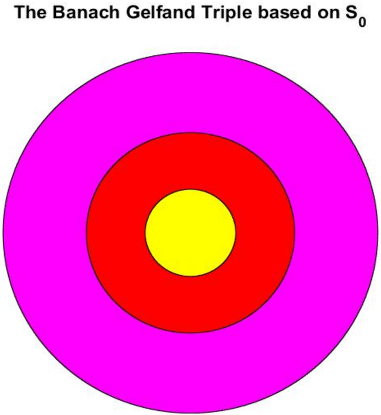
Theorem

(Balian-Low) *If $\mathcal{G}(g, 1, 1)$ constitutes a Riesz basis for $\mathbf{L}^2(\mathbb{R})$, then*

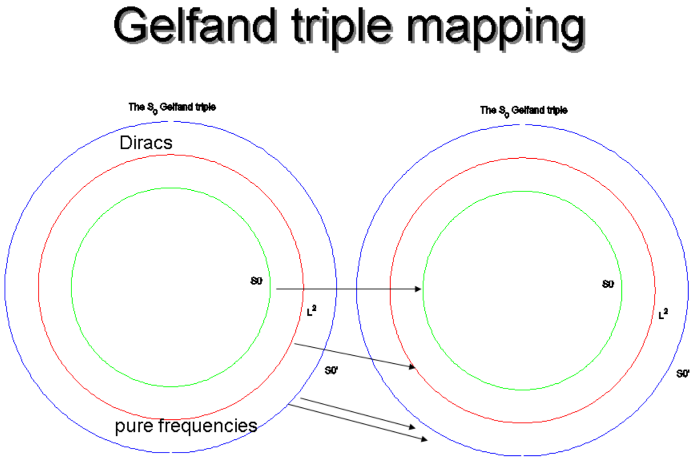
$$\int_{\mathbb{R}^2} |g(t)|^2 t^2 dt \int_{\mathbb{R}^2} |\hat{g}(\omega)|^2 \omega^2 d\omega = \infty\,.$$

The Balian-Low Theorem reveals a form of uncertainty principle and has inspired fundamental research.

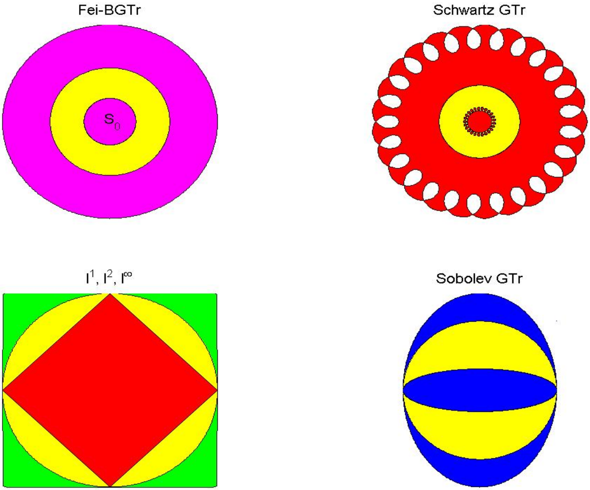
The Banach Gelfand Triple



Banach Gelfand Triple (auto)morphism



Various Gelfand Triples in the literature



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

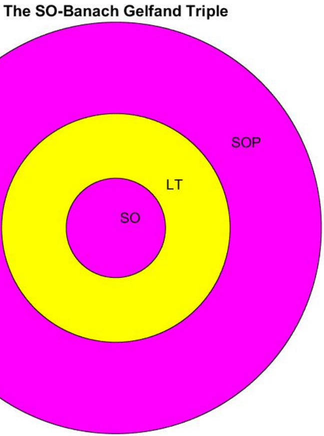
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

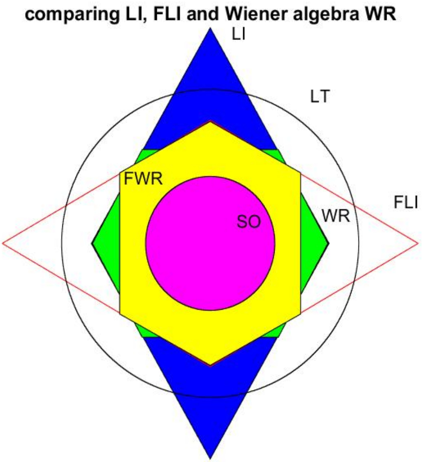
1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then automatically w^*-w^*--continuous!

The $\boldsymbol{S}_0(\mathbb{R}^d)$-Banach Gelfand Triple

From now on we will mostly focus on the Banach Gelfand triple based on the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$.

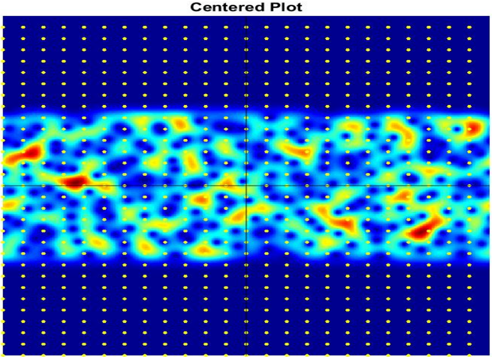


Another comparison of function spaces



Sampling the spectrogram (STFT)

Yellow points indicate sampling set corresponding to a Gabor frame. You see the spectrogram of some band-limited signal with respect to a discrete Gaussian.



What we have learned in this course

Starting from the end:

- Banach Gelfand Triples appear in many places and are in fact almost as useful as the usual Gelfand triple $\mathcal{S}(\mathbb{R}^d) \subset \mathbf{L}^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, except for PDE;
- Banach Gelfand Triples appear in many places and are in fact almost as useful as the usual Gelfand triple $\mathcal{S}(\mathbb{R}^d) \subset \mathbf{L}^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, except for PDE;
- Diagrams help to describe frames and Riesz basis at this level, corresponding to sampling and atomic decomposition;
- Many function spaces on LCA groups, like $\left(\mathbf{S}_0(G), \|\cdot\|_{\mathbf{S}_0}\right)$ and $\left(\mathbf{S}_0'(G), \|\cdot\|_{\mathbf{S}_0'}\right)$ can be introduced and used;
- We have a generalized Fourier transform, Shannon Sampling Theorems and Kernel Theorems