NuHAG **Numerical Harmonic Analysis Group**

A Timely Approach to Fourier Analysis

Hans G. Feichtinger, Univ. Vienna & TUM
hans.feichtinger@univie.ac.at
www.nuhag.eu

Fourth Harish-Chandra Lecture

.

Harish Chandra Research Center, Allahabad Jan. 25th, 2018

Preview on the talk

1. A quick historical "tour d'horizon";
2. Fourier Analysis in Mathemtics or Applications;
3. Lebesgue Theory, Tempered Distributions;
4. What is the Short-Time Fourier transform;
5. Application aspect (MP3), mobile communication;
6. Time-Frequency and Gabor Analysis;
7. Mathematical questions: convergence, continuity;
8. The role of the Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$.

Historical Aspects: the timeline

- 1822: Jean Baptiste **Fourier** suggest: periodic functions (relevant for heat equation) have a "Fourier series expansion";
- Dirichlet, Riemann, Fejer study the convergence issue (pointwise, using integrals, summability);
- H. Lebesgue develops the "correct integral" allowing to get the Banach spaces $\boldsymbol{L}^1(\mathbb{R}^d)$, $\boldsymbol{L}^2(\mathbb{R}^d)$; around 1904;
- early 20th century: beginning linear *functional analysis*, especially *Hilbert spaces*, Riesz representation theorem;
- A. Weil and I. Gelfand: Abstract Harmonic Analysis;
- L. Schwartz: theory of tempered distributions;
- Cooley-Tukey: FFT algorthm (fast Fourier transform);

Current situation: Theory versus Applications

If you look at the present situation Fourier analysis is an indispensable (mostly implicit) part of our life
(see my talk about FA in the 21st century, :))
We use it in our mobile phone to compress data, to transmit these data, we store, transmit and manipulate music and images and we are happy about medical image processing methods (tomography).
But also within the natural sciences many models (e.g. Schrödinger equation) are only understandable with the help of Fourier based mathematics (and then simulations).
Optical devices are not understandable without Fourier theory!

Diagnosis of the State of Affairs

Despite the arguable importance of Fourier Analysis in so many branches it has developed into a variety of almost disjoint areas of expertise, putting us into a questionable position:

- Is the knowledge that we produce not relevant for those who *have to make use of Fourier Analysis*?
- Is the mathematics that is used in applications (e.g. electrical engineering or theoretical physics) good enough to allow them to do what they want to do?
- Applied scientist often rely on their intuition which allows them to ensure that they stay on the track despite a bad road;
- Pure mathematicians are proud of the precision of their arguments and the complexity of their constructions, but sometimes they build highways to nowhere;

What is the level of precision required

According to my *personal view* we are facing this situation:

- The mathematical branch has become too complicated for the applied scientists so that they have establised their own (*questionable?*) standards;
- Sometimes their students are told, that the (pedantic) mathematicians would know how to fix these problems, but that this is not so important for the applications;
- But in case they would see the mathematicians probably one would try to teach them the properties of Lebesgue integrals and distribution theory (a bit e.g. in the Stanford course), at least the mysterious/ingenious Dirac “delta-function”.
- mostly **people do not know the other side!**

The Fourier Integral and Inversion

If we define the Fourier transform for functions on $\mathbb{R}^d$ using an integral transform, then it is useful to assume that $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, i.e. that f belongs to the space of Lebesgues integrable functions.

$$\hat{f}(\omega) = \int_{\mathbb{R}^d} f(t) \cdot e^{-2\pi i \omega \cdot t} \, dt \tag{1}$$

The inverse Fourier transform then has the form

$$f(t) = \int_{\mathbb{R}^d} \hat{f}(\omega) \cdot e^{2\pi i t \cdot \omega} \, d\omega, \tag{2}$$

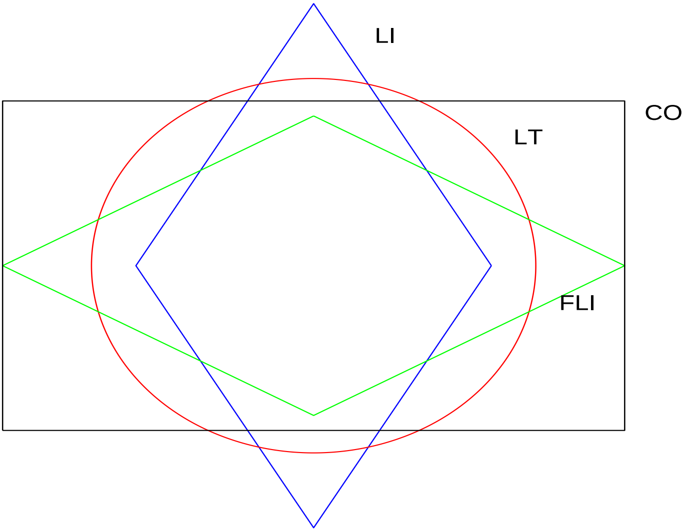
Strictly speaking this inversion formula only makes sense under the additional hypothesis that $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$. One often speaks of Fourier analysis followed by Fourier inversion as a method to build f from the pure frequencies (Fourier synthesis).

The classical situation with Fourier

Unfortunately the Fourier transform does not behave well with respect to $\boldsymbol{L}^1$, and a lot of functional analysis went into fighting the problems (or should we say symptoms?)

1. For $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ we have $\hat{f} \in \boldsymbol{C}_0(\mathbb{R}^d)$ (but not conversely, nor can we guarantee $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$);
2. The Fourier transform f on $\boldsymbol{L}^1(\mathbb{R}^d) \cap \boldsymbol{L}^2(\mathbb{R}^d)$ is isometric in the $\boldsymbol{L}^2$-sense, but the Fourier *integral!* may not apply;
3. Convolution and pointwise multiplication correspond to each other, but sometimes the convolution may have to be taken as improper integral, or using summability methods;
4. $\boldsymbol{L}^p$-spaces have traditionally a high reputation among function spaces, but tell us little about $\hat{f}$.

A schematic description of the situation



Mathematical difficulties

There are quite a few technical issues involved which have stimulated detailed the development mathematical analysis:

1. The Fourier integrals require f to belong to $\boldsymbol{L}^1(\mathbb{R}^d)$, but this does *not imply* $\hat{f} \in \boldsymbol{L}^1(\mathbb{R}^d)$ (box-function $\to$ $SINC \notin \boldsymbol{L}^1$);
2. Another strong argument is the use of convolutions:

$$f * g(x) = \int_{\mathbb{R}^d} g(x-y)f(y)dy.$$

   How could it be done without integrability??
3. The Plancherel Theorem, tells us that the FT is *unitary* on $\boldsymbol{L}^2(\mathbb{R}^d)$, but $\boldsymbol{L}^2(\mathbb{R}^d)$ is NOT a part of $\boldsymbol{L}^1(\mathbb{R}^d)$.

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \quad (3)$$

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X + Y$ has probability density distribution $f * g = g * f$.

Banach algebras

Theorem

Endowed with the bilinear mapping $(f,g) \to f * g$ the Banach space $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ becomes a commutative Banach algebra with respect to convolution.

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f, g \in \boldsymbol{L}^1(\mathbb{R}), \tag{4}$$

implies

Theorem

The Fourier algebra, defined as $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \boldsymbol{L}^1(\mathbb{R})\}$, with the norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ is a Banach algebra, closed under conjugation, and dense in $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (continuous functions, vanishing at infinity).

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{5}$$

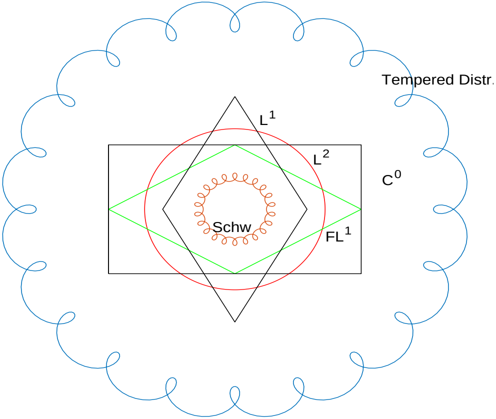
This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

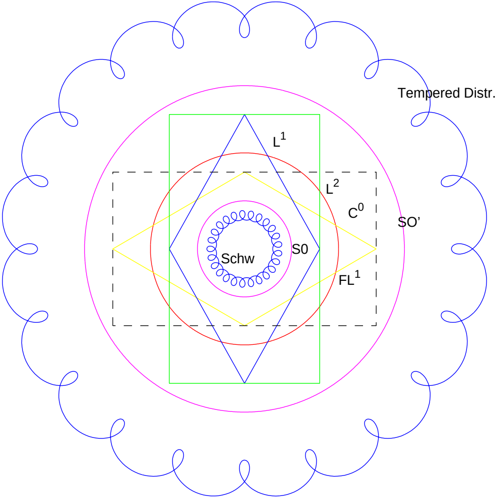
The classical setting of test functions & distributions

Inside of all the spaces is $\mathcal{S}(\mathbb{R}^d)$, but everything happens inside $\mathcal{S}'(\mathbb{R}^d)$!



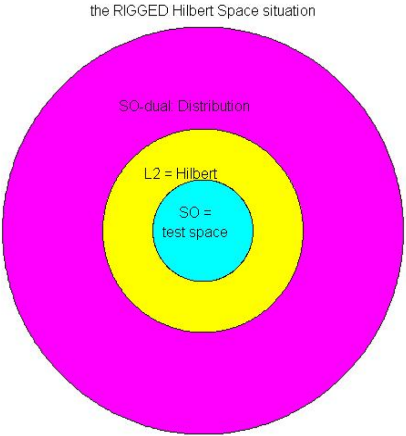
Overall it will be explained, that the **distributional view-point is nowadays more important** than the fine analysis of $\boldsymbol{L}^p$-spaces using Lebesgue integration methods. The setting of the *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ appears to be highly suitable for many applications.
There are many open questions related to time-frequency and Gabor analysis. In addition the computational side of Harmonic Analysis is not yet well integrated into the overall investigations in the area. Therefore the idea of *Conceptual Harmonic Analysis*, which includes (and integrates) both Abstract Harmonic Analysis and Numerical Harmonic Analysis, should be developed further.

A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

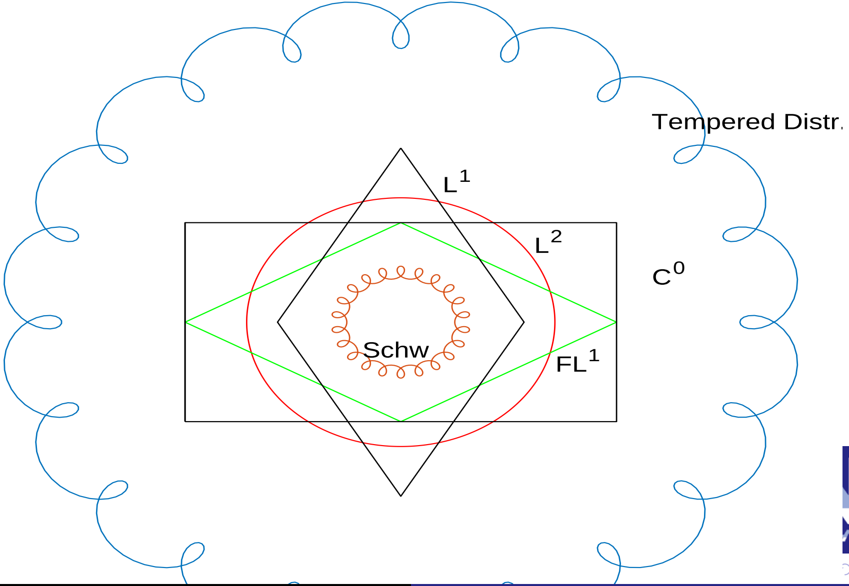
Laurent Schwartz Theory of Tempered Distributions

Laurent Schwartz is mostly known for having introduced the space of tempered distributions, a topological vector space of **generalized functions** or **distributions** which is invariant under the Fourier transform.

He starts out by defining the so-called *Schwartz space of rapidely decreasing functions*, consisting of all infinitely differentiable functions on $\mathbb{R}^d$ which decay faster at infinity than any polynomial. This space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is naturally endowed with a countable family of semi-norms, turning the space into a *nuclear Frechet space*.

The topological dual of $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, i.e. the collection of all linear functionals σ on $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ satisfying the continuity assumption $f_n \to f_0$ in $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ implies $\sigma(f_n) \to \sigma(f_0)$ in $\mathbb{C}$, constitutes $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, the space of tempered distributions.

The classical setting of test functions & distributions



Fourier Transforms of Tempered distributions

The Fourier transform $\hat{\sigma}$ of $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ is defined by the simple relation

$$\hat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

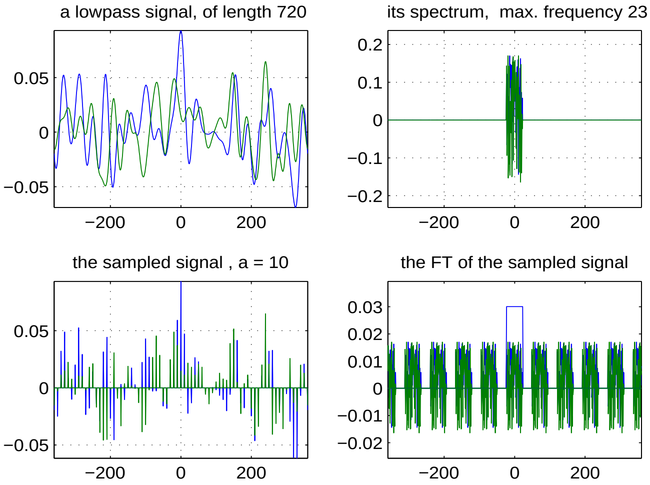
His construction *vastly extends the domain of the Fourier transform* and allows even polynomials to have a Fourier tranform. Among the objects which can now be treated are also the Dirac measures δ_x, as well as Dirac combs $⧢ = \sum_{k \in \mathbb{Z}^d} \delta_k$.
Poisson's formula, which expresses that one has for $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n), \tag{6}$$

can now be recast in the form

$$\widehat{⧢} = ⧢.$$

A Visual Proof of Shannon's Theorem



Shannon's Sampling Theorem

It is kind of clear from this picture that one can recover the spectrogram of the original function by isolating the central copy of the periodized version of $\widehat{f}$ by multiplying with some function $\hat{g}$, with g such that $\hat{g}(x) = 1$ on spec(f) and $\hat{g}(x) = 0$ at the shifted copies of $\widehat{f}$. This is of course only possible if these shifted copies of spec(f) do not overlap, resp. if the sampling is dense enough (and correspondingly the periodization of $\widehat{f}$ is a course one. This conditions is known as the *Nyquist criterion*. If it is satisfied, or supp$(f) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) T_{\alpha k} g(x), \quad x \in \mathbb{R}^d.$$

The Banach space $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$, which will serve our purpose. Its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

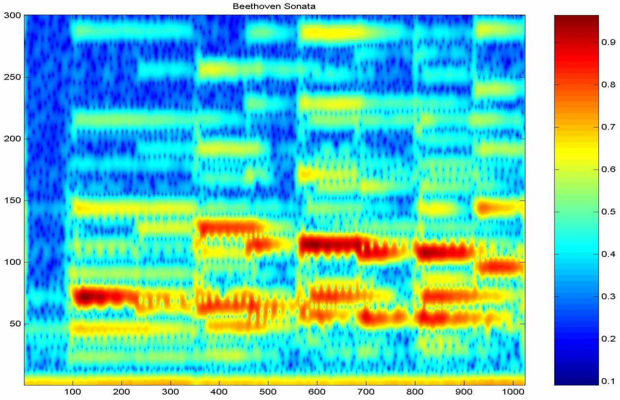
$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

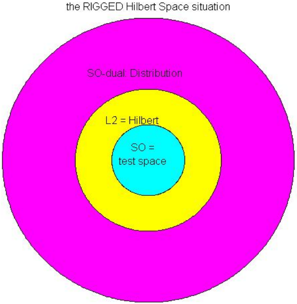
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

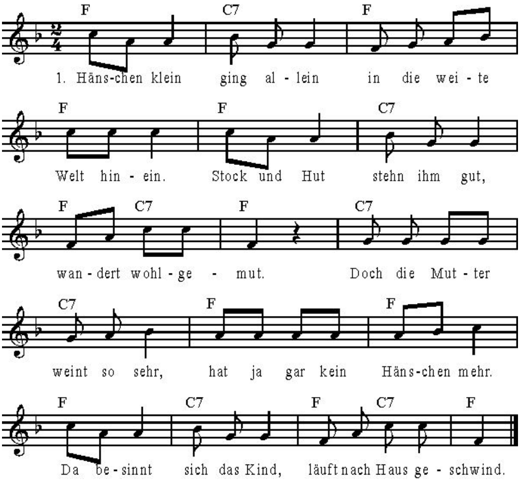
Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{7}$$

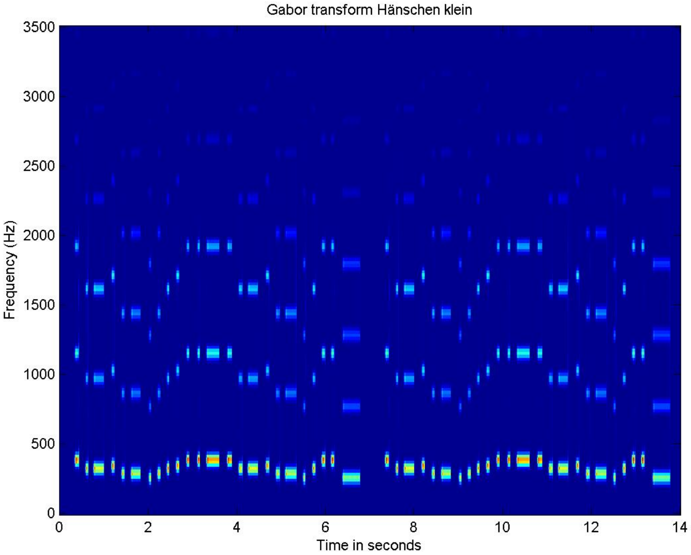
is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Time-Frequency and Musical Score

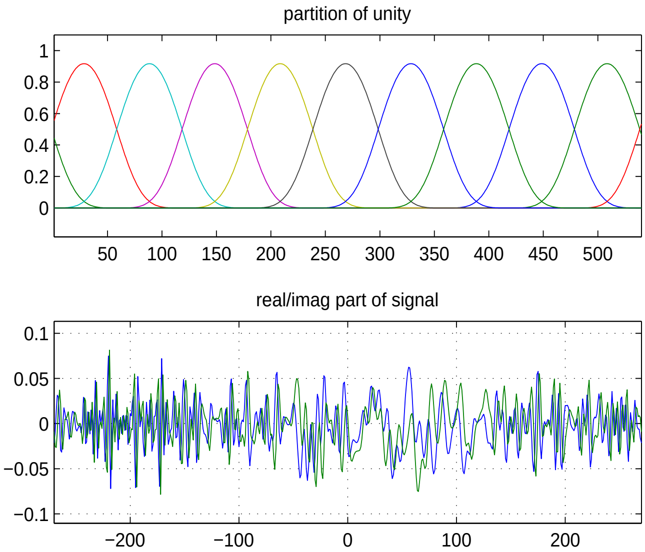
Time-Frequency Analysis and Music



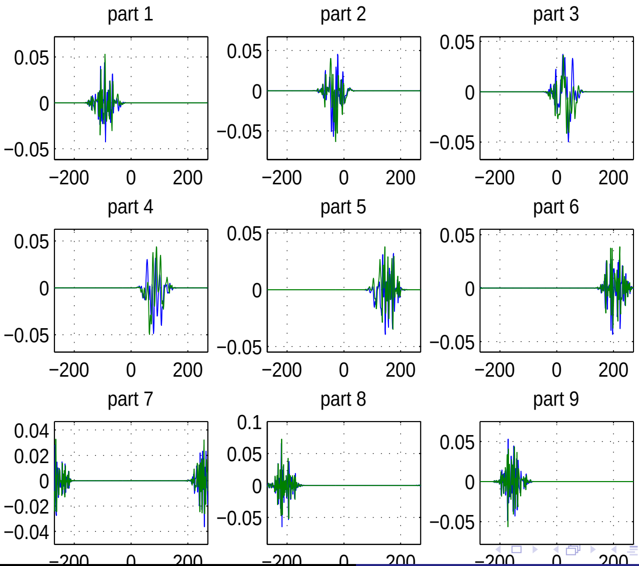
The Short-Time Fourier Transform of this Song



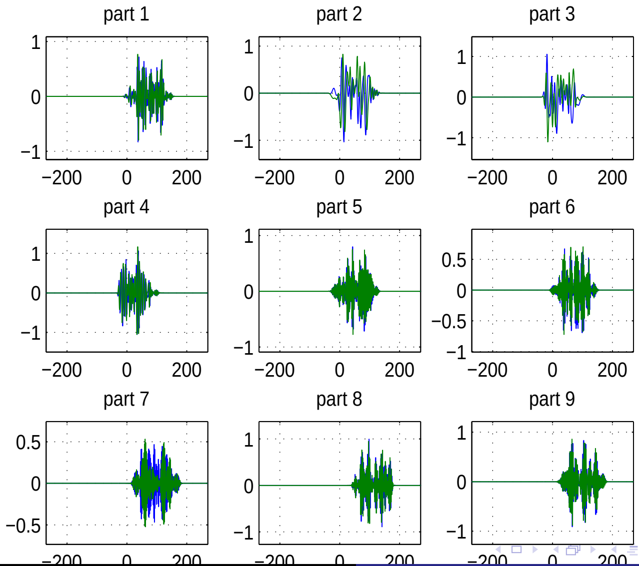
Motivated by MUSICAL SCORE one could do ?



... and cut the signal into pieces

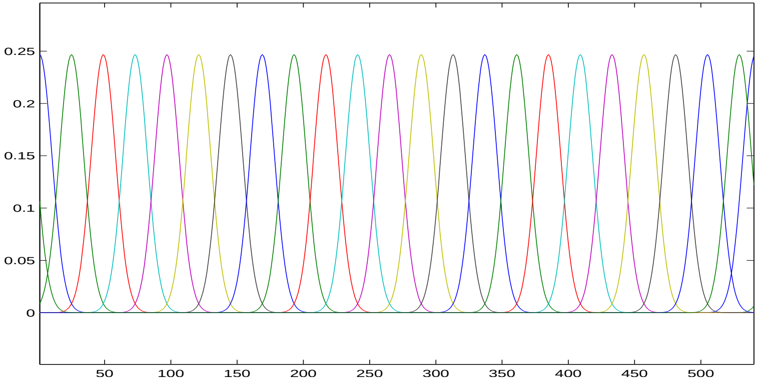


… and do localized spectra

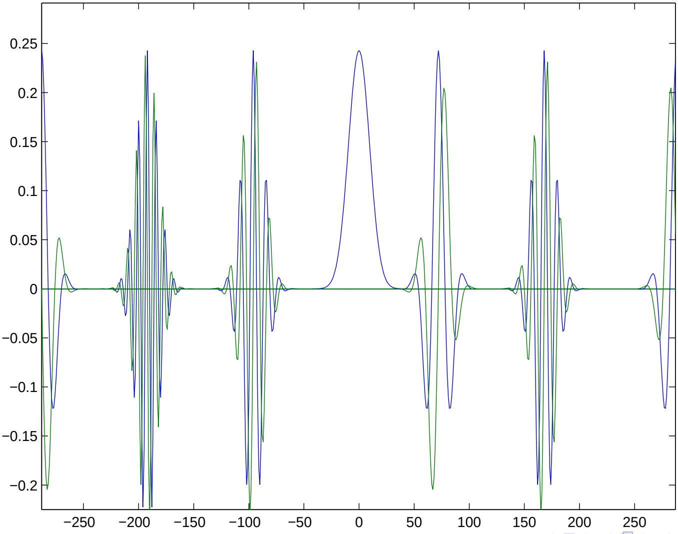


D.Gabor's suggestion of 1946

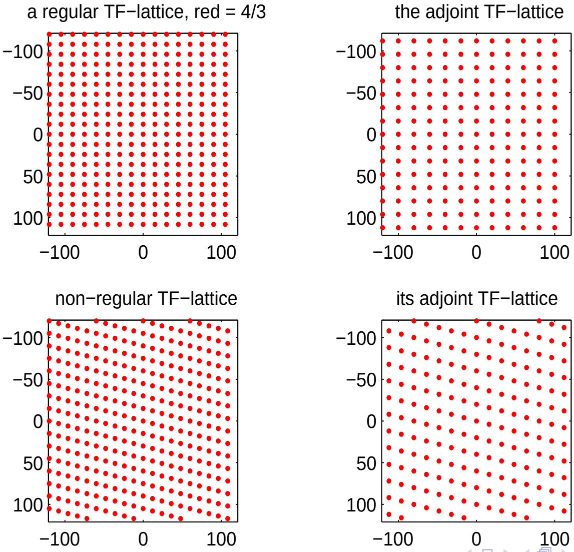
Choose the Gauss-function, because it is the unique minimizer to the *Heisenberg Uncertainty Relation* and choose the critical, so-called von-Neumann lattice, which is simply $\mathbb{Z}^2$.



The Gaborian Building blocks



Phase space lattices/ time-frequency plane



The Key Players (why is it called TF-analysis)

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^{d^d}$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(t, \omega) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \; \lambda = (t, \omega);$$

Modern Viewpoint, Time-Frequency Analysis; I

Today's Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \mathcal{S}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal f resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda)g \rangle$.
We speak of *tight Gabor frames* (g_λ) if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!!

Modern Viewpoint II

Another basic fact is that for each $g \in \boldsymbol{S}(\mathbb{R}^d)$ one can find, if Λ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\widetilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \tag{8}$$

$\widetilde{g}$ can be found as the solution of the (positive definite) linear system $S\widetilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\widetilde{g}$ instead of g for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.
Thus (8) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).

Modern Viewpoint III

The possibility of having such *tight Gabor frames* is resulting from the continuous reconstruction formula, valid for arbitrary $\boldsymbol{L}^2$-atoms g. Writing again for $\lambda = (t, \omega)$ and $\pi(\lambda) = M_\omega T_t$, and furthermore $g_\lambda = \pi(\lambda) g$ we have in fact for any $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda \, d\lambda.$$

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \boldsymbol{L}^2. \tag{9}$$

This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*.

Modern Viewpoint IV

There is a similar representation formula at the level of operators, where we also have a continuous representation formula, valid in a strict sense for *regularizing operators*, which map w^*-convergent sequences in $\boldsymbol{S}_0'(\mathbb{R}^d)$ into norm convergent sequences in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle T, \pi(\lambda) \rangle_{\mathcal{HS}} \pi(\lambda) d\lambda. \qquad (10)$$

It establishes an isometry for Hilbert-Schmidt operators:

$$\|T\|_{\mathcal{HS}} = \|\eta(T)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}, \quad T \in \mathcal{HS},$$

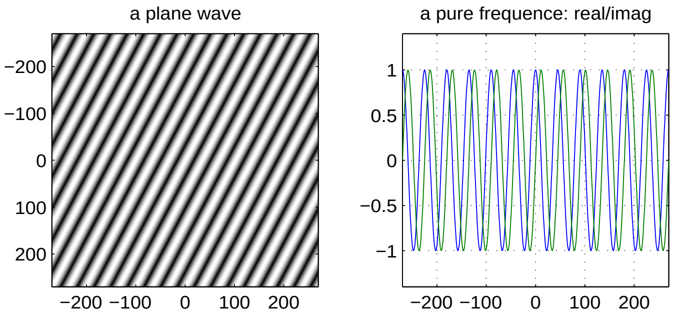
where $\eta T = \langle T, \pi(\lambda) \rangle_{\mathcal{HS}}$ is the *spreading function* of the operator T. The proof is similar to the proof of Plancherel's theorem.

Gabor Riesz bases and Mobile communication

Another usefulness of “sparsely distributed” Gabor systems comes from mobile communication:

1. Mobile channels can be modelled as slowly varying, or underspread operators (small support in spreading domain);
2. TF-shifted Gaussians are joint approximate eigenvectors to such systems, i.e. pass through was some attenuation only;
3. underspread operators can also be identified from transmitted pilot tones;
4. Communication should allow large capacity at high reliability.

2D-Gabor Transform: Plane Waves



2D-Gabor Analysis: Test Images

2D-Gabor Transform: Test-Images 2

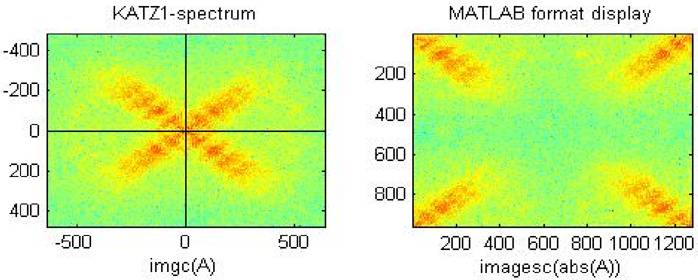
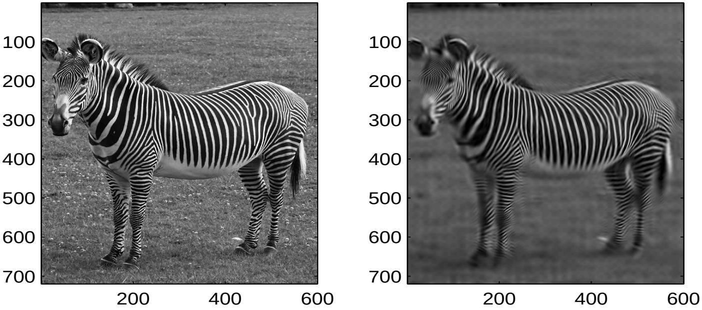
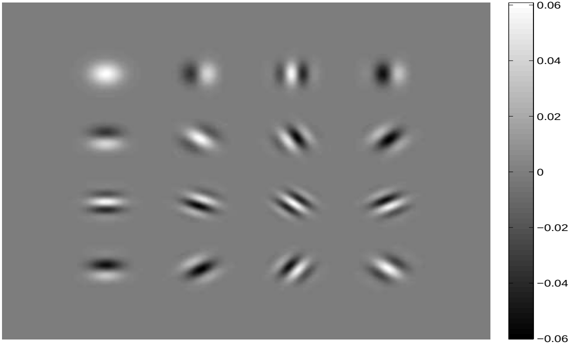


Image Compression: a Test Image



Showing the elementary $2D$-building blocks



(regular) Gabor families: Frames and Riesz bases

Definition

Given a pair (g, Λ), consisting of a vector $g \in \ell^2(G)$ and a lattice $\Lambda \lhd G \times \widehat{G}$ we call the family $(g_\lambda)_{\lambda \in \Lambda}$ a Gabor frame, if the family spans all of $\ell^2(G)$. It is called a Gaborian Riesz basis (resp. Riesz basic sequence) if it is a linear independent set.

There are - for people in numerical analysis - quality measures for the quality of such families, in the sense of a conditioning of the problem, thus being a quotient of two relevant singular values of associated operators, we don't go into details here.
Both situations are of practical relevance!

Usefulness and applications of Gabor frames:

The question of Gabor frames is of interest, when a signal (say some audio signal, or some image, cf. introduction) is to be *decomposed* into meaningful elementary building blocks, somehow like *transcription*. Ideally the distribution of *energy* in the signal goes over into an equivalent energy distribution. AND WHAT can we do with this:

a) contributions may be irrelevant (or disturbing) and can be eliminated (the bird contributing to the open air classical concert): denoising of signals

b) signals can be separated in a TF-situation

c) unimportant, small contributions can be omitted (+ masking effect): allows for efficient lossy compression schemes >> MP3.

Applications of Gabor Riesz bases:

Of course Gabor Riesz bases (for subspaces) will correspond to lattices Λ with at most N points. Ideally, the Gram matrix of the corresponding system is diagonal dominant (there is the so-called piano-reconstruction theorem).

They are very useful in mobile communication. The fact, that smooth envelopes (as used for Gabor frames), multiplied with pure frequencies are at least approximate eigenvectors for so-called *slowly varying channels* makes them useful for mobile communication. The physical assumption of limited multi-path propagation (variable kernels over time) and Doppler (due to movement) related to underspread operators, i.e. to matrices whose spreading function is supported on a given rectangular domain.

Applications of Gabor Riesz bases:

The information, encoded as a collection of coefficients which we will call (c_{λ°) are used to form a linear combination of the elements of our Gaborian Riesz basis. I.e. the sender *plays slowly a melody on the piano*.
Assume we are able to estimate the approximate eigenvalues $(d_{\lambda^\circ}$ of the involved building blocks $(g_{\lambda^\circ}$, the approximate eigenvector property of these building blocks implies that the receiver obtains $\sum_{\lambda^\circ} c_{\lambda^\circ} d_{\lambda^\circ} g_{\lambda^\circ}$. Knowing the factors (d_{λ°) (by sending so-called pilot tones) and the biorthogonal basis the receiver can then (approximately) recover the set of coefficients (c_{λ°) sent by the sender.
In other words, *the receiver listens to the music behind a wall, knowing e.g. that higher frequencies are absorbed more (or less) then others and figures out, what has been played.*

This greatly simplifies the calculation of (minimal norm) coefficients for the given signal. In fact, it is found that the solution $\widetilde{g}$ of the simple (positive definite) linear equation

$$S\widetilde{g} = g \quad \text{resp.} \quad \widetilde{g} = S^{-1}g,$$

spans the *dual Gabor frame*. In fact FFT-based methods can be applied to efficiently calculate these coefficients, once $\widetilde{g}$ is given. Sometimes alternative sets of coefficients are equally useful. For the solution of the above equations various iterative methods, e.g. conjugate gradients, can be applied .

It is clear, that one actually would like to build an arbitrary signal f given the pair (g, Λ) (in the frame case), or at least do the best approximation of f by linear combinations from the Gabor family in the Riesz basis case. In both cases one has a number of choices, but the canonical one (related to `PINV` resp. to the associated MNLSQ-problem is the one usually preferred.
The appropriate coefficients are then obtained by taking scalar products with respect to the correspondig "dual" family, which is numerically efficiently implemented by doing a sampled STFT (using FFT-based methods).

Operating on the audio signal: filter banks

Finally let us operate on the Gabor coefficients

Definition

Let g_1, g_2 be two L^2-functions, Λ a TF-lattice for $\mathbb{R}^d$, i.e. a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Furthermore let $\mathbf{m} = (\mathbf{m}(\lambda))_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the **Gabor multiplier** associated to the triple (g_1, g_2, Λ) with (*strong* or) **upper symbol m** is given as

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle\, \pi(\lambda)g_2. \qquad (11)$$

g_1 is called the *analysis* window, and g_2 is the synthesis window. If $g_1 = g_2$ and $\mathbf{m}$ is real-valued, then the Gabor multiplier is self-adjoint. Since the constant function $\mathbf{m} \equiv 1$ is mapped into the Identity operator if $g_1 = g_2$ is a Λ-tight Gabor atom this is often the preferred choice.

The family of projection operators (P_λ)

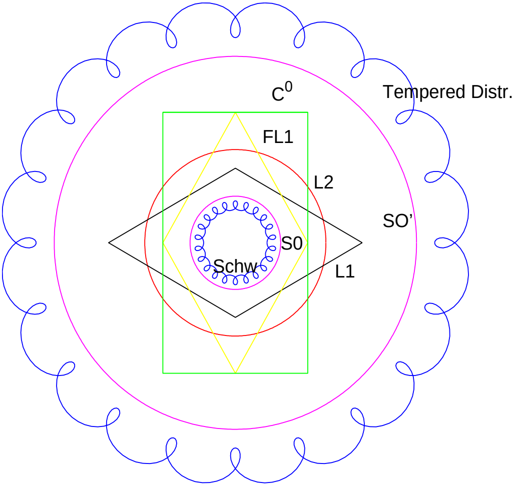
Theorem

Assume that (g,Λ) generates an S_0-Gabor frame for $\mathbf{L}^2(\mathbb{R}^d)$, with $\|g\|_2 = 1$, and write P_λ for the projection $f \mapsto \langle f, \pi(\lambda)g\rangle\pi(\lambda)g$.
i) Then the family $(P_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span within the Hilbert space $\mathcal{HS}$ of all Hilbert-Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ if and only if the function $H(s)$, defined as the Λ-Fourier transform of $\left(|STFT_g(g)(\lambda)|^2\right)_{\lambda\in\Lambda}$ is does not have zeros.
ii) An operator T belongs to the closed linear span of this Riesz basis if and only if it belongs to $\mathcal{GM}_2$, the space of Gabor multiplier with $\ell^2(\Lambda)$-symbol.
iii) The canonical biorthogonal family to $(P_\lambda)_{\lambda\in\Lambda}$ is of the form $(Q_\lambda)_{\lambda\in\Lambda}$,

$$Q_\lambda = \pi(\lambda)\circ Q\circ\pi^{-1}(\lambda) \text{ for } \lambda\in\Lambda,$$

for a uniquely determined Gabor multiplier $Q \in \mathcal{B}$.
iv) The best approximation of $T \in \mathcal{HS}$ by Gabor multipliers based on the pair (g,Λ) is of the form

Summarizing the situation: test functions & distributions



Kernel Theorem for the Banach Gelfand triple

See talk in Chennai, Jan. 22nd, 2018 >>
separate PDF file!

A few relevant references

K. Gröchenig: Foudations of Time-Frequency Analysis, Birkhäuser, 2001.

H.G. Feichtinger and T. Strohmer: Gabor Analysis, Birkhäuser, 1998.

H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, Birkhäuser, 2003.

G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.

I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.

Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.

See also www.nuhag.eu/talks.

Added in proof! last night

Time-Frequency Analysis and Black Holes

Breaking News

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

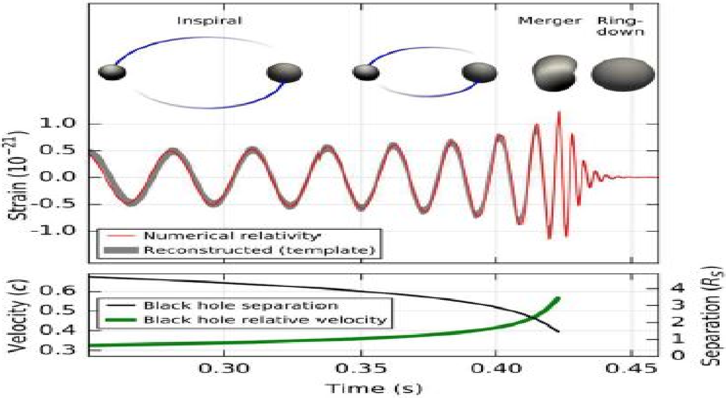
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

THANK YOU

Thank you for your attention It was an honour to present the fourth Harish-Chandra Lecture at HRI

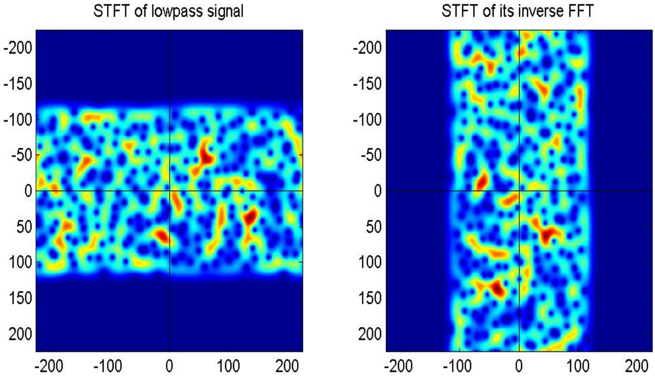
There is an upcoming book project (2019?) on this subject.

More at www.nuhag.eu

in particular at

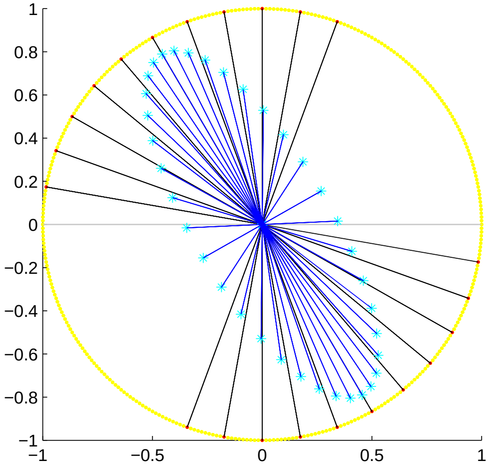
www.nuhag.eu/talks

Fourier transform as rotation in TF-plane



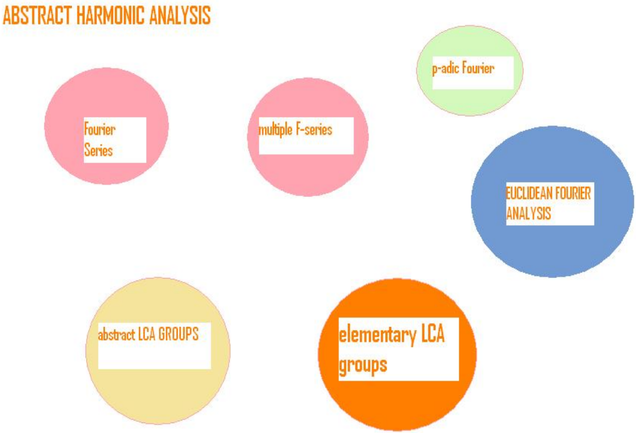
The action of a corresponding frame multiplier

The effect of a frame multiplier in the plane:

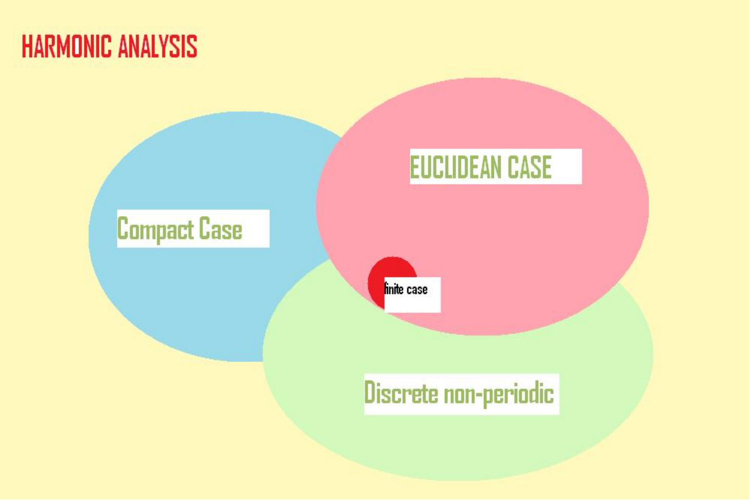


Operating on the audio signal: filter banks

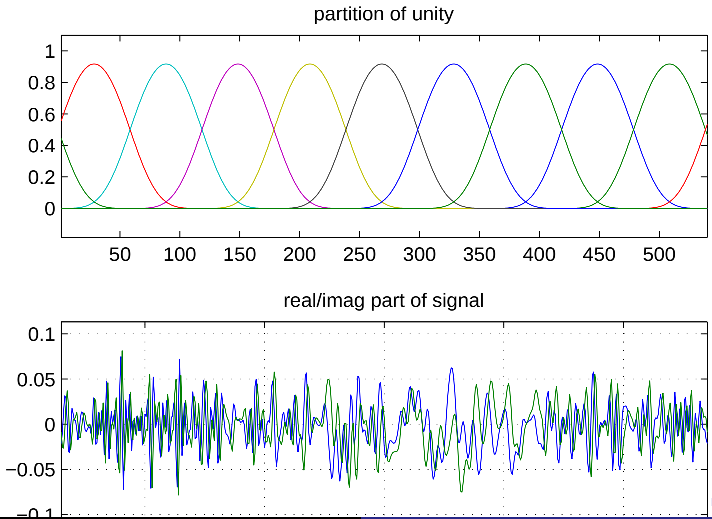
Abstract Harmonic Analysis: Fourier Transforms



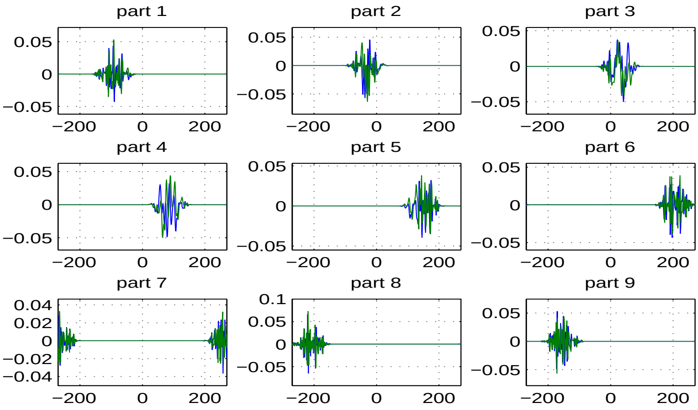
The classical view on the Fourier Transform



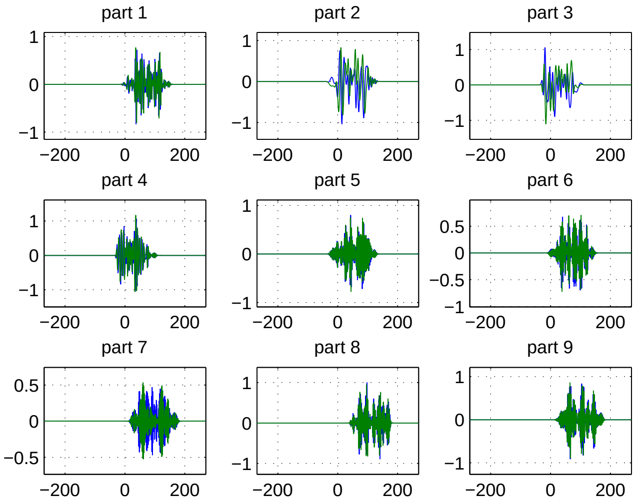
The idea of a "localized Fourier Spectrum"



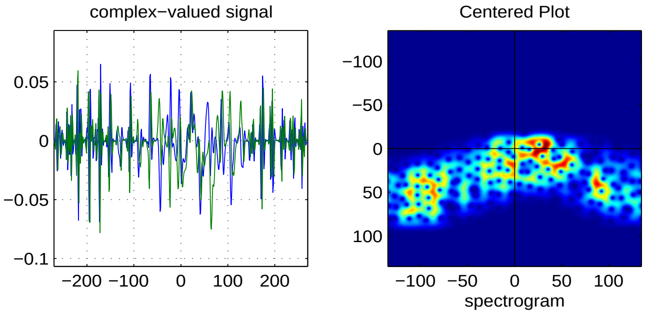
The localized Fourier transform (spectrogram)



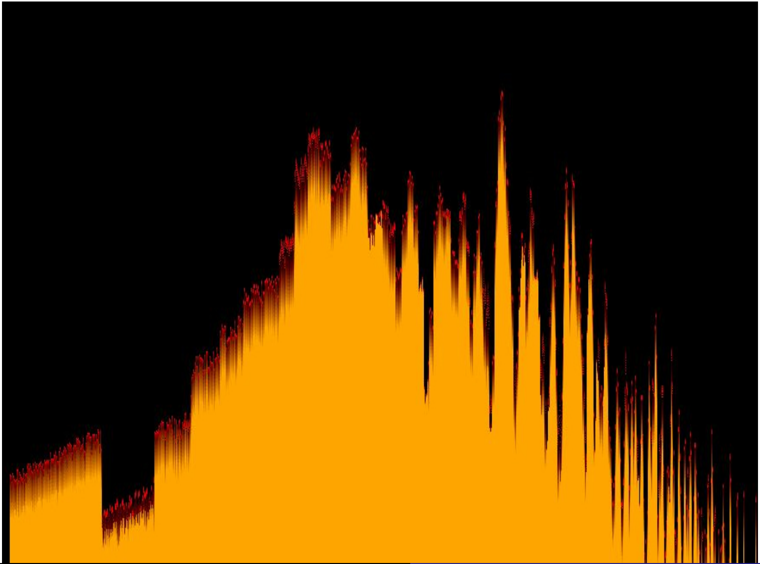
Spectral decomposition: variable bandwidth



STFT of a function of “variable band-width”



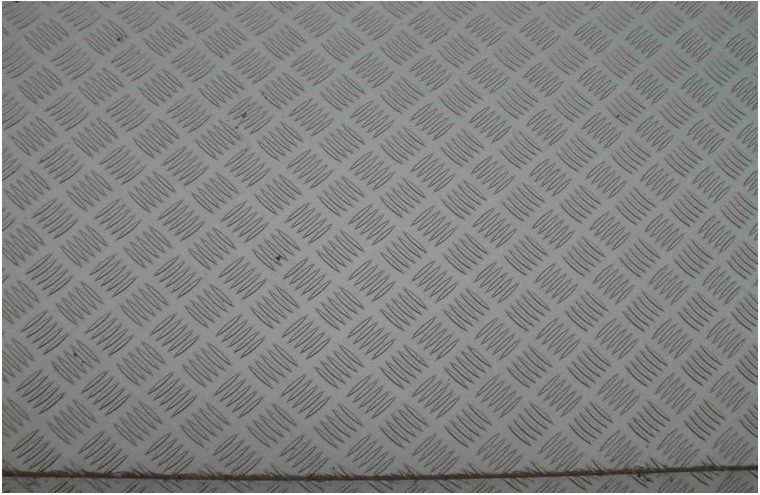
Another (Standard) representation of a Musical STFT



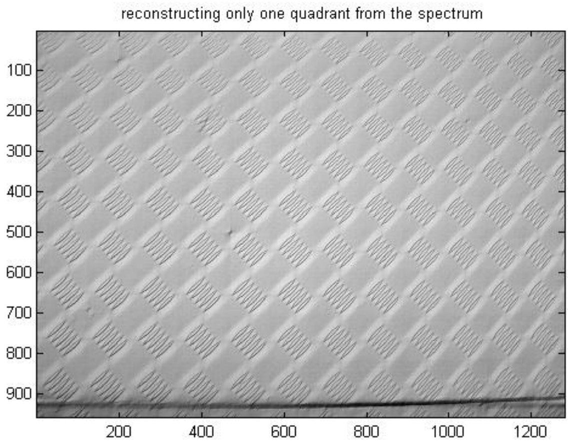
2D-Gabor Transform

2D-Gabor Transform

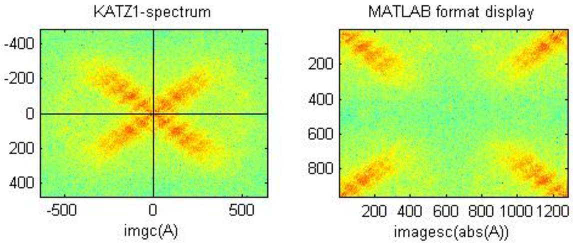
2D-Gabor Transform



2D-Gabor Transform



2D-Gabor Transform



2D-Gabor Transform: Plane Waves

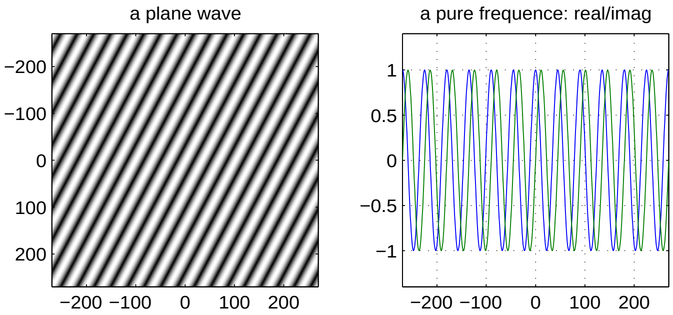


Image Compression