NuHAG Numerical Harmonic Analysis Group

Function spaces for time-frequency analysis: the usefulness of a Banach Gelfand Triple

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Key aspects of my talk

1. Browse the (long-standing) **history of Fourier Analysis**
2. Describing basic **time-frequency and Gabor analysis**
3. Which questions do we need to treat in this setting
4. Which function spaces are suited best
5. Definition and properties of **modulation spaces**
6. The Banach Gelfand-Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$

Probably to be modified later on!!

Overall it will be explained, that the distributional view-point is nowadays more important than the fine analysis of $\boldsymbol{L}^p$-spaces using Lebesgue integration methods. The setting of the *Banach Gelfand Triple* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ appears to be highly suitable for many applications.

There are many open questions related to time-frequency and Gabor analysis. In addition the computational side of Harmonic Analysis is not yet well integrated into the overall investigations in the area. Therefore the idea of *Conceptual Harmonic Analysis*, which includes (and integrates) both Abstract Harmonic Analysis and Numerical Harmonic Analysis, should be developed further.

The influence of the window length

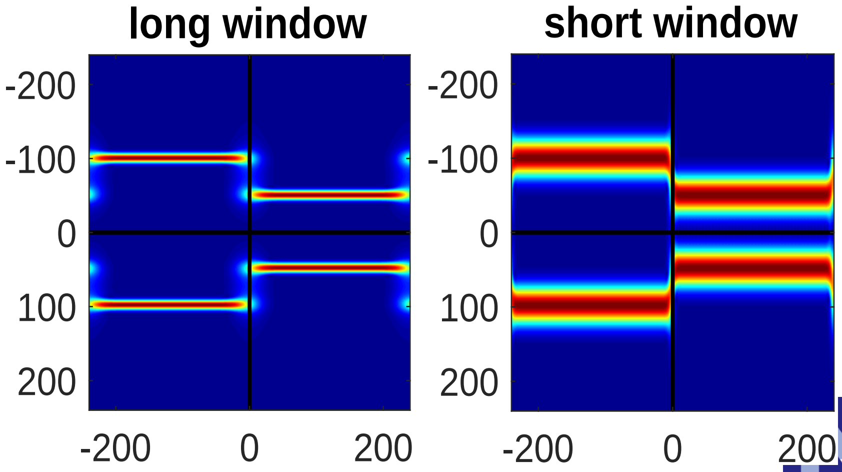


Figure: STFTshortlong1r.eps

But one can take a maximum over 3 windows

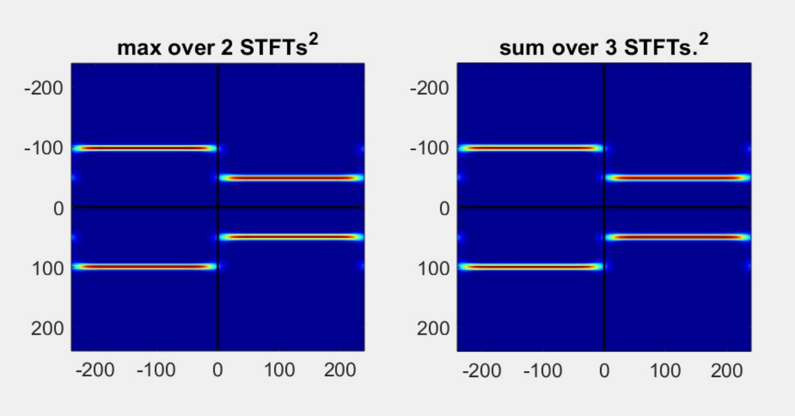


Figure: STFsumcum3.jpg CROPPED!!

... and remove small amplitudes by threshold

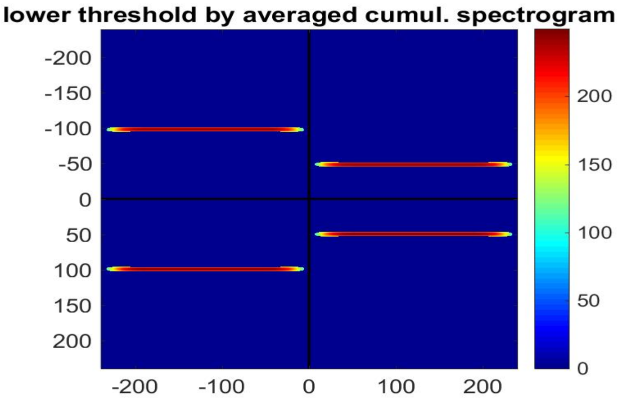


Figure: STFTmaxcumtresh.jpg

Given a lowpass signal and a Gaussian window...

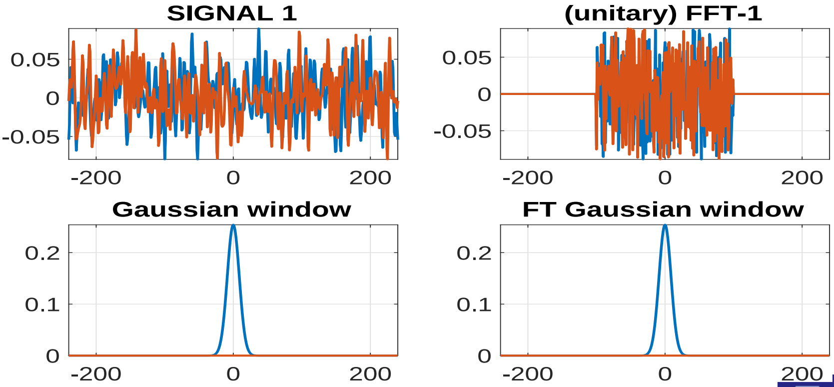
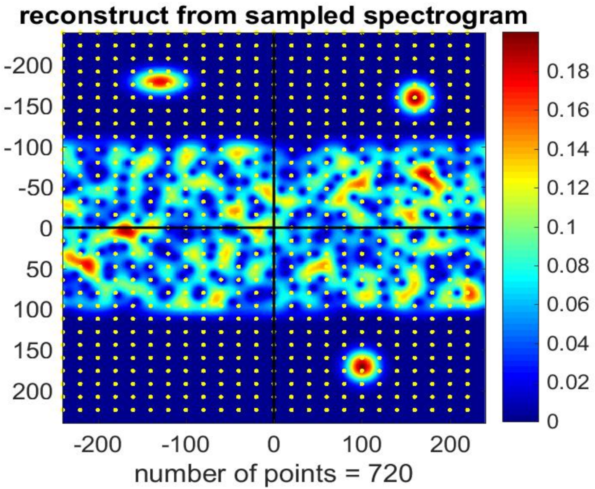


Figure: siglowgaus1.eps

Spectrogram (low pass signal) with lattice



Effect of time-frequency shift on functions

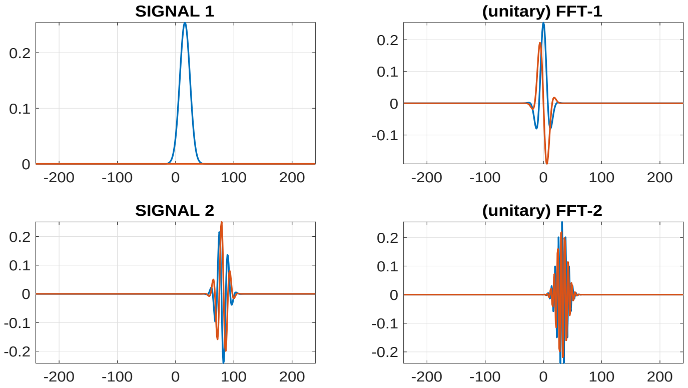


Figure: rotmoddemo1.eps

The discussion in the discrete case

The MATLAB experiments represent in a numerical way the linear algebra situation for signals (finite, discrete ones) over the group $\mathcal{G} = \mathbb{Z}_n$. We work with $n = 480$ and thus a lattice with lattice constants $a = 20$ and $b = 16$ (in the frequency direction), hence with $n/a = 24$ vertical columns with $n/b = 30$ points per column, providing us with $720 = n^2/(a * b)$ **Gabor atoms**, has a **redundancy** (factor) of $3/2$ (available vectors compared to the dimension of $\mathbb{C}^n = \ell^2(\mathbb{Z}_n)$).
Each of these vectors $g_\lambda = \pi(\lambda), \lambda \in \Lambda$ is a TF-shifted version of the original window, typically centered at zero both in time and frequency, often symmetric and real-valued. The natural choice is some (periodized and sampled) Gauss-function. Then the Gaussian STFT for these building blocks has a radial symmetric shape with the peak exactly at the given lattice point λ.

Representing elements of the Hilbert space $L^2(\mathbb{R}^d)$

This family of 720 vectors within $\mathbb{C}^n$ (with $n = 480$) generate this finite dimensional space if and only if the linear system, which is naturally formed in the search of coefficients is *consistent*. All the possible coefficients of this consistent system differ by some coefficient sequence in the null-space of the synthesis mapping (which ought to be 720-480 = 240-dimensional). Hence it makes sense to search for the solution of the MNLSQ-problem at hand, i.e. we search for the set of coefficents which is of minimal norm among all the coefficients which deliver a valid (norm-convergent) representation using the given *Gabor frame*.

It turns out that the mapping from signals to MNLSQ coefficients can be obtained by the so-called *dual Gabor family* , which is representing the `PINV`, the *pseudo-inverse* resp. Moore-Penrose inverse. We call the corresponding atom the *dual Gabor atom*, and use the symbol $\widetilde{g}$ for this (uniquely determined) function.

The Gabor frame operator

For the infinite dimensional situation the problem appears to be a bit harder, because it is not enough to know that the closed linear span of the atoms is the full Hilbert space (now $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$), but one requires an extra conditions, the so-called frame condition:

Definition

A Gabor family $(g_\lambda)_{\lambda\in\Lambda}$ is called a **Gabor frame** if there exist positive constants $A, B > 0$ such that

$$A\|f\|_2 \leq \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2 \leq B\|f\|_2, \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

We do not go into technical details, but want to just settle a few things. First of all the definition ensures that the range of the so-called *coefficient mapping* within $\ell^2(\mathbb{Z}^{2d})$ is closed!

Moore-Penrose inverse for Hilbert spaces

This *characteristic property* allows to apply the same geometric approach. The lower estimate implies that the *coefficient mapping*

$$C_{g,\Lambda} : f \to (\langle f, \pi(\lambda)g \rangle)_{\lambda \in \Lambda}$$

is injective. It establishes in fact an isomorphism between the range of this mapping and the domain, the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$. Once one has understood that the range of this mapping is just the orthogonal complement to the nullspace of synthesis mapping (the adjoint to the coefficient mapping), which is of the form

$$R = R_{g,\Lambda} : (c_\lambda)_{\lambda \in \Lambda} \mapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda,$$

mapping back from $\ell^2(\Lambda)$ to $\boldsymbol{L}^2(\mathbb{R}^d)$, one can understand the possible solution of the solution, namely the MNLSQ solution makes sense geomerically.

A long list of natural questions:Gabor frames

1. When is a given family $(g_\lambda)_{\lambda \in \Lambda}$ generating a **Gabor frame**?
2. When is it generating a **Gaborian Riesz basis**?
3. Are there Gaborian Riesz bases for $\boldsymbol{L}^2(\mathbb{R}^d)$?
4. How can one compute the Moore-Penrose inverse and is it of Gaborian form? (Yes!)
5. What can one say about the **canoncial dual window** $\widetilde{g}$?
6. What about *tight Gabor frames* (with $A = B$)?
7. Is there a continuous dependence of the dual window on the ingredients?

A long list of natural questions: Gabor multipliers

1. What are the properties of Gabor multipliers (with symbol $(\mathbf{m}(\lambda))_{\lambda\in\Lambda})$ $G_{g_1,g_2,\Lambda,\mathbf{m}}$?
2. Starting e.g.from tight Gabor families $(g_\lambda)_{\lambda\in\Lambda}$, what can one say about the eigenvalue behaviour?
3. Is the representation of a GM on the symbol unique?
4. How do ingredients change with the lattice?
5. What is the behaviour for increasing density
6. Which operators (and how) can be approximated by Gabor multipliers?
7. and many more questions.

The answers will make heavy use of the space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

We will see that most of the answers require the use of a certain **Banach space of test functions** which was found by the speaker in 1979 in the study of so-called **Segal algebras** on LCA groups. It was shown to be the smallest Segal algebra with the extra property of being isometric invariant under the modulation operators

$$M_s : f \mapsto \chi_s \cdot f, \quad \text{with } \chi_s(t) = e^{2\pi i s t}.$$

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

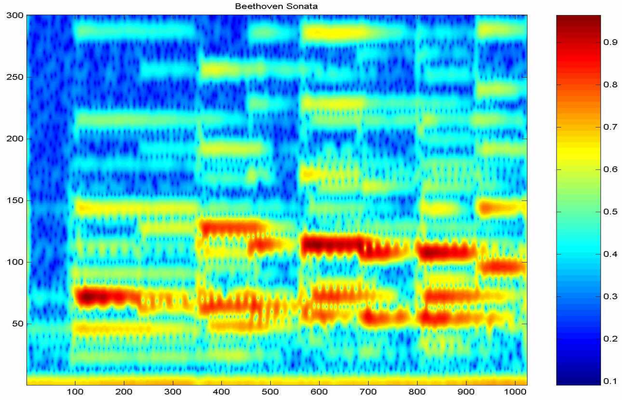
$$(T_x f)\widehat{\ } = M_{-x}\hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d))$, and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$*, then the following holds:*

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$*, and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$*, and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

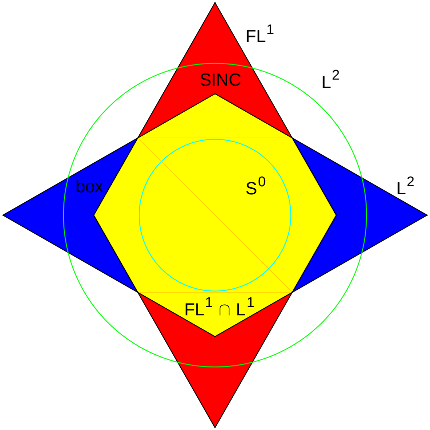


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\mathbf{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

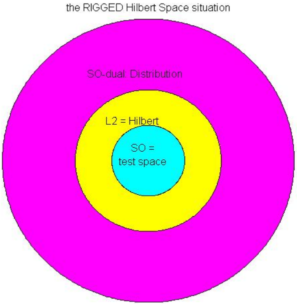
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{1}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

An alternative form of Admissibility

For any $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ reconstruction of any $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ from its short-time Fourier transform $V_g(f)$ is possible via V_g^*, the adjoint mapping, resp. the continuous integral version of the synthesis operator, but only in the *weak sense*. For example, one can obtain the coordinates of f in an ONB $(h_n)_{n\geq 0}$ by computing

$$\langle f, h_n \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \langle \pi(\lambda) g, h_n \rangle d\lambda = \int_{\mathbb{R}^{2d}} V_g(f)(\lambda) \overline{V_g(h_n)} d\lambda.$$

Since V_g is bounded from $\boldsymbol{L}^2(\mathbb{R}^d)$ to $\boldsymbol{L}^2(\mathbb{R}^{2d})$ Cauchy-Schwarz guarantees the existence of these integrals.

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (Fourier inversion) and the new formula formula While the building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters χ_s. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.

The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, in fact any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), hence any classical summability kernel is OK: see Ferenc Weisz: Inversion of the short-time Fourier transform using Riemannian sums for example (2007).

An alternative form of Admissibility

The above situation suggests that one might be able to use *Riemannian sums* to these integrals in order to recover f (or $\langle f, h_n \rangle$) approximately, but even for the boundedness of $R_{g,\Lambda}$ resp. $C_{g,\Lambda}$ one needs extra assumptions on g. It does not make sense to ask for this boundedness individually, both respect to Λ and g, but rather look of a simple and *universal* answer:

Lemma

Assume that g belongs to $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, then the operators $C_{g,\Lambda}$ and hence its adjoint $R_{g,\Lambda}$ are uniformly bounded by constants depending only on the size of the fundamental domain of Λ and $\|g\|^2_{\boldsymbol{S}_0}$. Normalized appropriately one can even have uniform boundedness over all lattices Λ.

Chaos otherwise

According to a result by F/Janssen from 2000: feja00 H. G. Feichtinger and A. J. E. M. Janssen. Validity of WH-frame bound conditions depends on lattice parameters. *Appl. Comput. Harmon. Anal.*, 8(1):104–112, 2000.

one can have boundedness for all rational lattices and unboundedness for all rational multiples of some irrational lattice, and in fact *not even locally (in the domain of lattice parameters) uniform boundedness* (!) over the rational ones.
Such a problem cannot happen for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$, because then the so-called Bessel bounds (even as operators between Banach Gelfand triples) are uniformly bounded over "compact sets" of lattices.

The Gaussian case, $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$

It is generally true, even for $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ and not just the Gauss function $g_0(t) = e^{-\pi|t|^2}$, that a lattice of the form will never create a Gabor frame (lack of density of atoms, the *undersampling (!) case*), if $a \cdot b > 1$ OR $a \cdot b > 1$. So one may hope for the case of $a \cdot b < 1$ one can prove that for this case one has Gabor frames. This is in fact true as it is known for a long time. It is also true that the dual window $\widetilde{g}$, has to be a Schwartz function, even it might not look like this.
Recently Gröchenig & Stöckler have shown that a an infinite, parameterized family of functions of *totally positive type* shares this property with the Gauss function.
Surprisingly B-spline functions, which are compactly supported, require not only the natural restrictions on the time-shift parameter $a > 0$ but also cannot generate Gabor frames for specific lattices with $a \cdot b < 1$ (see J. Lemvig). e

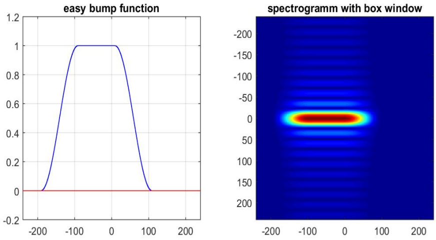
Is there an orthonormal Gaborian Basis for $(L^2(\mathbb{R}), \|\cdot\|_2)$?

Yes of course. One can take the indicator function $\mathbf{1}_{[0,a]}$ for any $a > 0$, take all of its translates along $a\mathbb{Z}$ and then do a Fourier expansions (in the sense of a-periodic functions) of each of the pieces.

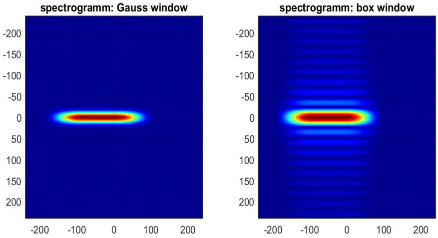
But how does the spectrogram (STFT) of a nice function are rather broad (spread out in the frequency direction, no integrable, even if the signal f is a nice bump function!). This is due to the bad decay properties of $\mathcal{F}(box) = SINC$ in this case.

Bad spread of spectrogram in frequency direction

This slide reminds us, why it is not a good idea to use the indicator function of some interval as Gabor atom!



Ideal concentration for Gaussian Window



The Ron-Shen Duality Theory

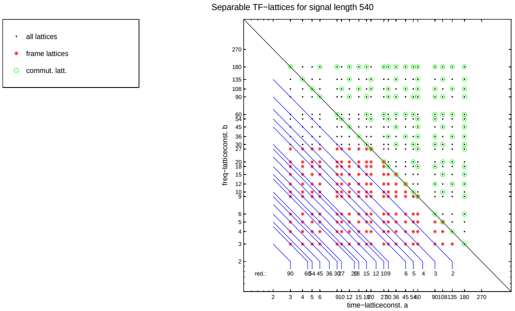


Figure: lower part: small lattice parameters > frames, red > 1; upper part: large frame constants > Gaborian Riesz bases, In the middle: **critical line, redundancy = 1.**

The Ron-Shen duality explained

The Ron-Shen duality (later formulated for general lattices by Feichtinger-Kozek) states:
If there is a lattice such that the regular Gabor system generated from (g, Λ), where g is the Gabor atom, is a Gabor frame, then (and only then) is the adjoint Gabor system (g, Λ°) a Gaborian Riesz basis. Moreover the dual Gabor atom (resp. the generator of the biorthogonal Gaborian RBS) are the same (up to normalization), and furthermore the condition number of the frame operator and the Gram matrix (of the RBS) are the same.
Let us mention that for $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$ one has $\Lambda^\circ = 1/b \cdot \mathbb{Z}^d \times 1/a \cdot \mathbb{Z}^d$.

So what are we looking for?

Within our landscape of Gabor lattices we look out for lattices of *not too high redundancy* which allows us to build good Gabor frames (with well TF-concentrated dual window, or even better well TF-concentrated tight Gabor frames) which do not have too high redundancy, i.e. corresponding to lattices near the critical line. For mobile communication we search for Gaborian Riesz basis of "high spectral efficiency" (so high redundancy, coming close to the critial line from above, while again still having good biorthogonal generators).
Of course in each case one can also consider (the rich family) of non-separable lattices, i.e. general lattices within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$, not just those of the form $a\mathbb{Z}^d \times b\mathbb{Z}^d$.

What is the problem with Gabor's suggestion?

Formally the technical problem with Gabor's idea of using a maximally TF-localized window (namely the Gauss function g_0, with $g_0 = e^{-\pi|t|^2}$, which is a minimizer to the Heisenberg uncertainty relation) is the Balian- Low theorem. In fact, while *most likely*, formulated in a modern terminology, D. Gabor was *hoping* to suggest a Riesz basis obtained from a family of TF-shifts of the Gauss-function along the integer lattice $\mathbb{Z}^2$, i.e. with $a = 1 = b$, the analysis in the 80th showed that it is neither a frame nor a Riesz basic sequence, so of course *not a Riesz basis*. What has been overlooked by D. Gabor (at least there is no indication that he was aware of this problem) that the more one comes to the critical lattice (e.g. by letting $a = b$ tend to the critical value $a = 1$) the more delocalized (in the TF-sense) the dual window is, i.e. the optimal localization of the Gabor atoms is in sharp contrast with the significant unsharpness of the overall system (Gabor and dual Gabor frame!).

Bastiaan's dual window γ

The usual formulations of early Gabor analysis suggested that Gabor analysis was that Gabor expansions are potentially quite important (because they provide information about the TF-concentration of signals, concentrated to the Neumann-lattice). The fact, that for $\Lambda = \mathbb{Z}^{2d}$ the closure of the Gabor family generated by the (symmetric!) Gauss-function within $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ has co-dimension ≥ 1 is an indication that it is not just numerical instability (as it was believed for a long time), but inappropriate for the expansion of arbitrary signals.
Recall, in the $\boldsymbol{L}^2$-setting they form a *total family* which did not look so bad.

Continuous dependence of Gabor atoms on the lattice

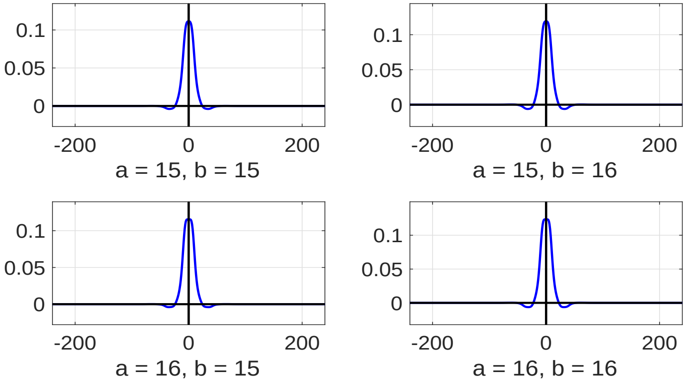


Figure: Dual Gabor windows $\widetilde{g}$ for (a, b): $(15, 15)$,$(15, 16)$, $(16, 15)$, or $(16, 16)$, are very similar, taking variable redundancy into account. Even in $\boldsymbol{S}_0$-norm only ca. $7, 5\%$ relative error.

Dual windows are automatically in $S_0(\mathbb{R}^d)$

It is well known, that a linear mapping between vector spaces which is invertible (as a mapping between sets) has an inverse which is automatically linear.
If furthermore domain and target spaces are Banach spaces the ordinary invertibility combined with boundedness of the forward mapping automatically implies boundedness of the inverse mapping (hence one has an isomorphism).
One of the deep and somehow surprising results in Gabor analysis is the following obtained by Gröchenig/Leinert (AMJ, 2004):

Theorem

Assume that (g, Λ) generates a Gabor frame with atom $g \in \mathbf{S}_0(\mathbb{R}^d)$., i.e.assume that the Gabor frame operator $S_{g,\Lambda}$ is invertible on $\left(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Then it is a Banach Gelfand Triple isomorphism. Hence $\widetilde{g} \in \mathbf{S}_0(\mathbb{R}^d)$ as well.

Robustness under small changes of lattice parameters

One of the surprising results in Gabor analysis is the robustness towards changes in the window *AND THE LATTICE!*. It is not surprising to find that one may allow $\boldsymbol{S}_0$-perturbation of a window $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ which generates a Gabor (g, Λ), because clearly the invertibility of the frame operator (on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$) is of course robust to modifications of an operator *with respect to the operator norm on* $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$. Since the $\boldsymbol{S}_0$-norm of an atom controls both the norm of the analysis and the synthesis operator changes of the window only change the frame operator.

One version (still relying on the control of the operator in the operator norm) is the following: Assume that every lattice points is modified by a small amount (*jitter error*). For small jitter the resulting family is still a Gabor frame and is the original dual still an *approximate dual family*.

Continuous dependence of Gabor atoms on the lattice II

The situation is quite different if one changes the lattice, because then the difference (in the operator norm) between to very similar lattices is not controllable by a simple jitter error, and the change is in fact not continuous in the operator norm, but only in the strong operator topology.
Whereas the orthonormal basis arising from $g = \mathbf{1}_{[0,1)}$ with $\Lambda = \mathbb{Z}^d$ is highly sensitive to any change of the lattice constants the $\boldsymbol{S}_0$-setting is quite convenient.

Theorem

Assume that (g, Λ_0), with $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ generates a Gabor frame. Then there is an open set of lattices containing Λ_0 such that (g, Λ) generates Gabor frames for Λ near Λ_0 (matrix convergence). Moreover, the dual atom depends continuously (in the $\boldsymbol{S}_0$-norm) on the lattice.

Functions, Distributions, Signal Expansions

As a *unifying principle* that allows me to explain the relevant points in the historical development of Fourier Analysis in the last 200 years as well as for a better understanding of how we should teach finally Fourier Analysis in the 21st century I want to focus on the following aspects:

1. What is a function?
2. What does it mean to represent a function on the basis of its Fourier coefficients (e.g. Fourier series expansion, ...)
3. How have these concepts changed over time and what was the effect on the understanding of Fourier Analysis?

Plancherel's Theorem: Unitarity Property of FT

Using the density of $\mathbf{L}^1(\mathbb{R}) \cap \mathbf{L}^2(\mathbb{R})$ in $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ it can be shown that the Fourier transform extends an a natural and unique way to $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$:

Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $\left(\mathbf{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, i.e. one has

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \mathbf{L}^2(\mathbb{R}),$$

$$\langle f, g\rangle = \langle \hat{f}, \hat{g}\rangle, \quad f, g \in \mathbf{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the "continuous representation" formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{2}$$

This looks like a perfect orthogonal expansion, but unfortunately the "building blocks" $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Topics of Abstract Harmonic Analysis

The central theme of Harmonic Analysis (according to my advisor Hans Reiter [1921-1992]) was the study of the Banach algebra $(\boldsymbol{L}^1(G), \|\cdot\|_1)$, in particular the structure of closed ideals. One of the central questions is the question of **spectral synthesis**.
This is a rather involved topic, roughly described as follows:
Can one approximate - in a suitable weak sense - a function f from finite linear combinations of pure frequencies of those frequencies which are found "in the signal" f via spectral analysis?
In other words, one considers only this frequencies χ_s, such that s belongs to the support of $\widehat{f}$ (i.e. s can be approximated by values s_n with $\widehat{f}(s_n) \neq 0$). Then one expect to approximate f weakly by trigonometric polynomials $t_k(x) = \sum_{k=1}^{n} c_k \chi_{s_k}$. The failure of spectral synthesis for $\mathbb{R}^3$ is due to L.Schwartz [1915 - 2002].

Laurent Schwartz Theory of Tempered Distributions

Laurent Schwartz is mostly known for having introduced the space of tempered distributions, a topological vector space of **generalized functions** or **distributions** which is invariant under the Fourier transform.

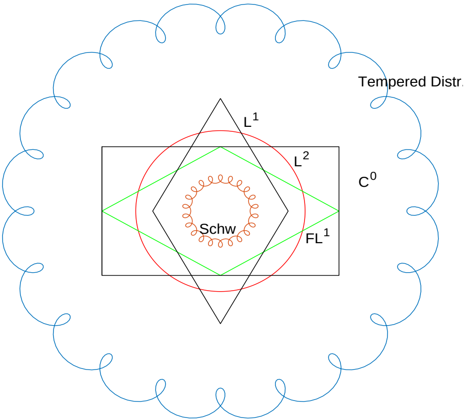
He starts out by defining the so-called *Schwartz space of rapidely decreasing functions*, consisting of all infinitely differentiable functions on $\mathbb{R}^d$ which decay faster at infinity than any polynomial.

This space $\mathcal{S}(\mathbb{R}^d)$ is naturally endowed with a countable family of semi-norms, turning the space into a *nuclear Frechet space*.

The topological dual of $\mathcal{S}(\mathbb{R}^d)$, i.e. the collection of all linear functionals σ on $\mathcal{S}(\mathbb{R}^d)$ satisfying the continuity assumption $f_n \to f_0$ in $\mathcal{S}(\mathbb{R}^d)$ implies $\sigma(f_n) \to \sigma(f_0)$ in $\mathbb{C}$, constitutes $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

The classical setting of test functions & distributions

$$(L^1(\mathbb{R}^d), \|\cdot\|_1), (L^2(\mathbb{R}^d), \|\cdot\|_2), (C_0(\mathbb{R}^d), \|\cdot\|_\infty), \mathcal{S}(\mathbb{R}^d), \mathcal{S}'(\mathbb{R}^d), (\mathcal{F}L^1($$



Fourier Transforms of Tempered distributions

The Fourier transform $\hat{\sigma}$ of $\sigma \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ is defined by the simple relation

$$\hat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

His construction *vastly extends the domain of the Fourier transform* and allows even polynomials to have a Fourier tranform. Among the objects which can now be treated are also the Dirac measures δ_x, as well as Dirac combs $⧢ = \sum_{k \in \mathbb{Z}^d} \delta_k$.
Poisson's formula, which expresses that one has for $f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n), \qquad (3)$$

can now be recast in the form

$$\widehat{⧢} = ⧢$$

Sampling and Periodization on the FT side

The convolution theorem, can then be used to show that sampling corresponds to periodization on the Fourier transform side, with the interpretaton that

$$\sqcup\!\sqcup \cdot f = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

In fact, we have

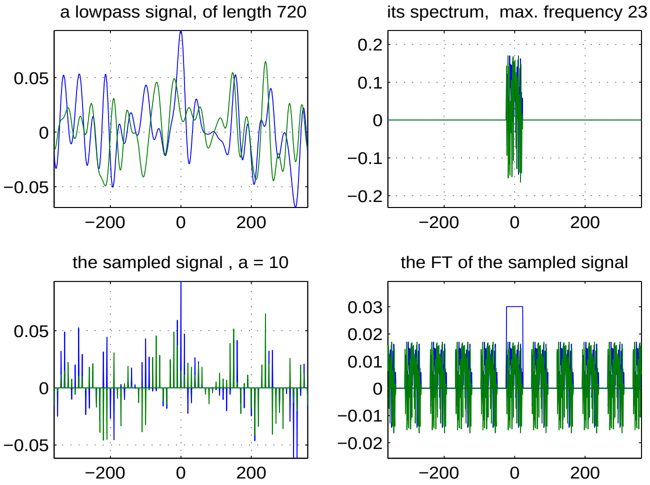
$$\widehat{\sqcup\!\sqcup \cdot f} = \widehat{\sqcup\!\sqcup} * \widehat{f} = \sqcup\!\sqcup * \widehat{f}.$$

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).

Recovery from Samples

If we try to recover a real function $f : \mathbb{R} \to \mathbb{R}$ from samples, i.e. from a sequence of values $(f(x_n))_{n \in I}$, where I is a finite or (countable) infinite set, we cannot expect perfect reconstruction. In the setting of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ any sequence constitutes only set of measure zero, so knowing the sampling values provides *zero information* without side-information.
On the other hand it is clear the for a *(uniformly) continuous* function, so e.g. a continuous function supported on $[-K, K]$ for some $K > 0$ piecewise linear interpolation (this is what MATLAB does automatically when we use the PLOT-routine) is providing a good (in the uniform sense) approximation to the given function f as long as the maximal distance between the sampling points around the interval $[-K, K]$ is small enough.
Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.

A Visual Proof of Shannon's Theorem

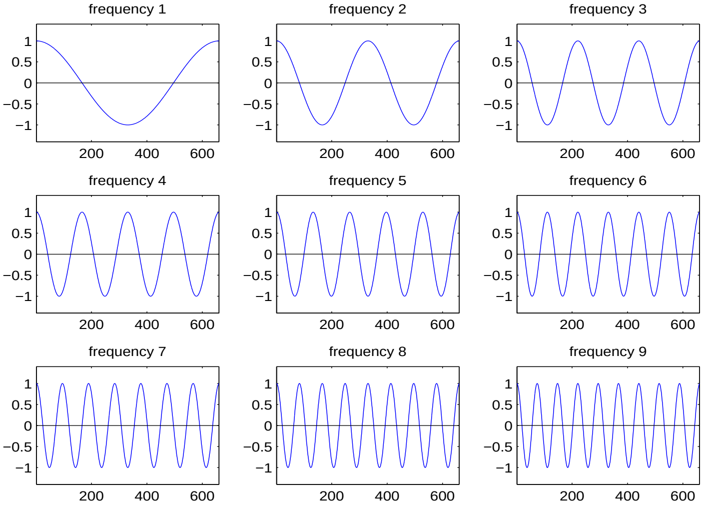


Shannon's Sampling Theorem

It is kind of clear from this picture that one can recover the spectrogram of the original function by isolating the central copy of the periodized version of $\widehat{f}$ by multiplying with some function $\hat{g}$, with g such that $\hat{g}(x) = 1$ on spec(f) and $\hat{g}(x) = 0$ at the shifted copies of $\widehat{f}$. This is of course only possible if these shifted copies of spec(f) do not overlap, resp. if the sampling is dense enough (and correspondingly the periodization of $\widehat{f}$ is a course one. This conditions is known as the *Nyquist criterion*. If it is satisfied, or supp$(f) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) T_{\alpha k} g(x), \quad x \in \mathbb{R}^d.$$

Building blocks for Discrete Cosine Transform DCT



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as 5/7 is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

The Banach space $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ or $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$), which will serve our purpose. Its dual space $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(S_0, L^2, S_0')(\mathbb{R}^d)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

Behavior under Fourier transform

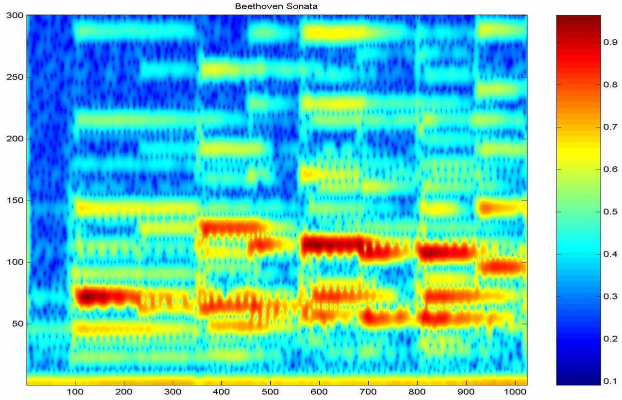
$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \ \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallestspace with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

$\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is also invariant under automorphism, such as dilations or rotations (but not isomerically), and even under the "Fractional Fourier Transform". and metaplectic transformations.

Basic properties of $M^\infty(\mathbb{R}^d) = S_0'(\mathbb{R}^d)$, Ctd.

It is probably no surprise to learn that the dual space of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$, i.e. $S_0'(\mathbb{R}^d)$ is the *largest* (reasonable) Banach space of distributions (in fact local pseudo-measures) which is isometrically invariant under time-frequency shifts $\pi(\lambda)$, $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. As an amalgam space one has $S_0'(\mathbb{R}^d) = W(\mathcal{F}L^1, \ell^1)' = W(\mathcal{F}L^\infty, \ell^\infty)(\mathbb{R}^d)$, the space of translation bounded quasi-measures, however it is much better to think of it as the modulation space $M^\infty(\mathbb{R}^d)$, i.e. the space of all tempered distributions on $\mathbb{R}^d$ with bounded Short-time Fourier transform (for an arbitrary $0 \neq g \in S_0(\mathbb{R}^d)$).
Consequently norm convergence in $S_0'(\mathbb{R}^d)$ is just uniform convergence of the STFT, while certain atomic characterizations of $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$ imply that w^*-convergence is in fact equivalent to locally uniform convergence of the STFT.

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

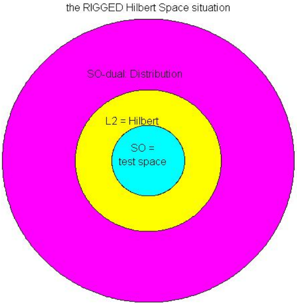
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

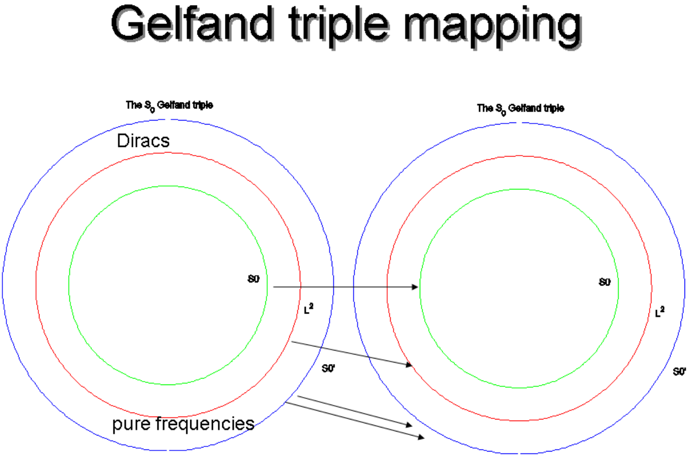
1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



A pictorial presentation of the BGTr morphism



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

Banach Gelfand Triple (S_0, L^2, S_0'): BASICS over $\mathbb{R}^d$

Let us collect a few facts concerning this Banach Gelfand Triple (BGTr), based on the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$:

- $S_0(\mathbb{R}^d)$ is dense in $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, in fact within any $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p < \infty$ (or in $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$);
- Any of the L^p-spaces, with $1 \leq p \leq \infty$ is continuously embedded into $S_0'(\mathbb{R}^d)$;
- Any translation bounded measure belongs to $S_0'(\mathbb{R}^d)$, in particular any Dirac-comb $⧢_\Lambda := \sum_{\lambda \in \Lambda} \delta_\lambda$, for $\Lambda \lhd \mathbb{R}^d$.
- $S_0(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any $\sigma \in S_0'(\mathbb{R}^d)$ there exists a sequence of test functions s_n in $S_0(\mathbb{R}^d)$ such that

$$(1) \quad \int_{\mathbb{R}^d} f(x) s_n(x) dx \to \sigma(f), \quad \forall f \in S_0(\mathbb{R}^d). \qquad (4)$$

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \tag{5}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Interpretation of key properties of the Fourier transform

Engineers and theoretical physicists tend to think of the Fourier transform as a change of basis, from the continuous, orthonormal system of Dirac measures $(\delta_x)_{x\in\mathbb{R}^d}$ to the CONB $(\chi_s)_{s\in\mathbb{R}^d}$. Books on quantum mechanics use such a terminology, admitting that these elements are "slightly outside the usual Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$", calling them "elements of the *physical Hilbert space*" (see e.g. R. Shankar's book on Quantum Physics). Within the context of BGTs we can give such formal expressions a meaning: The Fourier transform maps pure frequencies to Dirac measures:

$$\widehat{\chi_s} = \delta_s \quad \text{and} \quad \widehat{\delta_x} = \chi_{-x}.$$

Given the w^*-totality if both of these systems within $\boldsymbol{S}_0'(\mathbb{R}^d)$ we can now claim: The Fourier transform is the *unique* BGT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with this property!

There is just one Fourier transform

As a colleague (Jens Fischer) at the German DLR (in Oberpfaffenhausen) puts it in his writing: "There is just one Fourier Transform" And I may add: and it is enough to know about $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ in order to understand this principle and to make it mathematically meaningful.

In **engineering courses** students learn about discrete and continuous, about periodic and non-periodic signals (typically on $\mathbb{R}$ or $\mathbb{R}^2$), and they are treated separately with different formulas. Finally comes the DFT/FFT for finite signals, when it comes to computations. The all look similar.

Mathematics students learning Abstract Harmonic Analysis learn that one has to work with different LCA groups and their dual groups. Gianfranco Cariolaro (Padua) combines the view-points somehow in his book Unified Signal Theory (2011).

w^*-convergence justifies the various transitions!

Periodicity and Fourier Support Properties

The world of distributions allows to deal with continuous and discrete, periodic and non-periodic *signals* at equal footing. Let us discuss how they are connected.

The general Poisson Formula, expressed as

$$\mathcal{F}(\sqcup\!\sqcup_{\Lambda}) = C_{\Lambda} \sqcup\!\sqcup_{\Lambda^{\perp}} \tag{6}$$

can be used to prove

$$\mathcal{F}(\sqcup\!\sqcup_{\Lambda} * f) = C_{\Lambda} \sqcup\!\sqcup_{\Lambda^{\perp}} \cdot \mathcal{F}(f), \tag{7}$$

or interchanging convolution with pointwise multiplication:

$$\mathcal{F}(\sqcup\!\sqcup_{\Lambda} \cdot f) = C_{\Lambda} \sqcup\!\sqcup_{\Lambda^{\perp}} * \mathcal{F}(f). \tag{8}$$

I.e.: Convolution by $\sqcup\!\sqcup$ (corresponding to *periodization*) corresponds to pointwise multiplication (i.e. *sampling*) on the Fourier transform domain and *vice versa*.

Approximation by discrete and periodic signals

The combination of two such operators, just with the assumption that the sampling lattice Λ_1 is a subgroup (of finite index N) of the periodization lattice Λ_2 implies that

$$\sqcup\!\sqcup_{\Lambda_2} * [\sqcup\!\sqcup_{\Lambda_1} \cdot f] = \sqcup\!\sqcup_{\Lambda_1} \cdot [\sqcup\!\sqcup_{\Lambda_2} * f], \quad f \in \boldsymbol{S}_0(\mathbb{R}^d). \qquad (9)$$

For illustration let us take $d = 1$ and $\Lambda_1 = \alpha\mathbb{Z}$, $\Lambda_2 = N\alpha\mathbb{Z}$ and hence $\Lambda_1^\perp = (1/\alpha)\mathbb{Z}$. Then the periodic and sampled signal arising from equ. 9 corresponds to a vector $\mathbf{a} \in \mathbb{C}^N$ and the distributional Fourier transform of the periodic, discrete signal is completely characterized is again discrete and periodic and its generating sequence $\mathbf{b} \in \mathbb{C}^N$ can be obtained via the DFT (FT of quotient group), e.g. $N = k^2, \alpha = 1/k$, and period k.

Approximation by discrete and periodic signals 2

It is not difficult to verify that in this way, by making the sampling lattice more and more refined and periodization lattice coarser and coarser the resulting discrete and periodic versions of $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, viewed as elements within $\boldsymbol{S}_0'(\mathbb{R}^d)$, are approximated in a bounded and w^*-sense by discrete and periodic functions.
This view-point can be used as a justification of the fact used in books describing heuristically the continuous Fourier transform, as a limit of Fourier series expansions, with the *period going to infinity*.

Mutual w^*-approximations

The density of test functions in the dual space can be obtained in many ways, using so-called *regularizing operators*, e.g. combined approximated units for convolution and on the other hand for pointwise convolution, based on the fact that we have

$$(\boldsymbol{S}_0(\mathbb{R}^d) * \boldsymbol{S}_0'(\mathbb{R}^d)) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d), \quad \text{and} \tag{10}$$

$$(\boldsymbol{S}_0(\mathbb{R}^d) \cdot \boldsymbol{S}_0'(\mathbb{R}^d)) * \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d). \tag{11}$$

Alternatively one can take finite partial sums of the Gabor expansion of a distribution $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ which approximate σ in the w^*-sense (boundedly), for Gabor windows in $\boldsymbol{S}_0(\mathbb{R}^d)$.

On the other hand one can approximate test functions (in the w^*-sense) by discrete and periodic signals!

Approximation of Distributions by Test Functions

These properties of *product-convolution operators* or *convolution-product operators* can be used to obtain a w^*-approximation of general elements $\sigma \in \mathbf{S}_0'(\mathbb{R}^d)$ by test functions in $\mathbf{S}_0(\mathbb{R}^d)$. For example, one can take a Dirac family obtained by applying the compression operator

$$\mathrm{St}_\rho(g) := \rho^{-d} g(x/\rho), \quad \rho \to 0$$

in order to approximate σ by bounded and continuous functions of the form $\mathrm{St}_\rho(g_0) * \sigma$.
For the localization one can use the dilation operator

$$\mathrm{D}_\rho(h)(z) = h(\rho z), \quad \rho \to 0,$$

so altogether

$$\sigma = w^* - \lim_{\rho \to 0} \mathrm{D}_\rho g_0 \cdot [(\mathrm{St}_\rho g_0) * \sigma]$$

where all the functions on the right hand side belong to $\mathbf{S}_0(\mathbb{R}^d)$.

Generalized Stochastic Processes

The space of test functions is also very useful to model **Generalized Stochastic Processes** (GSPs) simple as bounded linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to some (abstract, or concrete) Hilbert space (of random variables): $\rho : f \to \rho(f) \in \mathcal{H}$.

Such GSPs have a natural *autocorrelation distribution* $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$, and its invariance properties correspond to e.g. wide-sense stationarity of the process itself.

There is also a Fourier transform $\hat{\rho}$ of such a process, and the autocorrelation of the $\hat{\rho}$ is just (the $2d$) Fourier transform of σ! The inverse Fourier transform is a very natural replacement for the "spectral representation" of a process.

Details can be found in paper with W. Hörmann (see his PhD thesis).

Modern Viewpoint I

Todays Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \mathcal{S}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal f resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda)g \rangle$.
We speak of *tight Gabor frames* (g_λ) if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!!

Modern Viewpoint II

Another basic fact is that for each $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ one can find, if Λ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\widetilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \qquad (12)$$

$\widetilde{g}$ can be found as the solution of the (positive definite) linear system $S\widetilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\widetilde{g}$ instead of g for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.
Thus (12) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).