FOURIER STANDARD SPACES and the Kernel Theorem

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OVERVIEW

We will concentrate on the setting of the LCA group $G = \mathbb{R}^d$, although all the results are valid in the setting of general **locally compact Abelian groups** as promoted by **A. Weil**.

Classical Fourier Analysis pays a lot of attention to $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ because these spaces (specifically for $p \in \{1, 2, \infty\}$) are important to set up the Fourier transform as an integral transform which also respects convolution (we have the convolution theorem) and preserving the energy (meaning that it is a unitary transform of the Hilbert space $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$).

Occasionally the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ is used and its dual $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, the space of tempered distributions (e.g. for PDE and the *kernel theorem*, identifying operators from $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ to $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ with their distributional kernels in $\boldsymbol{\mathcal{S}}'(\mathbb{R}^{2d})$).

OVERVIEW II

In the last 2-3 decades the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (equal to the modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1}))$ and its dual, $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ or $\boldsymbol{M}^\infty(\mathbb{R}^d)$ have gained importance for many questions of Gabor analysis or time-frequency analysis.

Fourier standard spaces is a new name for a class of Banach spaces sandwiched in between $\boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$, with *two module structures*, one with respect to the Banach convolution algebra $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$, and the other by pointwise multiplication with elements of the Fourier algebra $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.

As we shall point out there is a huge variety of such spaces, and many questions of Fourier analysis find an appropriate description in this context.

OVERVIEW III

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions (requiring to work with the space $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$), and many other spaces.

Within the family there are two subfamilies, namely the *Wiener amalgam spaces* and the so-called *modulation spaces*, among them the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

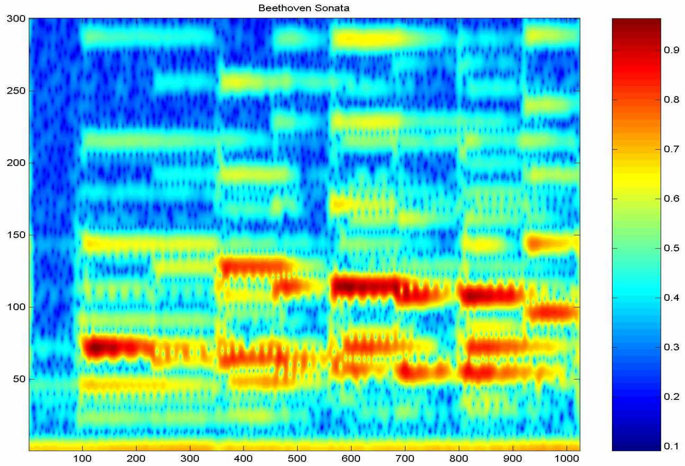
Behavior under Fourier transform

$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

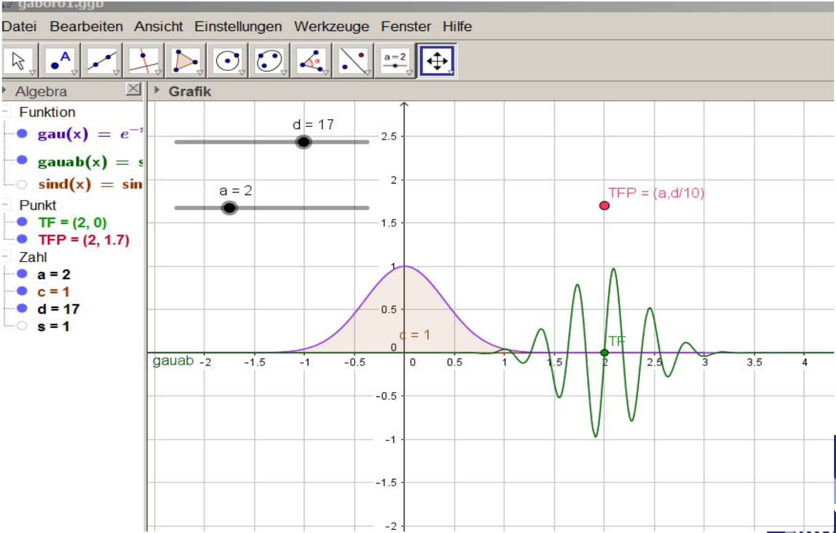
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2 = 1$ makes $f \mapsto V_g(f)$ isometric, hence we request this from now on. Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because $\forall h \in \mathcal{H}_1$

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \tag{1}$$

and thus by the *polarization principle* $T^*T = Id$.
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \tag{2}$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \tag{3}$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h \rangle = \langle V_g^*(V_g(f)), h \rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \quad (4)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda) g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle\, g_\lambda \, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \quad (5)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s \rangle\, \chi_s \, ds, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (6)$$

with $\chi_s(t) = exp(2\pi i \langle s, t \rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (6) (Fourier inversion) and the new formula formula (5). The building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters $\chi_s \notin \boldsymbol{L}^2(\mathbb{R}^d)$. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see [F. Weisz] Inversion of the short-time Fourier transform using Riemannian sums for example [7]).

Modulation spaces, in particular $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$

The reconstruction of f from its STFT (Short-time Fourier Transform) suggests that at least for "good windows" g one can control the smoothness (and/or decay) of a function or distribution by controlling the decay of $V_g(f)$ in the frequency resp. the time direction.

A polynomial weight depending on the frequency variable only can be used to describe Sobolev spaces, and (weighted) mixed-norm conditions can be used to define the (now classical) **modulation spaces** $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$.

We will put particular emphasis on the modulation spaces $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^{1,1} = \boldsymbol{M}^1$, characterized by the membership of $V_g(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$ and $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^{\infty,\infty} = \boldsymbol{M}^\infty$, with uniform convergence describing norm convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$, while pointwise convergence corresponds to the w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Wilson bases and modulation spaces

It is perhaps enlightening to know that the spaces $\boldsymbol{M}^p(\mathbb{R}^d)$ can be characterized with the help of so-called *Wilson bases* (also via local Fourier bases and of course via Gabor expansions).

Theorem

Assume that $(g_{k,n})_{k\in\mathbb{Z}^d, n\in\mathbb{N}^d}$ is a Wilson basis with generator $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$. Then a tempered distribution $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ belongs to $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$ if and only if the Wilson-coefficients of f belong to the corresponding weighted, mixed-norm sequence space, i.e. the following expression (equivalent norm for $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$) is finite

$$\Big[\sum_{n\in\mathbb{N}^d}\Big(\sum_{k\in\mathbb{Z}^d}|\langle f, g_{k,n}\rangle|^p\Big)^{q/p}(1+n)^{sq}\Big]^{1/q} < \infty. \tag{7}$$

Modulation spaces $M^p(\mathbb{R}^d)$ and Gabor analysis

Just as an alternative let us remind of the following situation concerning Gabor frames:

Theorem

Assume that (g, Λ) generators a Gabor frame with generator $g \in \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$, with dual Gabor atom $\widetilde{g}$. Then $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$ belongs to $\boldsymbol{M}^p(\mathbb{R}^d)$ if and only if one of the following expressions (equivalent norms) are finite:

1. $|V_g(f)|_\Lambda\|_{\ell^p}$;
2. $\|V_{gd}(f)|_\Lambda\|_{\ell^p}$.

Alternatively, $f \in \boldsymbol{M}^p(\mathbb{R}^d)$ if and only if it has an atomic representation of the form $\sum_{\lambda\in\Lambda} c_\lambda \pi(\lambda) g$, with $\mathbf{c} = (c_\lambda)_{\lambda\in\Lambda} \in \ell^p(\Lambda)$.

Banach Module Terminology

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ bilinear and associative, such that

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}, \tag{8}$$

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}. \tag{9}$$

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach ideal* in (or within, or of) a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$, and if in addition (8) is valid with respect to the internal multiplication inherited from $\boldsymbol{A}$.

Wendel's Theorem

Theorem

The space of $\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)$ *all bounded linear operators on* $\mathbf{L}^1(G)$ *which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with* $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$. *In terms of our formulas this means*

$$\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)(\mathbb{R}^d) \simeq (\mathbf{M}_b(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}_b}),$$

$$\text{via } T \simeq C_\mu : f \mapsto \mu * f, \quad f \in \mathbf{L}^1, \mu \in \mathbf{M}_b(\mathbb{R}^d).$$

Lemma

$$B_{\mathbf{L}^1} = \{f \in \mathbf{B} \mid \|T_x f - f\|_{\mathbf{B}} \to 0, \ for x \to 0\}.$$

Consequently we have $(\mathbf{M}_b(\mathbb{R}^d))_{\mathbf{L}^1} = \mathbf{L}^1(\mathbb{R}^d)$, *the closed ideal of absolutely continuous bounded measures on* $\mathbb{R}^d$.

Pointwise Multipliers

Via the Fourier transform we have similar statements for the Fourier algebra, involving the *Fourier Stieltjes algebra*.

$$\mathcal{H}_{\mathcal{F}\boldsymbol{L}^1}(\mathcal{F}\boldsymbol{L}^1, \mathcal{F}\boldsymbol{L}^1) = \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d)), \quad \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))_{\mathcal{F}\boldsymbol{L}^1} = \mathcal{F}\boldsymbol{L}^1. \quad (10)$$

Theorem

The completion of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *(viewed as a Banach algebra and module over itself) is given by*

$$\mathcal{H}_{\boldsymbol{C}_0}(\boldsymbol{C}_0, \boldsymbol{C}_0) = \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right).$$

On the other hand we have $(\boldsymbol{C}_b(\mathbb{R}^d))_{\boldsymbol{C}_0} = \boldsymbol{C}_0(\mathbb{R}^d)$.

Essential part and closure

In the sequel we assume that $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a Banach algebra with bounded approximate units, such as $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (with convolution), or $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ or $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ with pointwise multiplication.

Theorem

Let $\boldsymbol{A}$ be a Banach algebra with bounded approximate units, and $\boldsymbol{B}$ a Banach module over $\boldsymbol{A}$. Then we have the following general identifications:

$$(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{11}$$

or in a slightly more compact form:

$$\boldsymbol{B}_{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad {\boldsymbol{B}^{\boldsymbol{A}}}_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad {\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{12}$$

Essential Banach modules and BAIs

The usual way to define the *essential part* $\boldsymbol{B_A}$ resp. $\boldsymbol{B_e}$ of a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to some Banach algebra action $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \bullet \mathbf{b}$ is defined as the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ within $\left(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\right)$. This subspace has other nice characterizations using BAIs (bounded approximate units (BAI) in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$):

Lemma

For any BAI $(\mathbf{e}_\alpha)_{\alpha \in I}$ *in* $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ *one has:*

$$\boldsymbol{B_A} = \{\mathbf{b} \in \boldsymbol{B} \mid \lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b}\} \tag{13}$$

The Cohen-Hewitt Factorization Theorem

In particular one has: Let $(\mathbf{e}_\alpha)_{\alpha \in I}$ and $(\mathbf{u}_b)_{\beta \in J}$ be two bounded approximate units (i.e. bounded nets within $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ acting in the limit like an identity in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Then

$$\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b} \Leftrightarrow \lim_\beta \mathbf{u}_\beta \bullet \mathbf{b} = \mathbf{b}. \tag{14}$$

Theorem

(The Cohen-Hewitt factorization theorem, without proof, see [5]) Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI of size $C > 0$, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

Essential part and closure II

Having now Banach spaces of distributions which have two module structures, we have to use corresponding symbols. FROM NOW ON we will use the letter $\boldsymbol{A}$ mostly for pointwise Banach algebras and thus for the $\mathcal{F}\boldsymbol{L}^1$-action on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and we will use the symbol G (because convolution is coming from the integrated group action!) for the $\boldsymbol{L}^1$ convolution structure. We thus have

$$\boldsymbol{B}_{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad {\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad \boldsymbol{B}_{\boldsymbol{G}}^{\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}, \quad \boldsymbol{B}^{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}. \tag{15}$$

In this way we can combine the two operators (in view of the above formulas we can call them interior and closure operation) with respect to the two module actions and form spaces such as

$${\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{{\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}} \ldots$$

or changes of arbitrary length, as long as the symbols $\boldsymbol{A}$ and $\boldsymbol{G}$ appear in alternating form (at any position, upper or lower).

Combining the two module structures

Fortunately one can verify (paper with W.Braun from 1983, J.Funct.Anal.) that any "long" chain can be reduced to a chain of at most two symbols, the *last occurence of each of the two symbols being the relevant one!* So in fact all the three symbols in the above chain describe the same space of distributions.
But still we are left with the follwoing collection of altogether eight two-letter symbols:

$$\boldsymbol{B}_{GA}, \boldsymbol{B}_{AG}, \boldsymbol{B}_{A}{}^{G}, \boldsymbol{B}^{G}{}_{A}, \boldsymbol{B}_{G}{}^{A}, \boldsymbol{B}^{A}{}_{G}, \boldsymbol{B}^{AG}, \boldsymbol{B}^{GA} \tag{16}$$

and of course the four one-symbol objects

$$\boldsymbol{B}_{A}, \boldsymbol{B}_{G}, \boldsymbol{B}^{A}, \boldsymbol{B}^{G} \tag{17}$$

Some structures, simple facts

There are other, quite simple and useful facts, such as

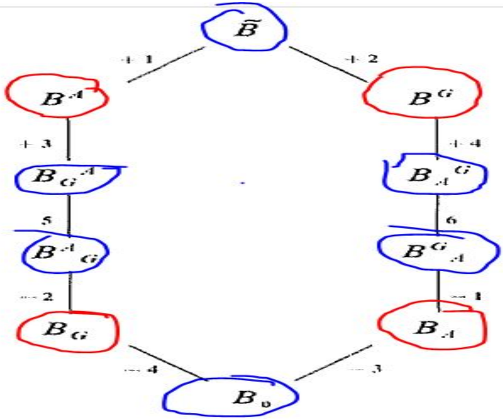
$$\mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, \boldsymbol{B}^2) = \mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, {\boldsymbol{B}^2}_{\boldsymbol{A}}) \tag{18}$$

which can easily be verified if ${\boldsymbol{B}^1}_{\boldsymbol{A}} = \boldsymbol{A} \bullet \boldsymbol{B}^1$, since then $T \in \mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, \boldsymbol{B}^2)$ applied to $\mathbf{b}^1 = \mathbf{a} \bullet \mathbf{b}^{1'}$ gives

$$T(\mathbf{b}^1) = T(\mathbf{a} \bullet \mathbf{b}^{1'}) = \mathbf{a} \bullet T(\mathbf{b}^{1'}) \in {\boldsymbol{B}^2}_{\boldsymbol{A}}.$$

The Main Diagram

This diagram is taken from [1].



Fourier Standard Spaces, FouSS

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, continuously embedded between $\boldsymbol{S}_0(G)$ and $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$, i.e. with

$$(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow (\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$$

is called a **Fourier Standard Space** on G (FSS of FoSS) if it has a *double module structure*: over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to *convolution* and over (the Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\widehat{G}))$ with respect to *pointwise multiplication*.

REMARK: One could unify this assumption by combining the two separate (commutative) group actions by the *integrated group action* of the *reduced Heisenberg group* $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$ under the *Schrödinger representation*: $\pi(t, s, \tau) = \tau M_s T_t$.

A sufficient setting is the following one:

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d,\ f \in \boldsymbol{B}. \tag{19}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous on often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty)$ or $(\boldsymbol{M}_b(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}_b})$).

An important fact concerning this family is the minimality property of the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$.
There is also a maximal space in the family of Fourier standard spaces, namely the dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ *resp.* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty)(\mathbb{R}^d)$.

The second claim even makes sense if FouSSs are defined as subspaces of the much larger space $\mathcal{S}'(\mathbb{R}^d)$ of tempered distributions!

Discussion of the Diagram

For each of the Fourier Standard Spaces the discussion of the above diagram makes sense. One may see that it can collaps totally to a single space, or that it has in fact a rich (like $(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty)$) or simple structure.

Theorem

A Fourier standard space is maximal, i.e. coicides with $\widetilde{\boldsymbol{B}} = \boldsymbol{B}^{AG} = \boldsymbol{B}^{GA}$ if and only if $\boldsymbol{B}$ is a dual space.

There is also a formula for the predual spaces, it is $((\boldsymbol{B}_0)')_0$, where $\boldsymbol{B}_0 = \boldsymbol{B}_{AG} = \boldsymbol{B}_{GA}$ is just the closure of $\mathcal{S}(\mathbb{R}^d)$ resp. $\boldsymbol{S}_0(\mathbb{R}^d)$ in $\boldsymbol{B}$.

Of course $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is minimal if and only if $\boldsymbol{S}_0(\mathbb{R}^d)$ is a dense subspace of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, resp. if it is a TF-homogeneous Banach space.

Discussion of the Diagram II

Theorem

A Fourier standard space is reflexive if and only if both the space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$ and its dual are both minimal and maximal. In other words, for the space itself and its dual the diagram is reduced to a single space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$.

Constructions within the FouSS Family

1. taking Fourier transforms;
2. *conditional dual spaces*, i.e. the dual space of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
3. with two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take intersection or sum
4. forming amalgam spaces $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. defining pointwise or convolution multipliers;
6. using complex (or real) interpolation methods, so that we get the (Fourier invariant) spaces $\boldsymbol{M}^{p,p} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$;
7. fractional invariant kernel and hull: For any given standard space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ we could define the largest Banach space inside of $\boldsymbol{B}$ which is invariant under all the fractional FTs, or the smallest such space which allows a continuous embedding of $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ into that space.

Constructions within the FouSS Family II

To explain the setting let us start with the familiar family of $\boldsymbol{L}^p$-spaces on a LCA group, say $G = \mathbb{R}^d$, and $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) = \left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$, for some $p \in [1, \infty)$. The $\left(\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ is well defined as the image of $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ under the Fourier transform, with transport of norm. It is another FouSS, even for $p > 2$ (because it is still well defined as a subspace of $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$).

It is a natural question to find the range of values (r, s) such that

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^r) \subseteq \mathcal{F}\boldsymbol{L}^p \subseteq \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^s).$$

Investigations by Peter Gröbner have shown (1992) that this is OK if and only iff $r \leq min(p, p')$ and $s \geq max(p, p')$.

Modulation spaces are Fourier Standard Spaces

The unweighted *modulation spaces* $\left(\boldsymbol{M}^{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,q}}\right)$ can be obtained by first forming the Wiener amalgams $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)$ and then taking the inverse Fourier transform of these spaces.

The above inclusion relations then translate into exact embedding conditions between $\boldsymbol{L}^p$-spaces and the corresponding modulation spaces.

Obviously there are natural embeddings between modulation spaces with parameters p_1, q_1 and p_2, q_2, with $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^1 = \boldsymbol{M}^{1,1}$ being the smallest one!

Constructions within the FouSS Family IV

There is a small body of literature (mostly papers by Kelly McKennon, a former PhD student of Edwin Hewitt) concerning spaces of "tempered elements". He has done the case starting $\boldsymbol{B} = \boldsymbol{L}^p(G)$, over general LC groups, but the construction makes sense if (and only if) one has a nice invariant space which happens not to be a convolution (or pointwise) algebra.
By *intersecting* the space with its own "multiplier algebra" one obtains an (abstract) Banach algebra, and often the Banach algebra homomorphism of this new algebra "are" just the translation invariant operators on the original spaces.
For the case of $\boldsymbol{B} = \left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ one would define

$$\boldsymbol{L}_p^t := \boldsymbol{L}^p \cap \mathcal{H}_G(\boldsymbol{L}^p, \boldsymbol{L}^p).$$

Tempered elements in L^p-spaces

understood as the intersection of two FouSSs, with the natural norm, which is the sum of the L^p-norm of f plus the operator norm of the convolution operator.

For $p > 2$ one has to be careful and has to define that operator norm only by looking at the action of $k \to k * f$ on $C_c(\mathbb{R}^d)$! (convolution in the pointwise sense might fail to exist, on more than just a null-set!).

However it is not a problem to approximate every element (in norm or even just in the w^*-sense) by test-functions in $S_0(\mathbb{R}^d)$ and then take the limit of the convolution products of the regularized expressions.

Constructions within the FouSS Family V

Another interesting result that came recently to my attention (thanks to Werner Ricker) provides an answer to the following question related to the Theorem of Hausdorff-Young:
We know, that one has that $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d) \subset \boldsymbol{L}^q(\mathbb{R}^d)$ for $1/p + 1/q = 1$, whenever $p \in [1, 2]$. But is there any strictly larger, solid Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (meaning pointwise $L^\infty(\mathbb{R}^d)$-module) such that it is still true that $\mathcal{F}(\boldsymbol{B}) \subset \boldsymbol{L}^q(\mathbb{R}^d)$).

The *answer* can be descibed as the FouSS (with natural norm):

$$\boldsymbol{B} = \mathcal{H}_{\boldsymbol{L}^\infty}(\boldsymbol{L}^\infty, \mathcal{F}\boldsymbol{L}^p).$$

In words: the pointwise multipliers from $\boldsymbol{L}^\infty(\mathbb{R}^d)$ to $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$.

Tensor products

Let us recall some basic terms concerning tensor products of functions or distributions (see [1], [2])
Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1) f^2(x_2), \;\; x_i \in \mathbb{R}^d, i = 1, 2.$$

For distributions this definition can be extended by taking w^*-limits or by duality, just like $\mu_1 \otimes \mu_2$ is defined, for two bounded measures $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$.
It is important to know that we have $\sigma_1 \otimes \sigma_2 \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ for any pair of distributions $\sigma_1, \sigma_2 \in \boldsymbol{S}_0'(\mathbb{R}^d)$.
In particular $\boldsymbol{S}_0'(\mathbb{R}^d) \widehat{\otimes} \boldsymbol{S}_0'(\mathbb{R}^d)$ is well defined and a (proper) subspace of $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Tensor product of FouSSps

Given two Banach spaces $\boldsymbol{B}^1$ and $\boldsymbol{B}^2$ embedded into $\boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$, $\boldsymbol{B}^1 \hat{\otimes} \boldsymbol{B}^2$ denotes their *projective tensor product*, i.e.

$$\left\{ f \mid f = \sum f_n^1 \otimes f_n^2, \; \sum \|f_n^1\|_{\boldsymbol{B}^1} \|f_n^2\|_{\boldsymbol{B}^2} < \infty \right\}; \qquad (20)$$

It is easy to show that this defines a Banach space of tempered distributions on $\mathbb{R}^{2d}$ with respect to the (quotient) norm:

$$\|f\|_{\hat{\otimes}} := inf\{\sum \|f_n^1\|_{B^1} \|f_n^2\|_{B^2}, ..\} \qquad (21)$$

where the infimum is taken over all admissible representations.

The Varopoulos algebra $V_0(\mathbb{R}^{2d})$ and bimeasures

For questions of harmonic analysis the so-called *Varopoulos algebra* $\boldsymbol{V}_0(\mathbb{R}^{2d}) := \boldsymbol{C}_0(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{C}_0(\mathbb{R}^d)$ plays an important role.
The dual space of this tensor product, which is a *proper subspace* of $\boldsymbol{C}_0(\mathbb{R}^{2d})$ is called the space of *bi-measures* $\boldsymbol{BM}(\mathbb{R}^{2d})$, which form again a Banach algebra with respect to convolution.
Their Fourier transforms (in the sense of $\boldsymbol{S}_0'$) are still well defined, and are bounded continuous functions, and one has again a *convolution theorem* (convolution goes into pointwise multiplication under the FT).
The space $\boldsymbol{BM}$ shares with $\boldsymbol{M}_b(\mathbb{R}^d)$ the property that the compactly supported elements are dense in the space, i.e. $\boldsymbol{B} = \boldsymbol{B}_{\boldsymbol{A}}$ in the spirit of the diagram.

The KERNEL THEOREM for *ScRd*

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{22}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (22) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)

One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert, [6]) is the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \tag{23}$$

with equivalence of the corresponding norms.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

This result is the "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\boldsymbol{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\boldsymbol{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x, y) = \delta_x(T(\delta_y), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ and the operator Gelfand triple around the Hilbert space $\mathcal{HS}$ of Hilbert Schmidt operators, namely $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$, where the first set is understood as the w^* to norm continuous operators from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\mathbb{R}^d)$, the so-called regularizing operators.

Spreading function and Kohn-Nirenberg symbol

1. For $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$ the *pseudodifferential operator* with *Kohn-Nirenberg symbol* σ is given by:

$$T_\sigma f(x) = \int_{\mathbb{R}^d} \sigma(x,\omega)\hat{f}(\omega)e^{2\pi i x\cdot\omega}\,d\omega$$

   The formula for the integral kernel $K(x,y)$ is obtained

$$\begin{aligned} T_\sigma f(x) &= \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} \sigma(x,\omega)e^{-2\pi i (y-x)\cdot\omega}\,d\omega \Big) f(y)dy \\ &= \int_{\mathbb{R}^d} k(x,y)f(y)dy. \end{aligned}$$

2. The *spreading representation* of T_σ arises from

$$T_\sigma f(x) = \iint_{\mathbb{R}^{2d}} \widehat{\sigma}(\eta,u)M_\eta T_{-u}f(x)\,du\,d\eta.$$

   $\widehat{\sigma}$ is called the spreading function of T_σ.

Further details concerning Kohn-Nirenberg symbol

(courtesy of Goetz Pfander (Eichstätt):)

- *Symmetric coordinate transform:* $\mathcal{T}_s F(x,y) = F(x+\frac{y}{2}, x-\frac{y}{2})$
- *Anti-symmetric coordinate transform:* $\mathcal{T}_a F(x,y) = F(x, y-x)$
- *Reflection:* $\mathcal{I}_2 F(x,y) = F(x,-y)$
- *partial Fourier transform in the first variable:* $\mathcal{F}_1$
- *partial Fourier transform in the second variable:* $\mathcal{F}_2$

The kernel $K(x,y)$ can be described as follows:

$$K(x,y) = \mathcal{F}_2\sigma(\eta, y-x) = \mathcal{F}_1^{-1}\widehat{\sigma}(x, y-x)$$
$$= \int_{\mathbb{R}^d} \widehat{\sigma}(\eta, y-x) \cdot e^{2\pi i \eta \cdot x} d\eta.$$

Kohn-Nirenberg symbol and spreading function II

operator H	$Hf(x)$
$\updownarrow$	$=$
kernel κ_H	$\int \kappa_H(x,s) f(s)\, ds$
$\updownarrow$	$=$
Kohn–Nirenberg symbol σ_H	$\int \sigma_H(x,\omega) \widehat{f}(\omega) e^{2\pi i x\cdot\omega}\, d\omega$
$\updownarrow$	$=$
time–varying impulse response h_H	$\int h_H(t,x) f(x-t)\, dt$
$\updownarrow$	$=$
spreading function η_H	$\int\int \eta_H(t,\nu) f(x-t) e^{2\pi i x\cdot\nu}\, dt\, d\nu$
	$=$
	$\int\int \eta_H(t,\nu) M_\nu T_t f(x), dt\, d\nu\,,$

Spreading representation and commutation relations

The description of operators through the spreading function and allows to understand a number of commutation relations.
If an operator is a limit (in the strong operator topology) of translation operators it is just a convolution operator with some $\tau \in \boldsymbol{S}_0'(\mathbb{R}^d)$, resp. its spreading representation is just an element concentrated on the *time axis* (more or less representing $\hat{\tau}$, the "individual frequency contributions".
Similarly, multiplication operators require just the use of modulation operators, so their spreading function is concentrated in the frequency axis of the TF-plane.
Finally typical *Gabor frame operators* arising from a family of Gabor atoms (g_λ), where $\lambda \in \Lambda$, some lattice within $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ typically commute with TF-shift operators, one can say that they are obtained by periodizing the projection operator $f \mapsto \langle f, g \rangle g$ along the lattice.

The symplectic Fourier transform

The *symplectic Fourier transform* connects the Kohn-Nirenberg symbol with the spreading function, i.e.

$$\mathcal{F}_s(\sigma(T)) = \eta(T) \quad \text{resp.} \quad \mathcal{F}_s(\eta(T)) = \sigma(T). \tag{24}$$

$$(\mathcal{F}_{symp} f)(k,l) = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} f(x,y) e^{-2\pi i (k \cdot y - l \cdot x)}; \quad f \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d). \tag{25}$$

It is completely characterized by its action on elementary tensors:

$$\mathcal{F}_{symp}(f \otimes \hat{g}) = g \otimes \hat{f}, \quad f, g \in \boldsymbol{S}_0(\mathbb{R}^d), \tag{26}$$

and extends from there in a unique way to a $w^* - w^*$ continuous mapping from $\boldsymbol{S}_0'(\mathbb{R}^{2d})$ to $\boldsymbol{S}_0'(\mathbb{R}^{2d})$, also ${\mathcal{F}_s}^2 = Id$.

Understanding the Janssen representation

The spreading representation of operators has properties very similar to the ordinary Fourier expansion for functions!
Periodization at one side corresponds to sampling on the transform side, if we understand "translation" either at the level of ordinary translation of the Kohn-Nirenberg symbol (which is the *symplectic Fourier transform* of the spreading function), OR by conjugation of an operator by the corresponding TF-shifts.
In other words: for any given operator T and $\lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ we can **define** [recall $\pi(x,\omega) = M_\omega T_x$ for $\lambda = (x,\omega)$]

$$\pi \otimes \pi^*(T) = \pi(\lambda) \circ T \circ \pi(\lambda)^*, \tag{27}$$

providing the important *covariance property* for KNS:

$$\sigma[\pi \otimes \pi^*(\lambda)(T)] = T_\lambda[\sigma(T)], \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d. \tag{28}$$

Periodization goes over to sampling

If we have a "nice operator" T_0 we can form its periodic version $\sum_{\lambda \in \Lambda} \pi \otimes \pi^*(\lambda)(T_0)$ and it is still a well defined operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$. Its KNS is just the Λ-periodization of T_0. Consequently its spreading function is obtained by sampling of $\eta(T) \in \boldsymbol{S}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, over the *adjoint lattice* Λ° and obtain in this case an ℓ^1-sequence.
The adjoint lattice Λ° can be characterized by the fact that

$$\mathcal{F}_s(\sqcup\!\sqcup_\Lambda) = C_\Lambda \sqcup\!\sqcup_{\Lambda^\circ}. \tag{29}$$

For the projection on the Gabor atom $P_g : f \mapsto \langle f, g \rangle g$ the spreading functions is essentially

$$[\eta(P_g)](\lambda) = Vg(g)(\lambda) = \langle g, \pi(\lambda) g \rangle, \quad \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d.$$

An important insight concerning the connection between the Gabor atom g, the TF-lattice $\Lambda \lhd \mathbb{R}^d \times \widehat{\mathbb{R}}^d$ and the quality of the resulting Gabor frame resp. Gabor Riesz basis (e.g.condition number) clearly comes from the *Janssen representation* of the *Gabor frame operator* for any $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$S_{g,\Lambda}(f) = \sum_{\lambda \in \Lambda} P_{g_\lambda}(f) = \sum_{\lambda \in \Lambda} \pi \otimes \pi^*(\lambda)[P_g]. \tag{30}$$

The periodization principle gives the Janssen representation

$$S_{g,\Lambda} = \eta^{-1}[\eta(S_{g,\Lambda})] = C_\Lambda \sum_{\lambda^\circ \in \Lambda^\circ} V_g(g))(\lambda^\circ)\pi(\lambda^\circ), \tag{31}$$

as an absolutely convergent sum of TF-shifts from Λ°.

Fourier Standard Spaces of Operators

The kernel theorem allows to identify many spaces of linear operators (with different forms of continuity) with suitable FouSSs over $\mathbb{R}^{2d}$.

For example, there are the so-called *Schatten classes* of operators on the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$ which are compact operators with singular values in ℓ^p, for $1 \leq p < \infty$. These spaces are *operator ideals* within $\mathcal{L}(\mathcal{H})$, i.e. they are Banach spaces, continuously embedded into the space of compact operators over the Hilbert space $\mathcal{H}$, as well as two-sided Banach ideals, i.e. whenever one has an operator T in such a space, and two bounded operators S_1, S_2 on $\mathcal{H}$, then $S_1 \circ T \circ S_2$ also belongs to that *operator ideal* and the operator ideal norm is bounded by the operator ideal norm of T multiplied with the operator norms of S_1 and S_2.

Spaces of Operators

Another family of operators are defined by their boundedness between certain FouSSs, e.g. an operator may be bounded from $\boldsymbol{L}^p(\mathbb{R}^d)$ (with $p \in [1, \infty)$) to some $\boldsymbol{L}^q(\mathbb{R}^d)$, with $1 \leq q \leq \infty$. Each of these operators has a kernel, so we can look at the set of all the kernels of bounded operators from $\boldsymbol{L}^p(\mathbb{R}^d)$ to $\boldsymbol{L}^q(\mathbb{R}^d)$ for p, q in the range described above, simply by testing the norm continuity on the dense subspace (of $\boldsymbol{L}^p(\mathbb{R}^d)$, for $p < \infty$) and embedding the target space into $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$.

Theorem

Consider the Banach space of operators $\mathbf{L}(\boldsymbol{L}^p, \boldsymbol{L}^q)$, with $1 \leq p, q < \infty$, which is isomorphic to a space of kernels in $\boldsymbol{S}_0'(\mathbb{R}^{2d})$, with the norm of the kernel being just the operator norm of the corresponding operator.
Then the space of kernels is ismorphic to the dual of the FouSS $\boldsymbol{L}^p(\mathbb{R}^d)\widehat{\otimes}\boldsymbol{L}^{q'}(\mathbb{R}^d)$.

Spaces of Operators II

Next we define the *Herz algebras* $\mathbf{A}^p(\mathbb{R}^d)$ via the "convolution tensor product: The dual space of the space of all (convolution) multipliers from $\mathbf{L}^p(\mathbb{R}^d)$ to $\mathbf{L}^p(\mathbb{R}^d)$ (for $1 < p < \infty$) can be identified with the dual space of the Herz algebra $\mathbf{A}_p(\mathbb{R}^d)$, given by

$$\mathbf{A}_p(\mathbb{R}^d) := \mathbf{L}^p(\mathbb{R}^d)\widehat{\circledast}\mathbf{L}^q(\mathbb{R}^d).$$

In the background of such a theorem stands the fact that a matrix commutes with (cyclic) translations if and only if it is constant along the side-diagonals. The kernels of such operators are constant along the main diagonal, respectively are a "moving average". Spectral synthesis results for the Fourier transform on $\mathbf{S}_0'(\mathbb{R}^d)$ then allow to derive this result.

Spaces of Operators III

Of course one can now start to combine the various items, especially concerning the "diagram" and the relation to Wiener amalgam spaces with these spaces.

One may ask, what about the density of $\boldsymbol{S}_0(\mathbb{R}^{2d})$ in such a space of operators (is it minimal, or a dual spaces, is it reflexive, and so on, what is the form the the general diagram).

One can also ask, what about decomposing the operator kernel into local patches of uniform size. Who is an ℓ^q-norm of the pieces related to the overall norm in such a FouSS of operators.

And furthermore one can switch to the set of *spreading symbols* or the *Kohn-Nirenberg* symbols of such FouSS of operators and ask similar questions.

Further information, LINKS

A lot of further material can be found through the NuHAG web-page, in particular at

`www.nuhag.eu/talks`

E.g. selecting one the follogin filters:

- BanGelTriples
- FeiTalks
- FeiConcept

or one of the (drafts of) lecture notes found at

`http://www.univie.ac.at/nuhag-php/home/skripten.php`

References

W. Braun and Hans G. Feichtinger.
Banach spaces of distributions having two module structures.
J. Funct. Anal., 51:174–212, 1983.

Elena Cordero, Hans G. Feichtinger, and Franz Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

Hans G. Feichtinger and Georg Zimmermann.
A Banach space of test functions for Gabor analysis.
In Hans G. Feichtinger and T. Strohmer, editors, *Gabor Analysis and Algorithms: Theory and Applications*, Applied and Numerical Harmonic Analysis, pages 123–170, Boston, MA, 1998. Birkhäuser Boston.

Karlheinz Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

Edwin Hewitt and Kenneth A. Ross.
Abstract Harmonic Analysis. Vol. II: Structure and Analysis for Compact Groups. Analysis on Locally Compact Abelian Groups.
Springer, Berlin, Heidelberg, New York, 1970.

Viktor Losert.
A characterization of the minimal strongly character invariant Segal algebra.
Ann. Inst. Fourier (Grenoble), 30:129–139, 1980.

Ferenc Weisz.
Inversion of the short-time Fourier transform using Riemannian sums.
J. Fourier Anal. Appl., 13(3):357–368, 2007.