Fourier Analysis in the 21st century

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The 23rd Srinivasa Rajan Memorial Lecture

Chennai, Univ. Madras, January 22nd, 2018

Key aspects of my talk

1. Browse the (long-standing) **history of Fourier Analysis**
2. Show large number of **applications** influencing our life
3. Discussing some of the mathematics behind it
   (take away the touch of mystery?)
4. Describing **time-frequency and Gabor analysis**
5. Suggesting ways to teach Fourier Analysis
   in the 21st century (outlook)

Personal Background

- Starting as a teacher student math/physics, Univ. Vienna
- PhD and habilitation (1974/1979)
  in **Abstract Harmonic Analysis**
- Establishing **NuHAG** (Numerical Harmonic Analysis Group)
- Reach out for applications (communication theory, image processing, astronomy, medicine, musicology,...)
- European projects (Marie Curie and EUCETIFA)
- Main interest: Function spaces, Fourier Transform
- Nowadays: formally retired, but teaching at ETH, DTU, TUM, with the goal of supporting the applied sciences
- As editor to JFAA also the perspective is sharpend.

Functions, Distributions, Signal Expansions

As a *unifying principle* that allows me to explain the relevant points in the historical development of Fourier Analysis in the last 200 years as well as for a better understanding of how we should teach finally Fourier Analysis in the 21st century I want to focus on the following aspects:

1. What is a function?
2. What does it mean to represent a function on the basis of its Fourier coefficients (e.g. Fourier series expansion, ...)
3. How have these concepts changed over time and what was the effect on the understanding of Fourier Analysis?
4. Which function spaces are relevant for a proper description of Fourier and time-frequency analysis?

Fourier history of in a nut-shell

1. 1822: J.B.Fourier proposes: Every periodic function can be expanded into a Fourier series using only pure frequencies;
2. up to 1922: concept of functions developed, set theory, Lebesgue integration, $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$;
3. first half of 20th century: Fourier transform for $\mathbb{R}^d$;
4. A. Weil: Fourier Analysis on Locally Compact Abelian Groups;
5. L. Schwartz: Theory of Tempered Distributions
6. Cooley-Tukey (1965): FFT, the Fast Fourier Transform
7. L. Hörmander: Fourier Analytic methods for PDE (Partial Differential Equations);

The Life of Fourier: 1768 - 1830

https://en.wikipedia.org/wiki/Joseph_Fourier

Classical Fourier Series

The classical approach to the theory of *FOURIER SERIES* appears in the following form: Looking at the partial sums of the (formally then infinite) Fourier series we expect them to approximate "any periodic function" in **some sense**[1]:

$$s_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} [a_n \cos(2\pi nx) + b_n \sin(2\pi nx)]. \qquad (1)$$

Assuming this is possible it is not so hard to find out, using the properties of the building blocks $(\cos(x), \sin(x)$, addition rules, derivatives, integration) that one can expect for any $z \in \mathbb{R}$:

$$a_n = \int_z^{z+1} f(x)\cos(2\pi nx)dx, \;\; b_n = \int_z^{z+1} f(x)\sin(2\pi nx)dx. \qquad (2)$$

[1] For simplicity we assum period 1!

What are the Ingredients and Questions 1

In my course on Fourier series I was taught (like many classical talks) that the representation (viewed for individual functions)

$$f(x) \approx \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(2\pi n x) + b_n \sin(2\pi n x)]. \qquad (3)$$

Fou

should be taken only as a "formal expression", which has to be formalized using various kinds of mysterious tricks! Despite their widespread use, we say:

But what does this mean?

What kind of concrete, mathematical questions should be asked? Why and how are summability methods saving the situation, and in which sense?

Until now Fourier series are seen as a mystery!

What are the Ingredients 2

First of all we have to note that by the time (1822!!, which was at the life-time of Carl Friedrich Gauss! [1777-1855]) the modern **concept of a function** was not available as it is now. Thanks to Leonhard Euler ([1707 - 1783]) the complex numbers and their connection to trigonometric functions had been known

$$e^{ix} = \cos(x) + i \sin(x), \quad i = \sqrt{-1}. \tag{4}$$

Eul

It was known what *polynomials* are and how to compute with them, and even to take "polynomial of infinite degree" (power series, with well defined regions of uniform convergence), hence **Taylor expansions** were known (going back to the English mathematician Brook Taylor [1685-1731]).

What are the Ingredients 3: the Integral

Of course the determination of the coefficients using **integrals** (over the period of the involved functions) is one of the corner-stones of the classical theory, raising some questions:

- What is the meaning of an integral *in the most general case*?
- What kind of functions can be integrated (over $[a, b]$?
- What can be said about the Fourier coefficients $(a_n)_{n \geq 0}$ or $(b_n)_{n \geq 1}$?

While the foundations of "Calculus" had been laid down by Isaac Newton [1642 - 1726] and Gottfried Wilhelm Leibniz [1646 - 1716] long before Fourier, it was **Bernhard Riemann** [1826 - 1866] who gave a clean definition and showed that e.g. every *continuous function* can be integrated over any interval $[a, b]$. He showed that the Fourier coefficients tend to zero $(n \to \infty)$.

What are the Ingredients 4: the Timeline

Isaac Newton [1642 - 1726]
Gottfried Wilhelm Leibniz [1646 - 1716]

AFTER FOURIER

Bernhard Riemann [1826 - 1866]
Karl Weierstrass [1815-1897]
Henri Leon Lebesgue [1875 - 1941]
Norbert Wiener [1894 - 1964]
Andre Weil [1906 1998]

What are the Ingredients 5: Perfect Integrals

By the beginning of the 20th century **Henri Leon Lebesgue** had developed his integral, and also given lectures on the application of this new techniques to trigonometric series.
He published a number of important papers between 1904 and 1907 in this direction.
From a modern (functional analytic) view-point his integral, which included the definition of the so-called *Lebesgue spaces* such as $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ or $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ (and of course later the $\boldsymbol{L}^p$-theory, duality etc.) opened the way to the field of (linear) **functional analysis**, which developed rapidly, the foundations being lead by e.g. David Hilbert [1982 - 1943], Friedrich Riesz [1880 - 1956] and Stefan Banach [1892 - 1945].

Convergence Issues: Pointwise

So let us return to the question of convergence: The key question being: **In which sense do the partial sums converge?**

In fact, it turned out that a more general problem appeared: What does convergence mean, and can one form classes of functions (nowadays Banach spaces or even topological vector spaces of such objects) such that one can guarantee convergence in those space in the corresponding norm (or topology).
The classical view-point was of course: Can one establish pointwise convergence (Dirichlet-conditions, J.P. Lejeune-Dirichlet [1805-1859])? Or uniform convergence at least for continuous functions (no, according to A.N.Kolmogorov [1903-1987], already in 1923 a found a counter-example and in 1926 he was able to prove that the Fourier series of an L^1-function can **diverge everywhere!**).

Convergence Issues: The idea of Summability

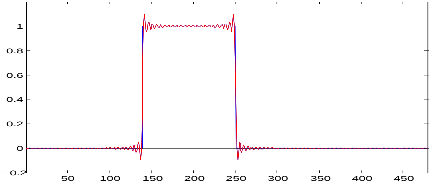
Of course one has to mention **Lipolt Fejer** [1880 - 1959] and a long list of names pursuing the problems related to **summability**.

The idea is to change the question from the question of convergence of the (partial sum) of the Fourier series to *the question of recovering a function from its Fourier coefficients*. For example, Fejer was suggesting to take (as a replacement for the ordinary partial sums) the *arithmetic means of the partial sums*. This was orginating from the idea of using so-called *Cesaro means*.

Fejer's Theorem of 1900 states that for every continuous periodic function f the (now known as) Fejer means of the Fourier series converges uniformly to f.

Gibb's Phenomenon: Behaviour at Jumps

Looking at the behaviour of Fourier series of a simple box-function one can also ask, how to best describe the convergence, noting that the over/undershooting peaks are just getting narrower as the number of terms increases.



Still it is true that this is the best approximations in the $\boldsymbol{L}^2$-sense, i.e. in the *quadratic mean*.

Dirichlet versus Fejer, zoom in

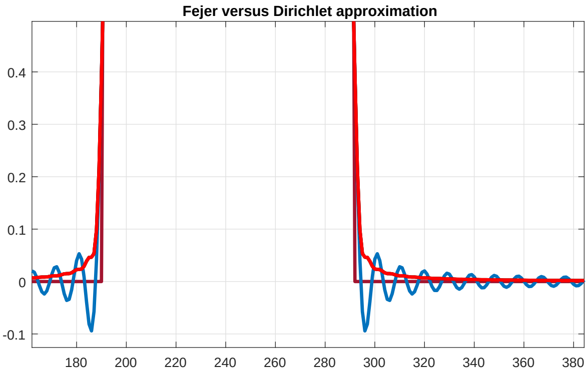


Figure: Closeup on Dirichlet or Fejer approximation to box fctn.

Fejer versus Dirichlet

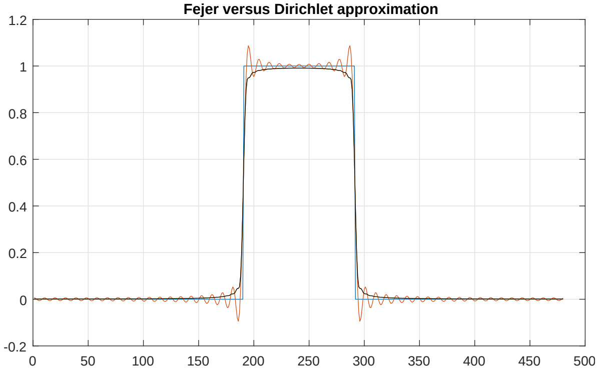


Figure: The box-car function and Gibbs phenomenon, compared .... with the smoothed version based on Fejer's summation.

Convergence Issues: the advent of Hilbert Spaces

Whereas convergence was at the beginning a difficult question it turned out that the setting of **Hilbert space** $(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2)$ with the inner product

$$\langle f, g \rangle := \int_0^1 f(t)\overline{g(t)}dt, \quad f, g \in \boldsymbol{L}^2(\mathbb{T}), \tag{5}$$

sca

allows to formulate the Fourier series expansions in the spirit of an orthonormal expansions. It became more convenient to use Euler's formula for a change of basis from trigonometric functions $\sin(x)$ and $\cos(x)$ to the *complex exponential functions*.

Orthogonal Expansions, ONBs in Hilbert Spaces

The pure frequencies $\chi_n(x) := exp(2\pi inx)$, $n \in \mathbb{Z}$ form a *complete orthonormal system* for the Hilbert space $\mathcal{H} = \left(\boldsymbol{L}^2(\mathbb{T}), \|\cdot\|_2\right)$:

$$f = \sum_{n \in \mathbb{Z}} \langle f, \chi_n \rangle \chi_n, \quad f \in \mathcal{H}, \tag{6}$$

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with unconditional convergence in the $\boldsymbol{L}^2$-norm

$$\|f\|_2 := \sqrt{\int_0^1 |f(x)|^2}.$$

The *coefficients* $c_n := \langle f, \chi_n \rangle$, $n \in \mathbb{Z}$ are uniquely determined and satisfy Parseval's equality:

$$\|f\|_2 = \sqrt{\sum_{n \in \mathbb{Z}} |c_n|^2}. \tag{7}$$

par

Almost everywhere convergence, Lusin's Conjecture

The convergence issue, in the form of Lusin's conjecture about the convergence of Fourier series in the *pointwise almost everywhere sense* was settled positively by Lennart in his famous Acta Mathematica paper of 1966. He showed that for every $f \in \boldsymbol{L}^2(\mathbb{T})$ the Fourier series is *almost everywhere convergent*.

Lennart Carleson On convergence and growth of partial sums of Fourier series. *Acta Math.*, 116:135–157, 1966.
This result was of course the counterpoint to Kolmogorov's negative results in the $\boldsymbol{L}^1$-setting (Kolmogorov was a student of Lusin).

Fourier Transform over the Real Line

The work of H.L. Lebesgue paved the way to a clean definition of the Fourier transform for "functions of a continuous variables" as an *integral transform* naturally defined on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$

$$\|f\|_1 := \int_{\mathbb{R}} |f(x)| dx, \quad f \in \boldsymbol{L}^1(\mathbb{R}). \tag{8}$$

LiR

The (continuous) Fourier transform for $f \in \boldsymbol{L}^1(\mathbb{R})$ is given by:

$$\hat{f}(s) := \int_{\mathbb{R}} f(x) e^{-2\pi i s x} dx, \quad s \in \mathbb{R}. \tag{9}$$

FTd

With this normalization the inverse Fourier transform looks similar, just with the conjugate exponent, and thus, *under the assumption that f is continuous and $\hat{f} \in \boldsymbol{L}^1(\mathbb{R})$* we have pointwise

$$f(t) = \int_{\mathbb{R}} \hat{f}(s) e^{2\pi i s t} ds. \tag{10}$$

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Plancherel's Theorem: Unitarity Property of FT

Using the density of $\boldsymbol{L}^1(\mathbb{R}) \cap \boldsymbol{L}^2(\mathbb{R})$ in $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ it can be shown that the Fourier transform extends an a natural and unique way to $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$:

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Theorem

The Fourier (-Plancherel) transform establishes a unitary automorphism of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$, i.e. one has

$$\|f\|_2 = \|\hat{f}\|_2, \quad f \in \boldsymbol{L}^2(\mathbb{R}),$$

$$\langle f, g \rangle = \langle \hat{f}, \hat{g} \rangle, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}).$$

In some sense *unitary* transformations of a Hilbert transform is like a change form one ONB to another ONB in $\mathbb{R}^n$.

The Continuous Superposition of Pure Frequencies

This impression is confirmed by the “continuous representation” formula, using $\chi_s(x) = e^{2\pi isx}$, $x, s \in \mathbb{R}$. Since we have

$$\hat{f}(s) = \langle f, \chi_s \rangle, \quad s \in \mathbb{R},$$

we can rewrite (formally) the Fourier inversion formula as

$$f = \int_{\mathbb{R}} \langle f, \chi_s \rangle \chi_s, \quad f \in \boldsymbol{L}^2(\mathbb{R}). \tag{11}$$

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This looks like a perfect orthogonal expansion, but unfortunately the “building blocks” $\chi_s \notin \boldsymbol{L}^2(\mathbb{R})$!! (this requires f to be in $\boldsymbol{L}^1(\mathbb{R})$).

Convolution and the Fourier Transform

Another important fact about the Fourier transform is the so-called **convolution theorem**, i.e. the Fourier transform converts convolution into pointwise multiplication.
Again it is natural to define convolution on $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$:

$$f * g(x) = \int_{\mathbb{R}} f(x-y)g(y)dy = \int_{\mathbb{R}} g(x-y)f(y)dy \quad xa.e.; \ (12)$$ con

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1, \quad f,g \in \boldsymbol{L}^1(\mathbb{R}).$$

For positive functions f, g one even has equality. This is relevant for the determination of probability distributions of a sum of *independent* random variables. Assume X has density f and Y has density g then the random variable $X+Y$ has probability density distribution $f * g = g * f$.

Banach algebras

lg1

Theorem

Endowed with the bilinear mapping $(f,g) \to f * g$ the Banach space $\left(\boldsymbol{L}^1(\mathbb{R}), \|\cdot\|_1\right)$ becomes a commutative Banach algebra with respect to convolution.

The **convolution theorem**, usually formulated as the identity

$$\widehat{f * g} = \hat{f} \cdot \hat{g}, \quad f,g \in \boldsymbol{L}^1(\mathbb{R}), \tag{13}$$

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implies

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Theorem

The Fourier algebra, defined as $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}) := \{\hat{f} \mid f \in \boldsymbol{L}^1(\mathbb{R})\}$, with the norm $\|\hat{f}\|_{\mathcal{F}\boldsymbol{L}^1} := \|f\|_1$ is a Banach algebra, closed under conjugation, and dense in $\left(\boldsymbol{C}_0(\mathbb{R}), \|\cdot\|_\infty\right)$ (continuous functions, vanishing at infinity).

Abstract Harmonic Analysis

Jumping into the 40th of the last century one can say that **Abstract Harmonic Analysis** was created, with $\mathbb{R}$ replaced by a general a general LCA (locally compact Abelian) group.
In engineering terminology this allows to discuss *continuous and discrete variables*, but also *periodic or non-periodic* functions as functions on different groups, such as $\mathcal{G} = \mathbb{R}^d, \mathbb{Z}^d, \mathbb{Z}_N, \mathbb{T}^k$ etc., their product being called *elementary groups*.
The fundamental fact in all these cases is the existence of an translation for functions, defined as

$$[T_z f](x) = f(x - z), x, z \in \mathcal{G},$$

and the existence of an invariant integral, the so-called *Haar measure* (Alfred Haar, [1885 - 1933]).

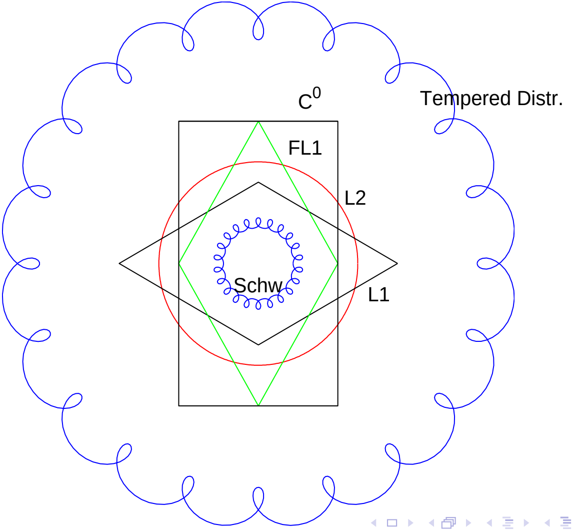
Topics of Abstract Harmonic Analysis

The central theme of Harmonic Analysis (according to my advisor Hans Reiter [1921-1992]) was the study of the Banach algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, in particular the structure of closed ideals. One of the central questions is the question of **spectral synthesis**.
This is a rather involved topic, roughly described as follows:
Can one approximate - in a suitable weak sense - a function f from finite linear combinations of pure frequencies of those frequencies which are found "in the signal" f via spectral analysis?
In other words, one considers only this frequencies χ_s, such that s belongs to the support of $\widehat{f}$ (i.e. s can be approximated by values s_n with $\widehat{f}(s_n) \neq 0$). Then one expect to approximate f weakly by trigonometric polynomials $t_k(x) = \sum_{k=1}^n c_k \chi_{s_k}$. The failure of spectral synthesis for $\mathbb{R}^3$ is due to L.Schwartz [1915 - 2002].

Laurent Schwartz Theory of Tempered Distributions

Laurent Schwartz is mostly known for having introduced the space of tempered distributions, a topological vector space of **generalized functions** or **distributions** which is invariant under the Fourier transform.
He starts out by defining the so-called *Schwartz space of rapidely decreasing functions*, consisting of all infinitely differentiable functions on $\mathbb{R}^d$ which decay faster at infinity than any polynomial. This space $\mathcal{S}(\mathbb{R}^d)$ is naturally endowed with a countable family of semi-norms, turning the space into a *nuclear Frechet space*.
The topological dual of $\mathcal{S}(\mathbb{R}^d)$, i.e. the collection of all linear functionals σ on $\mathcal{S}(\mathbb{R}^d)$ satisfying the continuity assumption $f_n \to f_0$ in $\mathcal{S}(\mathbb{R}^d)$ implies $\sigma(f_n) \to \sigma(f_0)$ in $\mathbb{C}$, constitutes $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions.

The classical setting of test functions & distributions



Fourier Transforms of Tempered distributions

The Fourier transform $\hat{\sigma}$ of $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ is defined by the simple relation

$$\hat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

His construction *vastly extends the domain of the Fourier transform* and allows even polynomials to have a Fourier tranform. Among the objects which can now be treated are also the Dirac measures δ_x, as well as Dirac combs $⊔⊔ = \sum_{k \in \mathbb{Z}^d} \delta_k$.
Poisson's formula, which expresses that one has for $f \in \mathcal{S}(\mathbb{R}^d)$

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n), \tag{14}$$

Poi

can now be recast in the form

$$\widehat{⊔⊔} = ⊔⊔$$

Sampling and Periodization on the FT side

The convolution theorem, can then be used to show that sampling corresponds to periodization on the Fourier transform side, with the interpretaton that

$$⧢ \cdot f = \sum_{k \in \mathbb{Z}^d} f(k)\delta_k, \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

In fact, we have

$$\widehat{⧢ \cdot f} = \widehat{⧢} * \widehat{f} = ⧢ * \widehat{f}.$$

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).

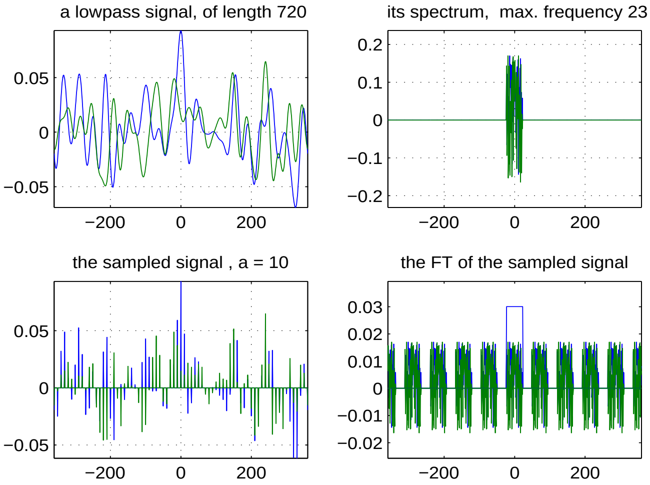
Recovery from Samples

If we try to recover a real function $f : \mathbb{R} \to \mathbb{R}$ from samples, i.e. from a sequence of values $(f(x_n))_{n \in I}$, where I is a finite or (countable) infinite set, we cannot expect perfect reconstruction. In the setting of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ any sequence constitutes only set of measure zero, so knowing the sampling values provides *zero information* without side-information.

On the other hand it is clear the for a *(uniformly) continuous* function, so e.g. a continuous function supported on $[-K, K]$ for some $K > 0$ piecewise linear interpolation (this is what MATLAB does automatically when we use the PLOT-routine) is providing a good (in the uniform sense) approximation to the given function f as long as the maximal distance between the sampling points around the interval $[-K, K]$ is small enough.

Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.

A Visual Proof of Shannon's Theorem



Shannon's Sampling Theorem

It is kind of clear from this picture that one can recover the spectrogram of the original function by isolating the central copy of the periodized version of $\widehat{f}$ by multiplying with some function $\hat{g}$, with g such that $\hat{g}(x) = 1$ on spec(f) and $\hat{g}(x) = 0$ at the shifted copies of $\widehat{f}$. This is of course only possible if these shifted copies of spec(f) do not overlap, resp. if the sampling is dense enough (and correspondingly the periodization of $\widehat{f}$ is a course one. This conditions is known as the *Nyquist criterion*. If it is satisfied, or supp$(f) \subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) T_{\alpha k} g(x), \quad x \in \mathbb{R}^d.$$

Lars Hörmander and PDE

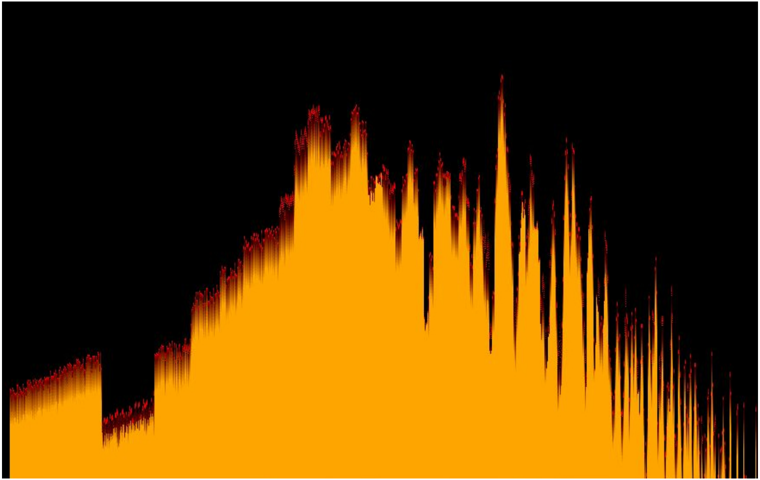
Just in order not to leave out an important mathematical applicatoin area of tempered distributions (and their generalizations) let us mention the work of **Lars Hörmander** [1931-2012]. He as well as **Elias Stein** (born also in 1931, most of the time in Princeton) have developed Fourier Methods tremendously into the multi-dimensional setting.
Their work cannot be summarized in a short talk, but it is clear that a modern theory of PDE (partial differential operators), or a fine analysis of functions or distributions on $\mathbb{R}^n$ (e.g. in the sense of *micro-local analysis*) cannot be thought without them.
Among the heroes of "modern Fourier analysis" let me mention two of Elias Stein's students: **Charles Feffermann** (born 1949) and **Terence Tao** (born 1975).
See: **Journal of Fourier Analysis and Applications** or **Applied and Computational Harmonic Analysis**.

Where do we use Fourier Analysis in our Daily Life?

Perhaps you have to think a bit? But there are MANY opportunities, and few activities do not involve the use of FFT-based technology.

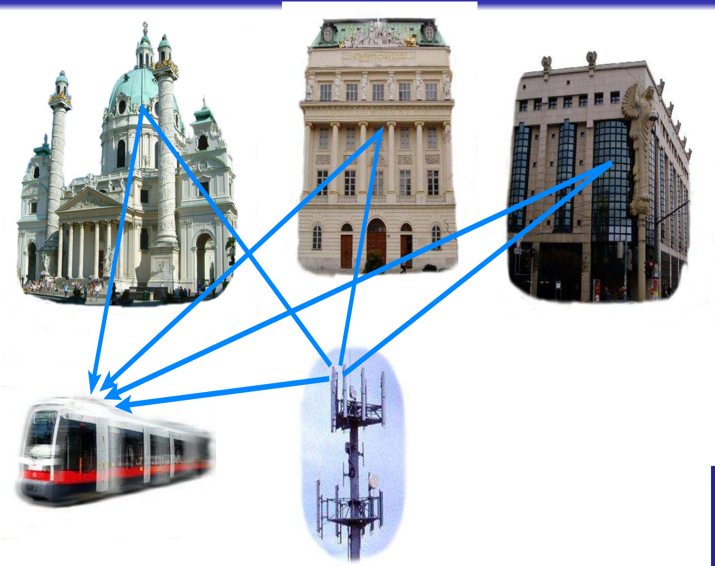
1. You use your mobile phone to communicate?
2. You listen to music? (MP3 or WAV-files);
3. You download images? (JPEG format);
4. Your computer communicates with your printer;
5. You watch digital videos?

Gabor Analysis in our kid's daily live (MP3)



Mobile Communication

Mobile Communication



Medical Imaging using Tomographs

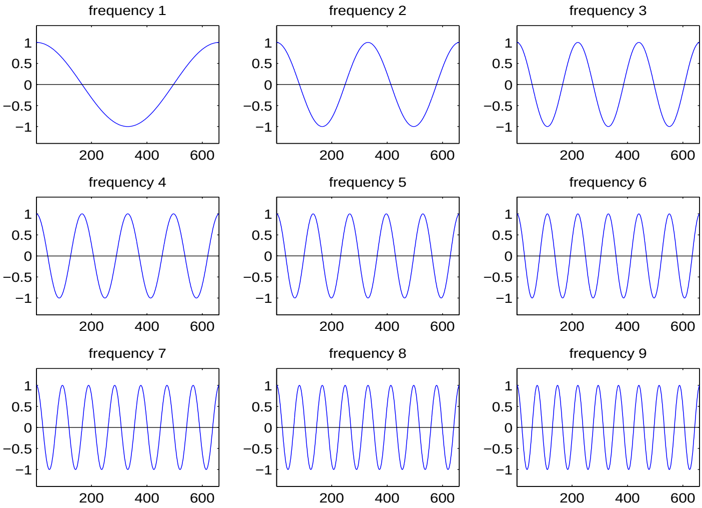
Medical Imaging using the Radon Transform

Tomography and the Radon Transform

Mathematical key idea behind tomography

- The tomographic device measures the attenuation of of X-rays through the tissue along many-many straight lines, between (rotating) X-ray source and sensor array;
- Different tissues have known absorption behaviour, thus attenuation indicates integrated density along lines;
- Mathematically speaking the task is the invert a **sampled Radon transform** which can be obtained from these data
- After regridding the data arising on a polar grid an IFFT2 provides one possible way to produce images (slices),
- Modern **Compressed Sensing** methods improve further

Building blocks for Discrete Cosine Transform DCT

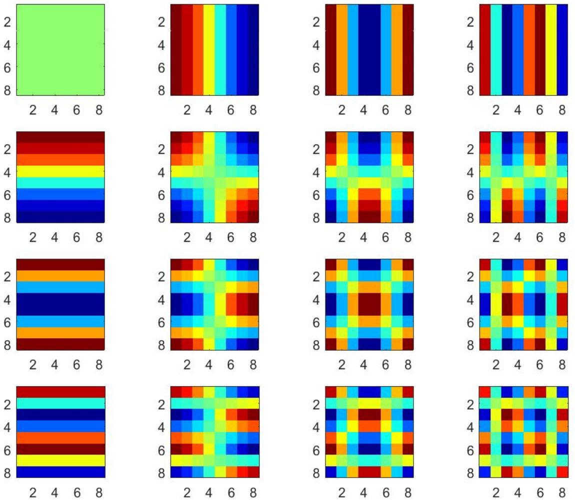


The JPEG compression

The widely used JPEG standard, established by the "Joint Photographic Experts Group" is based on the discrete cosine transform, a real version of the Fourier transform (real images give real coefficients).

- First a general image is decomposed into blocks of 8×8 pixels, (each of them in fact in the range of 0 to $255 = 2^8 - 1$, so one Byte or 8 Bits worth);
- Then depending on the chosen compression rate a fixed number of coefficients, from upper left to lower right corner (figure below) is stored and transmitted;
- Resynthesis from this set of coefficients provides the decoded image.

The building blocks for the Discrete Cosine Transform DCT2



On the Role of Fourier Analysis within Applied Sciences

Our talk would be endless if we had to explain the use of Fourier analysis within some modern disciplines, like PDE (Partial Diff. Equ.), hence Physics or (Theoretical) Chemistry and so on. Let us just mention:

1. Modern PDE or pseudo-differential operators
2. for Wavelet and Shearlet Theory
3. Fourier integral operators
4. Theory of Function Spaces (Sobolev, Triebel,...)
5. Schrödinger equation, quantum physics
6. Frames, Banach Frames, Riesz projection bases

From Linear Algebra to Fourier Analysis 1

Given the important role of the Fourier transform in *digital signal processing* (audio, images, video, etc.) and the closeness of the involved algorithms to those taught in Linear Algebra courses it would make sense to **teach Fourier analysis first in the discrete, finite** setting.

Here the DFT (Discrete Fourier Transform), realized as the FFT (Fast Fourier Transform) is just a change from the standard orthonormal basis of unit vectors (obtained by shifting the unit vector at zero along the finite group $\mathbb{Z}_N$).

The algebraic properties of unit roots of order N resp. those of polynomials imply many important properties of the DFT/FFT.

Basic Properties of the DFT

Key points implying some of the basic properties of the DFT are:

1. The orthogonality of (discrete) pure frequency vectors implies the energy preserving property;
2. The matrix realizing the DFT by matrix multiplication is the Vandermonde matrix for the collection of unit roots, starting from $1 = \omega^0$, taken in clockwise order;
3. Consequently we can interpret the mapping $\mathbf{a} \to \mathbf{b} := fft(\mathbf{a})$ as the mapping of coefficients of a polynomial to its values over the unit roots of order N;
4. pointwise multiplication (of these values) corresponds to the Cauchy product, describing the coefficients of the product polynomial, e.g. binomial law.

What makes the FFT so fast?

It is a nice mathematical feature that the group structure of $\mathbb{Z}_N$ allows to recursively compute FFTs of big length $2N$ by doing (up to some simple recomputation) two FFTs of length N. Practically speaking this comes done to the fact that one can easily combine the values of a polynomial with even terms only (half length) and then multiply the odd ones (with coefficients shifte to the even part) with x and then add:

$$1 + 2x + 3x^2 + 4x^3 + 5x^4 + 6x^5 + 7x^6 + 8x^7 =$$

$$(1 + 3x^2 + +5x^4 + 7x^6) + x(2 + 4x^2 + 6x^4 + 8x^6).$$

As a consequence FFTs of length 2^k, i.e. $k = 10$, then $2^{10} = 1024$ or $2^9 = 512$ are good numbers for image formats.

A good number for HiFi Music

When it comes to audio applications, the natural number 44100 is quite popular. It describes the number of samples taken per second for the recording of a CD or as a typical format for WAV-files.

But why 44100, and not e.g. 44000 or 50000?

First of all the fact the human beings never hear anything beyond 20kHz is the first step of the reasoning. Since $\sin(2 \cdot 20000\pi x)$ has 40000 zeros on any intervall of length 1 it is plausible, that the *Nyquist criterion* is satisfied for sampling rates better than 40000 samples per second.
BUT 44100 is such a nice number!! Because it can be written as $(2 \cdot 3 \cdot 5 \cdot 7)^2$, and it is thus an integer having a lot of (different) divisors! Still, the FFT of such a length is fast.

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Convolution and time-invariant linear systems

Aside from probability (cf. above) convolution has its role in *Linear Systems Theory*, in particular in the mathematical description of time-invariant linear systems, meaning linear operators T , mapping signals f to signals $g = T(f)$, with *time-invariance*:

$$T \circ T_x = T_x \circ T, \quad \forall x \in \mathcal{G}.$$

A non-trivial, although quite plausible result is the following one

hm1

Theorem (hgfei)

Any bounded and translation invariant operator from $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$ into itself (so-called BIBOS system) which commutes with translation is a moving average by some bounded measure, i.e. by some element in the dual space of $(C_0(\mathbb{R}^d), \|\cdot\|_\infty)$. In fact, $\mu(f) = Tf(0)$ describes the system, given by

Convolution and Fourier Stieltjes transforms

The fact, that the collection of all TILS is not only a Banach space, but also a Banach algebra under composition of operators (with the operator norm) allows to transfer this composition rule to the *generating measures*.

One can shown, that $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$ is then a Banach algebra with respect to convolution imposed in this way, containing $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ as a closed, translation-invariant ideal.

The collection of *characters*, i.e. the functions χ_s are the *joint eigenvectors* to all these operators, among them the translation operators. The Fourier transform extends to all of these characters, and is then often called Fourier Stietjes transform, again with

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2},$$

for $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$ (Convolution Theorem for measures).

Consequences an the Transfer Function

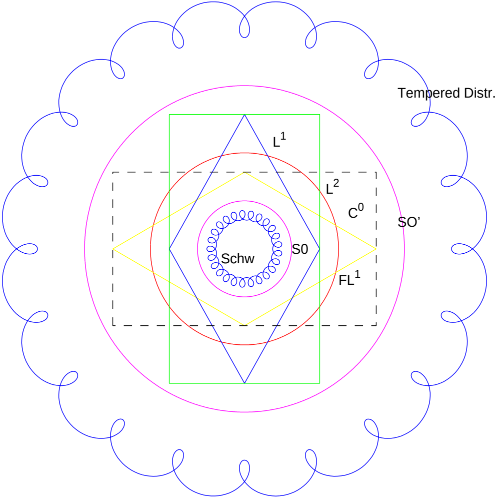
This approach to convolution can be carried out *without the use of measures theory* (!), details can be found in my course notes.

In an engineering terminology the measure μ describing the linear system T via $T(f) = \mu * f$ is the *impulse response of the system*. It can be obtained (!proof) as a w^*-limit of input functions tending to the Dirac measure, e.g. compressed, normalized (in the $\boldsymbol{L}^1$-sense) rectungular pulses.

The Fourier (Stieltjes) transform of the system T is know as the *transfer function* of the system T, and it is characterized by the eigen-vector property:

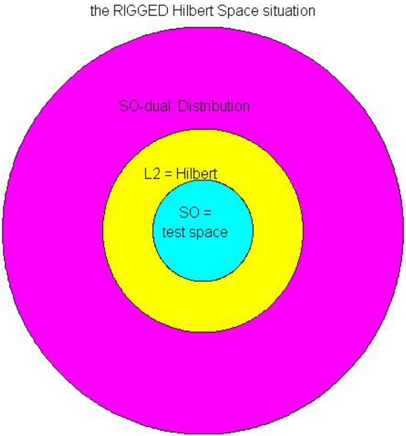
$$\mu(\chi_s) = \hat{\mu}(s)\chi_s, \quad s \in \mathbb{R}.$$

A schematic description: all the spaces



A schematic description: the simplified setting

Testfunctions ⊂ Hilbert space ⊂ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The way out: Test Functions and Generalized Functions

The usual way out of this problem zone is to introduce generalized functions. In order to do so one has to introduce test functions, and give them a reasonable topology (family of seminorms), so that it makes sense to separate the *continuous* linear functionals from the pathological ones. The "good ones" are admitted and called generalized functions, since most reasonable ordinary functions can be identified (uniquely) with a generalized function (much as $5/7$ is a complex number!).
If one wants to have Fourier invariance of the space of distributions, one must Fourier invariance of the space of test functions (such as $\mathcal{S}(\mathbb{R}^d)$). If one wants to have - in addition - also closedness with respect to differentiation one has to take more or less $\mathcal{S}(\mathbb{R}^d)$. BUT THERE IS MORE!

The Banach space $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ or $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$, which will serve our purpose. Its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).

Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts (II)

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) \, .$$

Behavior under Fourier transform

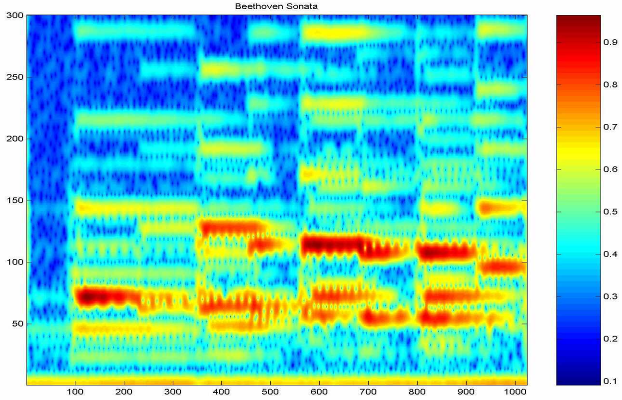
$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x, \omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

cSo

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\mathbf{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

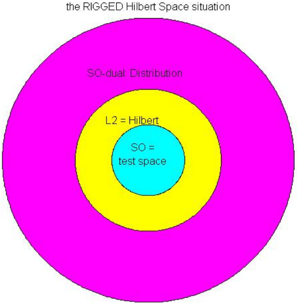
Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}'_1)$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}'_2)$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}'_1$ and $\mathbf{B}'_2$.

A schematic description: the simplified setting

In our picture this simple means that the inner "kernel" is mapped into the "kernel", the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\mathbf{S}_0(\mathbb{R}^d)$ to $\mathbf{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\mathbf{L}^2(\mathbb{R}^d)$ and $\mathbf{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\mathbf{S}_0'(\mathbb{R}^d)$ onto $\mathbf{S}_0'(\widehat{\mathbb{R}}^d)$.

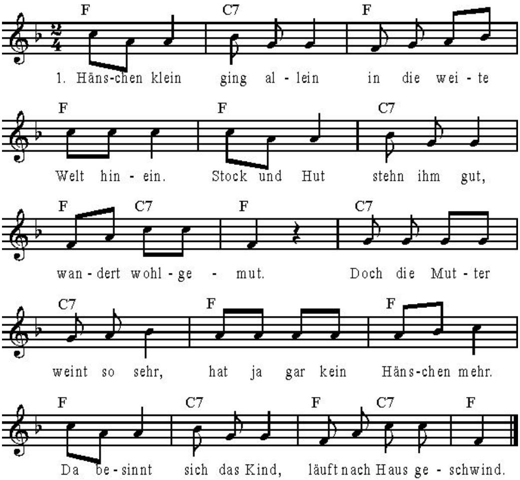
Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (15)$$

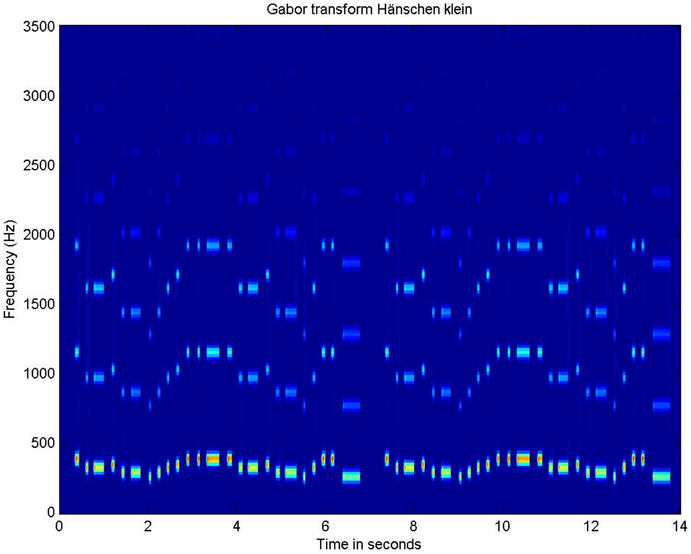
par

is valid for $(f, g) \in \mathbf{S}_0(\mathbb{R}^d) \times \mathbf{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

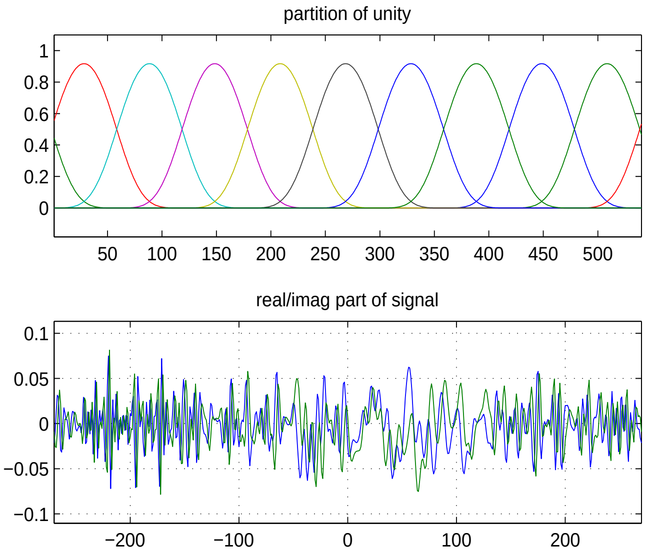
Time-Frequency Analysis and Music



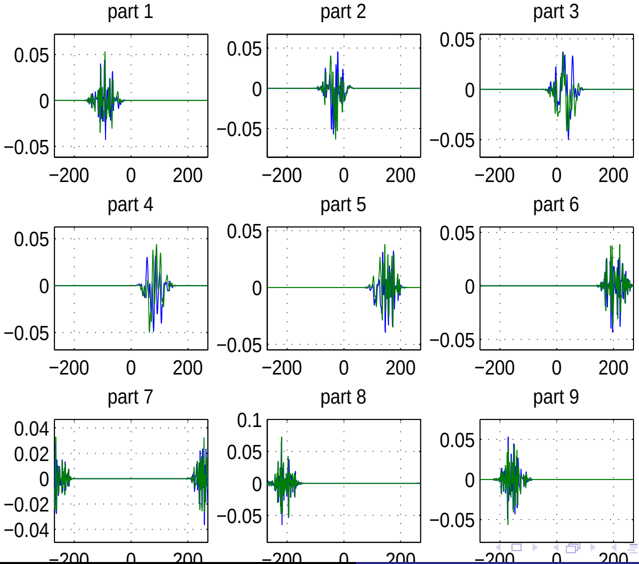
The Short-Time Fourier Transform of this Song



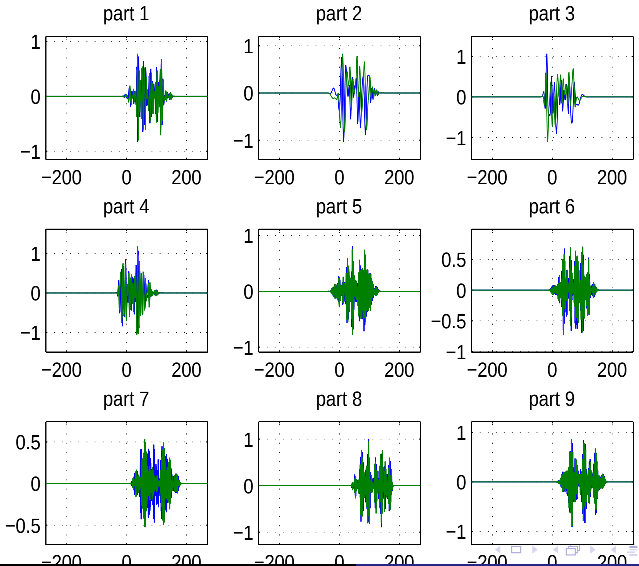
Motivated by MUSICAL SCORE one could do ?



... and cut the signal into pieces

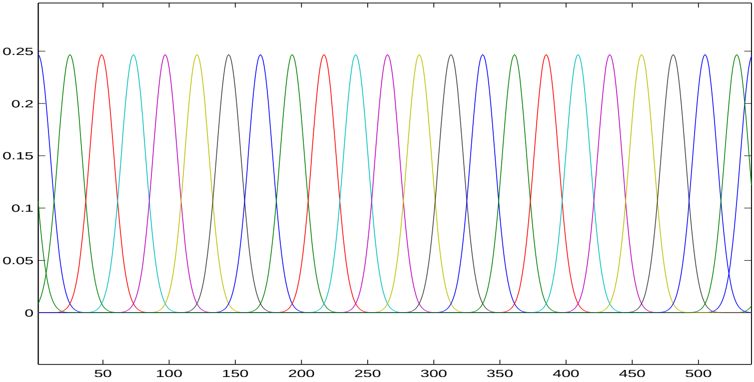


... and do localized spectra

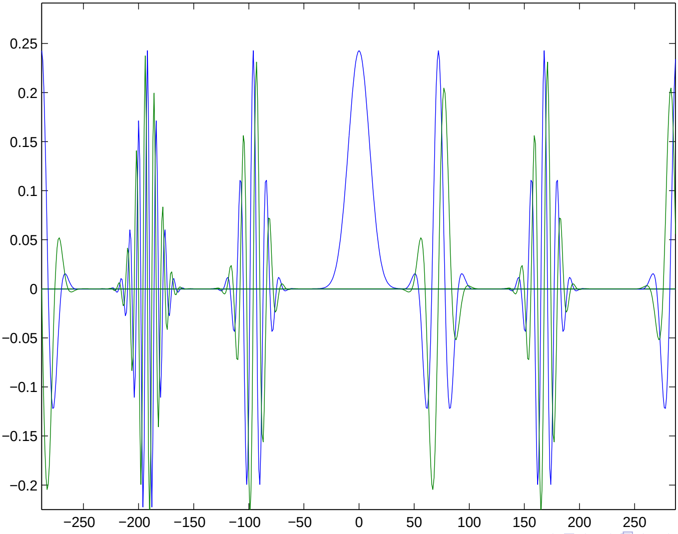


D.Gabor's suggestion of 1946

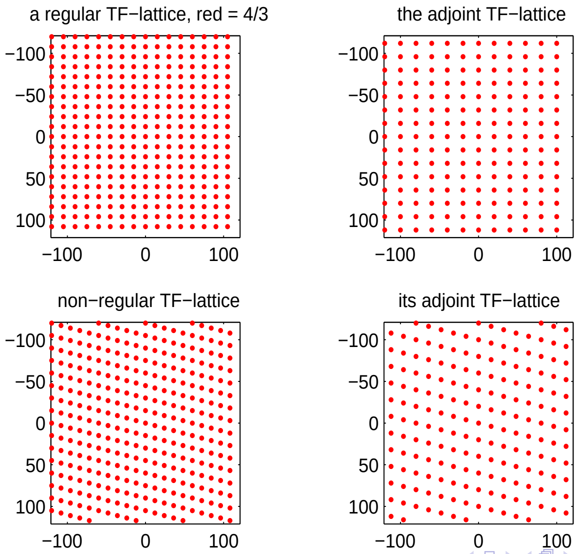
Choose the Gauss-function, because it is the unique minimizer to the *Heisenberg Uncertainty Relation* and choose the critical, so-called von-Neumann lattice, which is simply $\mathbb{Z}^2$.



The Gaborian Building blocks



Phase space lattices/ time-frequency plane



Modern Viewpoint I

Todays Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \mathcal{S}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal f resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda)g \rangle$.
We speak of *tight Gabor frames* (g_λ) if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!!

Modern Viewpoint II

Another basic fact is that for each $g \in \mathcal{S}(\mathbb{R}^d)$ one can find, if Λ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\widetilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \qquad (16)$$

gab

$\widetilde{g}$ can be found as the solution of the (positive definite) linear system $S\widetilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\widetilde{g}$ instead of g for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.

gabexpans0

Thus (16) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).

Modern Viewpoint III

The possibility of having such *tight Gabor frames* is resulting from the continuous reconstruction formula, valid for arbitrary $\mathbf{L}^2$-atoms g. Writing again for $\lambda = (t, \omega)$ and $\pi(\lambda) = M_\omega T_t$, and furthermore $g_\lambda = \pi(\lambda) g$ we have in fact for any $g \in \mathbf{L}^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda \, d\lambda.$$

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\mathbf{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \mathbf{L}^2. \qquad (17)$$

ene

This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*.

Modern Viewpoint IV

There is a similar representation formula at the level of operators, where we also have a continuous representation formula, valid in a strict sense for *regularizing operators*, which map w^*-convergent sequences in $\boldsymbol{S}_0'(\mathbb{R}^d)$ into norm convergent sequences in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle T, \pi(\lambda) \rangle_{\mathcal{HS}} \pi(\lambda) d\lambda. \tag{18}$$

spr

It establishes an isometry for Hilbert-Schmidt operators:

$$\|T\|_{\mathcal{HS}} = \|\eta(T)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}, \quad T \in \mathcal{HS},$$

where $\eta T = \langle T, \pi(\lambda) \rangle_{\mathcal{HS}}$ is the *spreading function* of the operator T. The proof is similar to the proof of Plancherel's theorem.

Gabor Riesz bases and Mobile communication

Another usefulness of "sparsely distributed" Gabor systems comes from mobile communication:

1. Mobile channels can be modelled as slowly varying, or underspread operators (small support in spreading domain);
2. TF-shifted Gaussians are joint approximate eigenvectors to such systems, i.e. pass through was some attenuation only;
3. underspread operators can also be identified from transmitted pilot tones;
4. Communication should allow large capacity at high reliability.

Things that you should forget! (dislearn!?)

The concept of *linear independence*

Definition

A set $M \subset \boldsymbol{V}$ within (any) vector space is linear independent if every finite subset $F \subset M$ is linear independent in the usual sense, i.e. if

$$\sum_{k=1}^{n} c_i f_i = 0 \quad \text{in} \quad \boldsymbol{V} \quad \Rightarrow \quad \vec{\mathbf{c}} = \vec{\mathbf{n}} \in \mathbb{C}^n. \tag{19}$$

cla

SHORTCOMING: Once transferring the question to infinite-dimensional spaces, in particular to normed spaces, one should adapt the concept by allowing "infinite linear combinations".

Note: There are books (cf. I.Singer) on the concept of bases in a Banach space. We would like to say that "every element is uniquely expanded into a series of elements using the elements of a basis", but *what does it mean "begin represented"?* Should we assume unconditional convergence, and/or norm convergence. Should conditional convergence in some weaker topology (e.g. pointwise convergence) be admitted? Due to the large variety of concepts even the notion of a basis in a Banach space appears to be non-trivial! (hence even more the concept of linear independence). Problem: How should one generalize this to infinite dimensional settings. Which sequences should be allowed. Exactly ℓ^2-sequences? Should this be done only for so-called *Bessel sequences* (f_i) which are such that the mapping

$$\mathbf{c} \mapsto \sum_{i \in I} c_i f_i$$

is bounded from $\ell^2(I)$ to some Hilbert space $\mathcal{H}$, implying unconditional convergence of the series. Or just (un?)conditional

Gabor's suggestion from 1946 (!)

A good example for problems with infinite dimensional spaces is the collection (let us call it call D. Gabor's classical family): Take the family of TF-shifted copies of the standard Gaussian (i.e. we take the density of the normal distribution, shift it by integers, and multiply it with pure frequencies which are compatible with the time-shifts), so each "atom" has a well-defined position on the integer grid $\mathbb{Z}$ and a well defined integer frequency, also in $\mathbb{Z}$ if we use the description of pure frequencies using complex exponential functions

$$e^{2\pi ikx} \quad = \quad \cos(2\pi kx) + i \cdot \sin(2\pi kx).$$

This family has the following properties:

- (pos0) the family is linear independent in the classical sense;
- (pos1) the family is *total*, i.e. the linear combinations of these building blocks allow to approximate any $f \in \boldsymbol{L}^2(\mathbb{R})$ to any precision $\varepsilon > 0$.
- (neg1a) if the required precision is increased, i.e. for $\varepsilon \to 0$ the corresponding coefficients do not converge, so there is no "final/limiting" set of coefficients.
- (neg1b) the set is *not minimal*, i.e. one can remove e.g. one element (!but not two!) such that the remaining set is still total.
- (neg2) If one wants to represent arbitrary elements from the Hilbert space $\boldsymbol{L}^2(\mathbb{R})$ one should not restrict the attention to coefficients from $\ell^2(\mathbb{Z}^{2d})$!

- (pos3) the building blocks are optimally localized in the TF-sense, because the Gauss-function is providing the minimizer (Fourier invariant) for the Heisenberg uncertainty relation.
- (neg3a) the coefficients can be obtained using a (quasi-) biorthogonal system, which can be "computed" (Bastiaans dual window), but it is in fact not anymore an $\boldsymbol{L}^2$-function, but only $\boldsymbol{L}^\infty(\mathbb{R})$.
- (neg3b) so strictly speaking we cannot even determine "the coefficients" by taking ordinary scalar products (should the be taken using summability methods?? and/or should we allow alternative forms of convergence??)

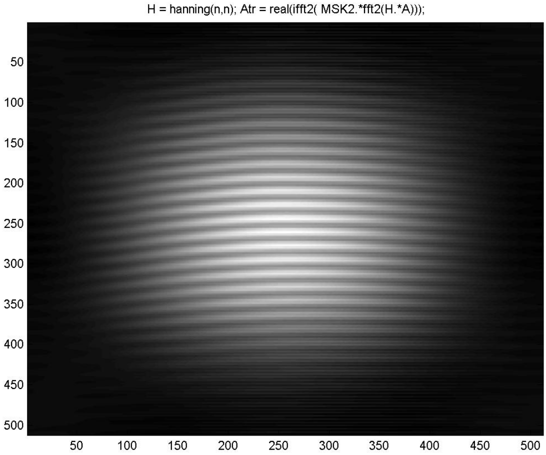
- Another well trained sentence is this one:

A **series** *is convergent if the sequence of partial sums is convergent.*

Coming to Fourier series this view-point brings a lot of trouble (or if you prefer: challenging mathematical problems, only resolved by Carleman in 1972!, after conjectures due to Lusin from around 1922).

In fact, the interpretation of a series (of function) in the *classical* (i.e. the pointwise almost everywhere) setting makes the problem a (very) hard one, while it is easily resolved if one puts oneself in the context of a Hilbert space setting, with convergence being taken in the quadratic mean (the L^2-norm).

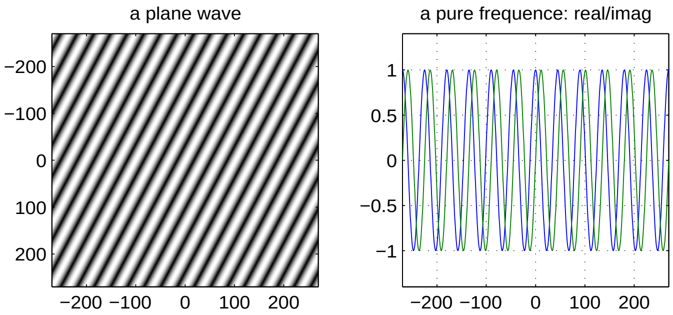
Pending application: Removing Fringes in Astronomy ? as part of an ongoing ESO project in AUSTRIA.



Motivation for compactness of musical description

1. It is localized (as opposed to the global Fourier transform)
2. Its building blocks are localized pure frequencies, hence approximate eigenvectors to slowly variant systems;
3. recall that the pure frequencies are a complete system of eigenvectors for the (commutative algebra) of translation operators;
4. one has to choose whether one wants to have redundant and generating families (frames), OR undersampled, linear independent families (Riesz bases), and one cannot have both, except with other undesirable properties (Balian-Low principle)

2D-Gabor Transform: Plane Waves



2D-Gabor Analysis: Test Images

2D-Gabor Transform: Test-Images 2

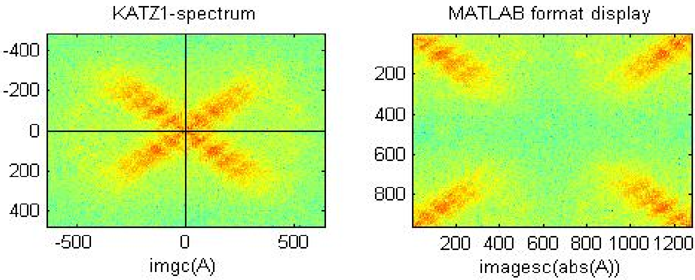
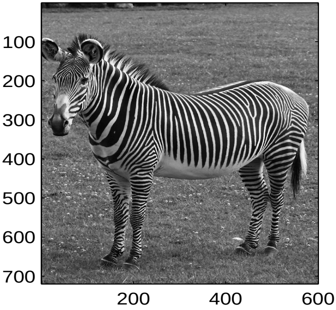
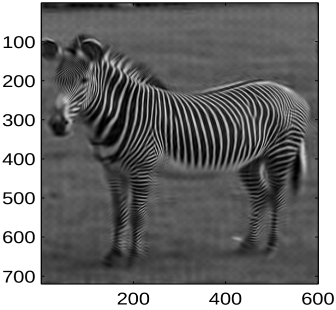
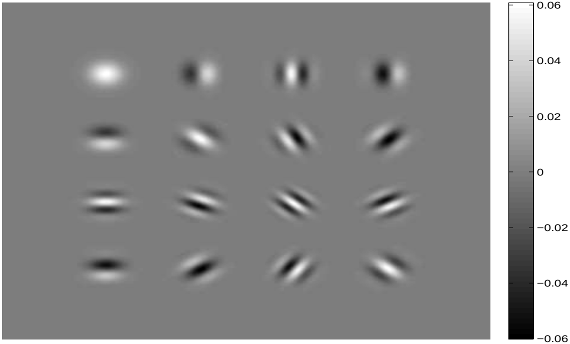


Image Compression: a Test Image





Showing the elementary $2D$-building blocks



(regular) Gabor families: Frames and Riesz bases

Definition

Given a pair (g, Λ), consisting of a vector $g \in \ell^2(G)$ and a lattice $\Lambda \lhd G \times \widehat{G}$ we call the family $(g_\lambda)_{\lambda \in \Lambda}$ a Gabor frame, if the family spans all of $\ell^2(G)$. It is called a Gaborian Riesz basis (resp. Riesz basic sequence) if it is a linear independent set.

There are - for people in numerical analysis - quality measures for the quality of such families, in the sense of a conditioning of the problem, thus being a quotient of two relevant singular values of associated operators, we don't go into details here.
Both situations are of practical relevance!

Usefulness and applications of Gabor frames:

The question of Gabor frames is of interest, when a signal (say some audio signal, or some image, cf. introduction) is to be *decomposed* into meaningful elementary building blocks, somehow like *transcription*. Ideally the distribution of *energy* in the signal goes over into an equivalent energy distribution. AND WHAT can we do with this:

a) contributions may be irrelevant (or disturbing) and can be eliminated (the bird contributing to the open air classical concert): denoising of signals

b) signals can be separated in a TF-situation

c) unimportant, small contributions can be omitted (+ masking effect): allows for efficient lossy compression schemes >> MP3.

Applications of Gabor Riesz bases:

Of course Gabor Riesz bases (for subspaces) will correspond to lattices Λ with at most N points. Ideally, the Gram matrix of the corresponding system is diagonal dominant (there is the so-called piano-reconstruction theorem).

They are very useful in mobile communication. The fact, that smooth envelopes (as used for Gabor frames), multiplied with pure frequencies are at least approximate eigenvectors for so-called *slowly varying channels* makes them useful for mobile communication. The physical assumption of limited multi-path propagation (variable kernels over time) and Doppler (due to movement) related to underspread operators, i.e. to matrices whose spreading function is supported on a given rectangular domain.

Applications of Gabor Riesz bases:

The information, encoded as a collection of coefficients which we will call (c_{λ°) are used to form a linear combination of the elements of our Gaborian Riesz basis. I.e. the sender *plays slowly a melody on the piano*.
Assume we are able to estimate the approximate eigenvalues $(d_{\lambda^\circ}$ of the involved building blocks $(g_{\lambda^\circ}$, the approximate eigenvector property of these building blocks implies that the receiver obtains $\sum_{\lambda^\circ} c_{\lambda^\circ} d_{\lambda^\circ} g_{\lambda^\circ}$. Knowing the factors (d_{λ°) (by sending so-called pilot tones) and the biorthogonal basis the receiver can then (approximately) recover the set of coefficients (c_{λ°) sent by the sender.
In other words, *the receiver listens to the music behind a wall, knowing e.g. that higher frequencies are absorbed more (or less) then others and figures out, what has been played.*

The (canonical) dual Gabor frame

This greatly simplifies the calculation of (minimal norm) coefficients for the given signal. In fact, it is found that the solution $\widetilde{g}$ of the simple (positive definite) linear equation

$$S\widetilde{g} = g \quad \text{resp.} \quad \widetilde{g} = S^{-1}g,$$

spans the *dual Gabor frame*. In fact FFT-based methods can be applied to efficiently calculate these coefficients, once $\widetilde{g}$ is given. Sometimes alternative sets of coefficients are equally useful.
For the solution of the above equations various iterative methods, e.g. conjugate gradients, can be applied .

dual atoms and Gabor multipliers

It is clear, that one actually would like to build an arbitrary signal f given the pair (g, Λ) (in the frame case), or at least do the best approximation of f by linear combinations from the Gabor family in the Riesz basis case. In both cases one has a number of choices, but the canonical one (related to `PINV` resp. to the associated MNLSQ-problem is the one usually preferred.

The appropriate coefficients are then obtained by taking scalar products with respect to the correspondig “dual” family, which is numerically efficiently implemented by doing a sampled STFT (using FFT-based methods).

Operating on the audio signal: filter banks

Finally let us operate on the Gabor coefficients

GMO

Definition

Let g_1, g_2 be two L^2-functions, Λ a TF-lattice for $\mathbb{R}^d$, i.e. a discrete subgroup of the phase space $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$. Furthermore let $\mathbf{m} = (\mathbf{m}(\lambda))_{\lambda\in\Lambda}$ be a complex-valued sequence on Λ. Then the **Gabor multiplier** associated to the triple (g_1, g_2, Λ) with (*strong* or) **upper symbol m** is given as

$$G_{\mathbf{m}}(f) = G_{g_1,g_2,\Lambda,\mathbf{m}}(f) = \sum_{\lambda\in\Lambda} m(\lambda)\langle f, \pi(\lambda)g_1\rangle\, \pi(\lambda)g_2. \qquad (20)$$

g_1 is called the *analysis* window, and g_2 is the synthesis window. If $g_1 = g_2$ and $\mathbf{m}$ is real-valued, then the Gabor multiplier is self-adjoint. Since the constant function $\mathbf{m} \equiv 1$ is mapped into the Identity operator if $g_1 = g_2$ is a Λ-tight Gabor atom this is often the preferred choice.

The family of projection operators (P_λ)

ual

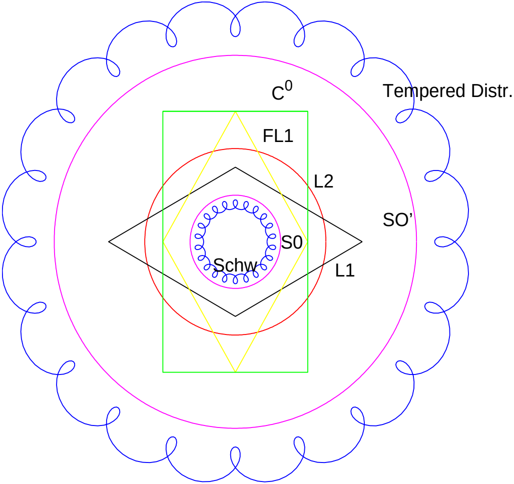
Theorem

Assume that (g, Λ) generates an S_0-Gabor frame for $\mathbf{L}^2(\mathbb{R}^d)$, with $\|g\|_2 = 1$, and write P_λ for the projection $f \mapsto \langle f, \pi(\lambda)g\rangle \pi(\lambda)g$.
i) Then the family $(P_\lambda)_{\lambda\in\Lambda}$ is a Riesz basis for its closed linear span within the Hilbert space $\mathcal{HS}$ of all Hilbert-Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ if and only if the function $H(s)$, defined as the Λ-Fourier transform of $\left(|STFT_g(g)(\lambda)|^2\right)_{\lambda\in\Lambda}$ is does not have zeros.
ii) An operator T belongs to the closed linear span of this Riesz basis if and only if it belongs to $\mathcal{GM}_2$, the space of Gabor multiplier with $\ell^2(\Lambda)$-symbol.
iii) The canonical biorthogonal family to $(P_\lambda)_{\lambda\in\Lambda}$ is of the form $(Q_\lambda)_{\lambda\in\Lambda}$,

$$Q_\lambda = \pi(\lambda) \circ Q \circ \pi^{-1}(\lambda) \text{ for } \lambda\in\Lambda,$$

for a uniquely determined Gabor multiplier $Q \in \mathcal{B}$.
iv) The best approximation of $T \in \mathcal{HS}$ by Gabor multipliers based on the pair (g, Λ) is of the form

Summarizing the situation: test functions & distributions



A few relevant references

K. Gröchenig: Foudations of Time-Frequency Analysis, Birkhäuser, 2001.

H.G. Feichtinger and T. Strohmer: Gabor Analysis, Birkhäuser, 1998.

H.G. Feichtinger and T. Strohmer: Advances in Gabor Analysis, Birkhäuser, 2003.

G. Folland: Harmonic Analysis in Phase Space. Princeton University Press, 1989.

I. Daubechies: Ten Lectures on Wavelets, SIAM, 1992.

Some further books in the field are in preparation, e.g. on modulation spaces and pseudo-differential operators.

See also www.nuhag.eu/talks.

Some recent comment on the Physics Nobel Prize 2017

Time-Frequency Analysis and Black Holes

Breaking News

Today, Oct. 3rd, 2017, the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!

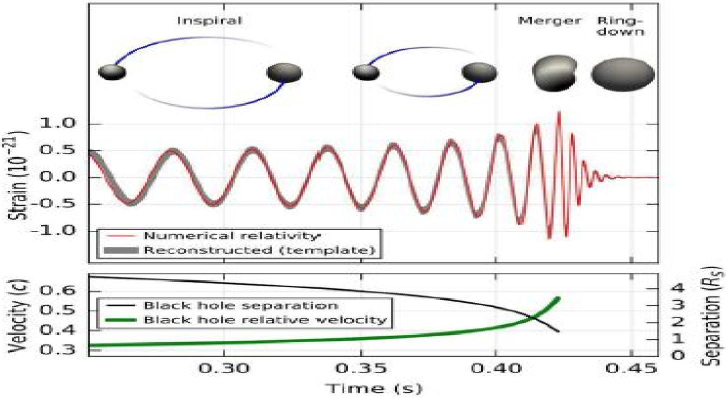
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal "needle in the haystack".

There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.

NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

THANK YOU

Thank you for your attention and the honour of giving this

16th Jarnik Lecture

at Charles University in the nice city of Prague!

More at www.nuhag.eu