Numerical Harmonic Analysis Group

Banach Frames for Banach Gelfand Triples

Hans G. Feichtinger, Univ. Vienna & TUM
hans.feichtinger@univie.ac.at
www.nuhag.eu

Kirori Mal College, Delhi University

.

January 31st, 2018

Personal Background

- Starting as a teacher student math/physics, Univ. Vienna
- PhD and habilitation (1974/1979) in **Abstract Harmonic Analysis**
- Establishing **NuHAG** (Numerical Harmonic Analysis Group)
- Reach out for applications (communication theory, image processing, astronomy, medicine, musicology,...)
- European projects (Marie Curie and EUCETIFA)
- Main interest: Function spaces, Fourier Transform
- Nowadays: formally retired, but teaching at ETH, DTU, TUM, Chennai; goal: supporting the applied sciences
- As editor to JFAA also the perspective is sharpened.

How I came to frames

After first contacts with the sampling theorem (Shannon) by reading the work of Paul Butzer (90 anniversary coming up) who was pushing for a cooperation between Electrical Engineers and Mathematicians I was getting interested in sampling, later especially Irregular Sampling (with T. Strohmer and K. Gröchenig). By starting with function spaces on groups, in particular Banach algebras under convolution, Segal algebras, Beurling algebras I came to define Wiener amalgam spaces and *modulation spaces* and slowly realized the connection to analysis of the Heisenberg group, short-time Fourier transforms I could see the connection between modulation spaces and Besov-Triebel-Lizorkin spaces, which resulted in a series of papers with "Charly Gröchenig" (now also at NuHAG, Vienna).

Families of Banach spaces

From Triebel and Peetre I could learn that is important to understand that *function spaces* (meaning mostly Banach spaces of [generalized] functions) come in families, and that interpolation theory helps to "connect" the end-parameters, e.g. $p \in [1, \infty]$ for $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$.
From H. Reiter I learned to relevance of convolution and Fourier transform, but also Segal and Beurling algebras.
The design of Coorbit theory together with K. Gröchenig (and the paper "Describing Functions: Atomic decompositions versus frames") was based on such ideas, maximal generality of the setting using "Banach function spaces" (solid BF-spaces, cf. Luxemburg/Zaanen) and can be viewed as part IV of our series.
By analogy we found that our methods should be applicable to the case of irregular sampling of band-limited and then spline-type functions. Again families of function spaces appeared.

Some relevant references

- H. Reiter. *Classical Harmonic Analysis and Locally Compact Groups*. Clarendon Press, Oxford, 1968.
- H. G. Feichtinger. Banach convolution algebras of Wiener type. In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.
- K. Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(3):1–41, 1991.
- H. G. Feichtinger and K. Gröchenig. Multidimensional irregular sampling of band-limited functions in L^p-spaces. *Conf. Oberwolfach Feb. 1989*, pages 135–142, 1989.

Atomic decomposition versus reconstruction

The two sides of frame theory are: Reconstruction from samples versus atomic [de]compositions of function spaces.
In the first case the analysis of coefficient operator is first and the question is how to *reconstruct a signal from a collection of linear measurements*. The second case ask for (minimal norm) coefficient representations of a given signal, using the frame elements (usually described as an overcomplete set of generators).

The connection between irregular sampling of band-limited functions and frame theory (but also with the Moore-Penrose inverse) became slowly clear around 1989/90 (one year visit to John Benedetto in College Park/MD).

Technical key aspects of my talk

1. The linear algebra setting:
   - generating families
   - linear independent sets
   - bases
2. The Hilbert space setting: Frames and Riesz bases
3. Banach frames
4. Banach Gelfand Triples
5. Frames for Banach Gelfand Triples
6. unconditional Banach frames for families
7. outlook on applications
8. continuous ONBs in physics $(\delta_x)_{x \in \mathbb{R}}$

Early References

- R. J. Duffin and A. C. Schaeffer. A class of nonharmonic Fourier series. *Trans. Amer. Math. Soc.*, 72:341–366, 1952.
- I. Daubechies, A. Grossmann, and Y. Meyer. Painless nonorthogonal expansions. *J. Math. Phys.*, 27(5):1271–1283, May 1986.
- K. Gröchenig. Describing functions: atomic decompositions versus frames. *Monatsh. Math.*, 112(3):1–41, 1991.
- O. Christensen. *An Introduction to Frames and Riesz Bases.* Applied and Numerical Harmonic Analysis. Birkhäuser Basel, Second edition, 2016.

Preview on Goals of the talk

The linear algebra setting starts from the concept of linear combinations in a vector spaces and soon is brought to the concept of bases (for subspaces). The SVD (singular value decomposition) Theorem shows that every linear mapping can be viewed as the natural identification of the row space with the column space of the given matrix $\mathbf{A}$. This is the geometry.
This situation can be described by means of commutative diagrams, or in the case of Hilbert spaces by a double inequality.
The Banach Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$ is very useful for any branches of time-frequency analysis and Banach frames involving this Banach Gelfand Triple (also Riesz projection sequences) are occurring in many places.

Connections to interpolation theory: retracts

A quite important reservoir for Banach frames, even for Banach frames for *families of Banach* appears in interpolation theory. It is/was there an important method to derive *interpolation results*, e.g. in order to obtain a characterization of complex interpolation spaces in terms of their parameters, just like the *scales of* $\boldsymbol{L}^p$*-spaces* are stable under complex interpolation (by using linear interpolation of the parameters $1/p$), see e.g. the book of Bergh/Löfström, or Bennett/Sharpley, and in particular the work of J. Peetre and H. Triebel, also Frazier/Jawerth.
They make heavy use of dyadic decompositions of elements in $\boldsymbol{L}^p(\mathbb{R}^d)$ on the Fourier transform side and thus identify Besov spaces $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ with closed, complemented subspaces of vector-valued version of $\boldsymbol{L}^p$-spaces.

What are functions or signals?

Although most of us believe that it is correct to think of functions in the way we have learned it and taught it to our students, it is still worthwhile to *reflect* a bit on this concept.
Often one finds (in mathematics as well as applications) the idea that a function as occurring in "nature" has to be modeled as an element of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (of equivalence classes of measurable functions, identifying those which coincide almost everywhere). But despite the rich structure of a Hilbert space and the claim that any "natural signal has at most finite energy" (and thus has to be square integrable) it is questionable whether it is actually a physically justified assumption rather than a *matter of mathematical convenience*, allowing to use orthogonality, angles between signals, projection onto closed subspace and more.

What are functions? Generalized functions?

I am convinced that one should allow other models for the functions, and in some sense could be even vague at this point, like physicists, with some more general frame-work than the one provided by Lebesgue's measure theory.

The theory of *generalized functions*, often called *distributions* takes the idea that functions are pointwise well defined and allows to describe a measure, a pseudo-measure or more generally a (e.g. tempered) distribution as a "reasonable linear measurement", hence as a continuous linear functional on a space of test functions. Already this allows to go outside of e.g.the Lebesgue space $\boldsymbol{L}^1(\mathbb{R}^d)$, as the dual space of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ can be characterized as the Banach space of all bounded, regular Borel measures, endowed with the total variation norm. Due to the Riesz representation theorem this identification is justified. We just *call the dual space* $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$.

Generators and linear independence

In principle one can draw most of the intuition concerning frames from linear algebra, by interpreting existing results, definitions and conventions more closer.

Writing n column vectors from $\mathbb{R}^m$ into an $n \times m$ Matrix $\boldsymbol{A}$ gives us the possibility describe any linear mapping T from $\mathbb{R}^n$ to $\mathbb{R}^m$ via matrix multiplication: $T(\mathbf{x}) = \boldsymbol{A} * \mathbf{x}, \mathbf{x} \in \mathbb{R}^m$.

This mapping is injective if and only if the column of $\boldsymbol{A}$ are *linear independent* and it is surjective if and only if they form a *generating family* of vectors, i.e.if their linear combinations allow to write any vector in $\mathbb{R}^m$ (in a possibly non-unique way).

But if every $\mathbf{y} \in \mathbb{R}^m$ can be written the set of all possible values $\{\mathbf{z} \,|\, \boldsymbol{A} * \mathbf{z} = \mathbf{y}\}$ forms an affine subspace, which has a unique element of minimal norm $\mathbf{z}_0$, which is perpendicular to the null-space of $\boldsymbol{A}$. This is the MNLSQ solution to the problem $\boldsymbol{A} * \mathbf{z} = \mathbf{y}$ (Minimum norm least squares solution).

The general situation, the four spaces

Even for a general matrix $\boldsymbol{A}$ we have a geometric interpretation. Given a matrix $\boldsymbol{A}$ and its transpose (transpose conjugate in the complex case, so let us write in the MATLAB spirit $\boldsymbol{A}'$) both of them have a null-space and a range space. Often the column-space of $\boldsymbol{A}'$ is described as the row-space of $\boldsymbol{A}$, but certainly it consists of vectors in $\mathbb{R}^m$.
It is now important to verify that the null-space of $\boldsymbol{A}$ and the range of $\boldsymbol{A}'$ are orthogonal complements of each other, and so is the null-space of $\boldsymbol{A}'$ and the column space of $\boldsymbol{A}$.
By Gauss elimination the row-space and the column-space have equal dimension, in fact the action of T on the orthogonal complement of its null-space still has the same range but obviously no null-space, hence T must act as bijection from row to column space!

Invertible matrices, pseudo-inverse

Clearly *invertible matrices* provide an ideal scenario. The give both good properties at once and thus can serve as coordinate systems in a given finite-dimensional vector space.
Next we may look into linear independent sets of vectors spanning certain (closed) subspaces and forming a basis there, resp. we can look at generating systems, even if the are not linear independent (and thus kind of redundant).
The corresponding matrices are exactly the matrices of maximal rank, i.e.with $\mathrm{rank}(\boldsymbol{A}) = \min(m, n)$.
In all the cases there is the *Moore-Penrose inverse* of $\boldsymbol{A}$, known as $\mathrm{pinv}(\boldsymbol{A})$ in MATLAB or $\boldsymbol{A}^+$ in the mathematical literature. It reverts the isomorphism between row and column space (with projections to zero for the rest). It describes exactly the mapping from $\mathbf{y}$ to $\mathbf{z}_0$ described above.

Quality of such matrices

For any such $m \times n$-matrix $\boldsymbol{A}$ one can determine to quality of the isomorphism by considering its condition number (Kato condition number). Obviously a generating system is not so nice if it is very expensive to generate certain unit vectors, i.e.if (geometrically speaking) all the column vectors belong "almost" to some proper subspace of $\mathbb{R}^m$.
Accordingly a linear independent system is not so good if at least one of its vectors is almost in the linear span of the other elements. We note that the SVD (singular value decomposition of any matrix) provides a way to describe the relevant isomorphis as a diagonal matrix with respect to suitable chosen ONBs for the row resp. column space of $\boldsymbol{A}$.

From finite to infinite situation

We see that matrices of maximal rank define via $\mathbf{A}$ resp. $\mathbf{A}'$ an injective embedding of the smaller of the two spaces (by dimensions) into the bigger, and we can express the quality of this embedding via the norm-equivalence, between the original norm and the embedded norm (on its isomorphic image in the larger vector space).
But due to finite-dimensionality being linear independent (or a Riesz basic sequence in modern terminology) resp. a generating system (i.e.a frame) is equivalent to have corresponding norm inqualities. In the first case the sequence space of possible coefficients is mapped into e.g.spline-functions, while in the second case a sequence of linear measurements is taken.

Connections to interpolation theory: rectracts

A quite important reservoir for Banach frames, even for Banach frames for *families of Banach* appears in interpolation theory. It is/was there an important method to derive *interpolation results*, e.g. in order to obtain a characterization of complex interpolation spaces in terms of their parameters, just like the *scales of* $\boldsymbol{L}^p$*-spaces* are stable under complex interpolation (by using linear interpolation of the parameters $1/p$), see e.g. the book of Bergh/Löfström, or Bennett/Sharpley, and in particular the work of J. Peetre and H. Triebel, also Frazier/Jawerth.
They make heavy use of dyadic decompositions of elements in $\boldsymbol{L}^p(\mathbb{R}^d)$ on the Fourier transform side and thus identify Besov spaces $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$ with closed, complemented subspaces of vector-valued version of $\boldsymbol{L}^p$-spaces.

Frames and Riesz Bases

A family is a Riesz basis in a Hilbert space if it behave like a finite linear independent sequence in a Hilbert space: The set of all linear combinations (infinite linear combinations with square summable coefficients) is a closed subspace (whose orthogonal complement is the null-space of the adjoint mapping). It always has a biorthogonal Riesz basis (obtained via the inverse Gram matrix).

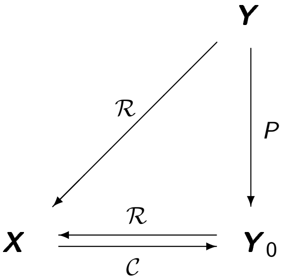
A family is a frame in a Hilbert spaces, if it behaves like a generating set (in the sense that the set of all linear combinations with square summable coefficient equals the whole Hilbert space. There is always a dual frame (obtained by inverse frame operator).

There is also a natural "duality between these problems: A rectangular $m \times n$ matrix A of *maximal rank* is either linear independent ($m \geq n$) or a generating set ($n \geq m$), and therefore its transpose (conjugate) is of the "other type".

Frames and Riesz Bases: Commutative Diagrams

Think of $\boldsymbol{X}$ as something like $\boldsymbol{L}^p(\mathbb{R}^d)$, and $\boldsymbol{Y} = \ell^p$:

Frame case: $\mathcal{C}$ is injective, but not surjective, and $\mathcal{R}$ is a left inverse of $\mathcal{R}$. This implies that $P = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$ in $\boldsymbol{Y}$:



Riesz Basis case: E.g. $\boldsymbol{X}_0 \subset \boldsymbol{X} = L^p$, and $\boldsymbol{Y} = \ell^p$:

Unconditional Banach Frames

A suggestion for “realistic Banach frames”:

Definition

A mapping $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ defines an unconditional (or solid) Banach frame for $\mathbf{B}$ w.r.t. the sequence space $\mathbf{Y}$ if

1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, with $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \|\|_{\mathbf{Y}})$ is a solid Banach space of sequences over I, with $\mathbf{c} \mapsto c_i$ being continuous from $\mathbf{Y}$ to $\mathbf{C}$ and solid, i.e. satisfying $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$;
3. finite sequences are dense in $\mathbf{Y}$ (at least W^*).

Corollary

By setting $h_i := \mathcal{R} e_i$ we have $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ unconditional in $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, hence $f = \sum_{i \in I} T(f)_i h_i$ as unconditionally convergent series.

Unconditional Banach Frames

A suggestion for "realistic Banach frames":

Definition

A mapping $\mathcal{C} : \mathbf{B} \to \mathbf{Y}$ defines an unconditional (or solid) Banach frame for $\mathbf{B}$ w.r.t. the sequence space $\mathbf{Y}$ if

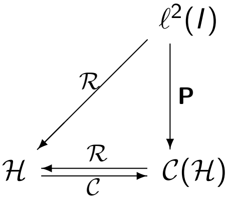
1. $\exists \mathcal{R} : \mathbf{Y} \to \mathbf{B}$, with $\mathcal{R} \circ \mathcal{C} = Id_{\mathbf{B}}$,
2. $(\mathbf{Y}, \|\|_{\mathbf{Y}})$ is a solid Banach space of sequences over I, with $\mathbf{c} \mapsto c_i$ being continuous from $\mathbf{Y}$ to $\mathbf{C}$ and solid, i.e. satisfying $\mathbf{z} \in \mathbf{Y}, \mathbf{x} : |x_i| \leq |z_i| \forall i \in I \Rightarrow \mathbf{x} \in \mathbf{Y}, \|\mathbf{x}\|_{\mathbf{Y}} \leq \|\mathbf{z}\|_{\mathbf{Y}}$;
3. finite sequences are dense in $\mathbf{Y}$ (at least W^*).

Corollary

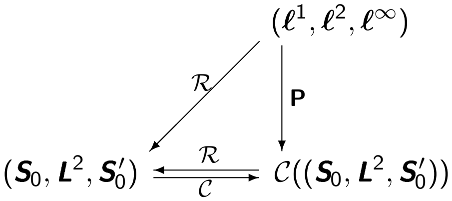
By setting $h_i := \mathcal{R}e_i$ we have $\mathcal{R}\mathbf{c} = \mathcal{R}(\sum e_i \mathbf{e}_i) = \sum_{i \in I} c_i h_i$ unconditional in $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, hence $f = \sum_{i \in I} T(f)_i h_i$ as unconditionally convergent series.

Frames described by a diagram

Similar to the situation for matrices of maximal rank (with row and column space, null-space of $\boldsymbol{A}$ and $\boldsymbol{A}'$) we have:
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Gelfand triples $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')$:



pline type Spaces: Reconstruction from Local Averages

thm

Theorem (Reconstruction from local averages, Aldroubi/Fei)

Let the weight m, the lattice Λ, and the generator ϕ in Wiener's algebra be given. Then there exists a density $\gamma = \gamma(\phi) > 0$ and $a_0 > 0$ such that any $f \in V_m^p(\phi)$ can be recovered from the data $\{\langle f, \psi_{x_j}\rangle : j \in J\}$ on any γ-dense set $X = \{x_j : j \in J\}$ and for any $0 < a < a_0$, by the following iterative algorithm :

$$f_1 = PA_X f, \quad f_{n+1} = PA_X(f - f_n) + f_n \ ,$$

where P is the operator onto $V_m^p(\phi)$. In this case, the iterate f_n converges to f in the $\mathbf{W}_m^p$-norm at a geometric rate, hence both in the $\mathbf{L}_m^p$-norm, and uniformly over compact sets. If furthermore $m(x) \geq 1$ for all $x \in \mathbb{R}^d$, then $\mathbf{L}_m^p \subset L^p(\mathbb{R}^d)$ and one has uniform convergence.

Riesz Projection bases for Spline-type spaces

Think of a transation invarant (say wavelet) closed subspace V with a Riesz (or even orthonormal) basis of the form $(T_\lambda \varphi)_{\lambda \in \Lambda}$. If φ is of some mild quality, namely $\varphi \in \boldsymbol{W}(Ł^2, \ell^1)$ then we have $\varphi * \varphi^* \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{S}_0(\mathbb{R}^d)$, hence the sampled autocorrelation function is in ℓ^1. The orthonormal projection from the Hilbert space $f \mapsto P_V$ is obtained by the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi.$$

But this mapping is not only well defined on $\boldsymbol{L}^2$, but also on a $\boldsymbol{L}^p$, for the full range of $1 \leq p \leq \infty$ and - again due to the properties of Wiener amalgams brings us for $f \in (\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ coefficients which are in $(\ell^1, \ell^2, \ell^\infty)$, which in turn implies that the function $\sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi$ is in $\boldsymbol{W}(\boldsymbol{C}^0, (\ell^1, \ell^2, \ell^\infty))$.

Riesz Projection bases for Spline-type spaces

Think of a transation invarant (say wavelet) closed subspace V with a Riesz (or even orthonormal) basis of the form $(T_\lambda \varphi)_{\lambda \in \Lambda}$. If φ is of some mild quality, namely $\varphi \in \boldsymbol{W}(Ł^2, \ell^1)$ then we have $\varphi * \varphi^* \in \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{S}_0(\mathbb{R}^d)$, hence the sampled autocorrelation function is in ℓ^1. The orthonormal projection from the Hilbert space $f \mapsto P_V$ is obtained by the mapping

$$f \mapsto \sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi.$$

But this mapping is not only well defined on $\boldsymbol{L}^2$, but also on a $\boldsymbol{L}^p$, for the full range of $1 \leq p \leq \infty$ and - again due to the properties of Wiener amalgams brings us for $f \in (\boldsymbol{L}^1, \boldsymbol{L}^2, \boldsymbol{L}^\infty)$ coefficients which are in $(\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty)$, which in turn implies that the function $\sum_{\lambda \in \Lambda} (\widetilde{\varphi} * f)(\lambda) T_\lambda \varphi$ is in $\boldsymbol{W}(\boldsymbol{C}^0, (\boldsymbol{\ell}^1, \boldsymbol{\ell}^2, \boldsymbol{\ell}^\infty))$.

REFERENCES

- H. G. Feichtinger and K. Gröchenig. Banach spaces related to integrable group representations and their atomic decompositions, I. *J. Funct. Anal.*, 86(2):307–340, 1989.
- E. Cordero, H. G. Feichtinger, and F. Luef. Banach Gelfand triples for Gabor analysis. In *Pseudo-differential Operators*, volume 1949 of *LN Math.*, pages 1–33. Springer, Berlin, 2008.
- Peter Balazs and Karlheinz Gröchenig. A guide to localized frames and applications to Galerkin-like representations of operators. In Isaac Pesenson, Hrushikesh Mhaskar, Azita Mayeli, Quoc T. Le Gia, and Ding-Xuan Zhou, editors, *Novel methods in harmonic analysis with applications to numerical analysis and data processing*, Appl. Numer. Harmonic Analysis series (ANHA). Birkhäuser/Springer, 2017.

Application to Gabor multipliers

Just an outline. **Clearly the description of Gabor systems (Gabor frames or Gaborian Riesz basic sequences) requires frame theory, in order to determine the dual frame resp. the dual Gabor atom $\widetilde{g}$ generating this dual Gabor frame. Reconstruction resp. atomic decomposition then describe the double usefulness of the dual Gabor atom. The Ron-Shen principle establishes an important link to Gaborian Riesz sequences (mobile communication).**

We also have a Banach Gelfand Triple of operators, namely $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$, **and via the** *kernel theorem* **identification with the space** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Gabor Multipliers

Starting from a tight Gabor atom we can study the properties of Gabor multipliers, i.e. weighted sums of projection operators on the family of Gabor atoms. Surprisingly these can be linear independent (especially at low redundancy) so that one can ask for the relationship between decay rate of coefficients and operator properties of those Gabor multipliers.

`Oral explanation of details!`

THANK YOU

Thank you for your attention

There is an upcoming book project (2019?) on this subject.

More at `www.nuhag.eu`

in particular at

`www.nuhag.eu/talks`