Numerical Harmonic Analysis Group

A Function Space defined by the Wigner Transform and is Applications

Hans G. Feichtinger, Univ. Vienna & TUM
hans.feichtinger@univie.ac.at
www.nuhag.eu

Eugene Wigner Institute, Budapest

.

JOINT presentation with Maurice de Gosson

Key aspects of my talk

1. Fourier Analysis is a classical topic
2. Time-Frequency Analysis
3. The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$
4. Various Typical Applications
5. The Idea of *Conceptual Harmonic Analysis*

Starting with my personal background

- My education at the University of Vienna started as a *teacher student* in Mathematics and Physics;
- Soon the "imprecision" in physics combined with the BOURBAKI-style (clear and well structured) introduction into analysis (including Lebesgue integrals and functional analysis) let me go more in the direction of (pure) mathematics;
- In my third year of studies (1971) my then advisor H. Reiter arrived in Vienna, so I became a "Harmonic Analyst";
- After my habilitation (1979) I learned about the connection between Fourier Analysis and signal processing in Heidelberg;
- Since that time I tried to look out for *real world applications* and call myself now an *application oriented* mathematician;
- One of my recent "hobbies" is to promote what I call CONCEPTIONAL HARMONIC ANALYSIS.

A personal background story

At some point in the early 80th I tried to connect with the applied people at TU Vienna, and (fortunately) I ended up contacting Franz Hlawatsch (Communication Theory Dept., TU Vienna). *During our first meeting* he explained to me, that he was working on the so-called *Pseudo-Wigner distribution*, which is a kind of smoothed version of the Wigner distribution, with the idea of reducing the so-called interference terms in a Wigner distribution. I told him that I was studying a certain function space (I called it $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$, because it was a special *Segal algebra*), given by:

$$\boldsymbol{S}_0(\mathbb{R}^d) = \{f \mid f = \sum_{n\geq 1} c_n M_{\omega_n} T_{x_n} g_0, \text{ with } \sum_{n\geq 1} |c_n| < \infty\}, \quad (1)$$

where $g_0(x) = exp(-\pi|x|^2)$ is the usual Gauss function.

The relevance for my future work

At this point none of us had an idea how close we had been in terms of the setting of our research (which was then going on for almost 20 years)! He had **two important questions**

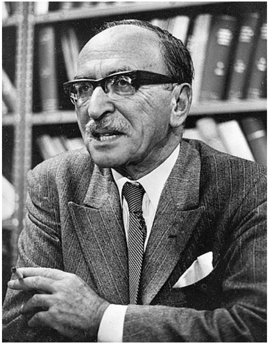
1. Do you really need all TF-shifts, D. Gabor has suggested to use only $x_n, \omega_n \in \mathbb{Z}^d$!
2. and: How do you compute the coefficients?

In fact, according to the claim made in D. Gabor's paper of 1946 [5] one *should expect* that an optimally centrated representation of any function, using integer TF-shifts of the Gauss function (achieving equality in the *Heisenberg Uncertainty Relation*!) should be possible. In fact, he only argued that a TF-lattice of the form $a\mathbb{Z} \times b\mathbb{Z}$ with $ab > 1$ is not comprehensive enough and $ab < 1$ produces linear dependencies.

Eugene Wigner

Eugene Paul Wigner (November 17, 1902 to January 1, 1995), was a Hungarian-American theoretical physicist, engineer, and mathematician. Nobel Prize in Physics in 1963 *for his contributions to the theory of the atomic nucleus and the elementary particles and symmetry principles.*

Dennis Gabor



Dennis Gabor (5 June 1900 to 9 February 1979) was a Hungarian-British electrical engineer and physicist, most notable for inventing holography, for which he later received the 1971 Nobel Prize in Physics.

$S_0(\mathbb{R}^d)$ via the Wigner transform

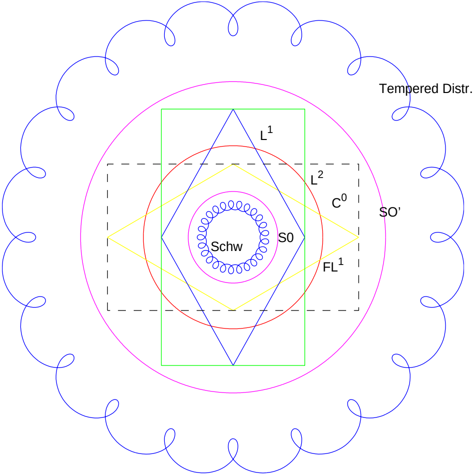
$$S_0(\mathbb{R}^d) = \left\{ f \in L^1(\mathbb{R}^d) \, : \, \int_{\mathbb{R}^{2d}} \|M_\omega f * f\|_1 \, d\omega < \infty \right\}. \tag{2}$$

Condition (2) is equivalent to the integrability of the Wigner functions over phase space $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$.
Here $M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)$, $t \in \mathbb{R}^d, \omega \in \mathbb{R}^d$, is the modulation operator and $*$ is the usual convolution of $L^1(\mathbb{R}^d)$-functions. Any non-zero function $g \in S_0(\mathbb{R}^d)$ defines a norm on $S_0(\mathbb{R}^d)$ via

$$\|f\|_{S_0,g} = \int_{\widehat{g}} \|E_\omega f * g\|_1 \, d\omega, \tag{3}$$

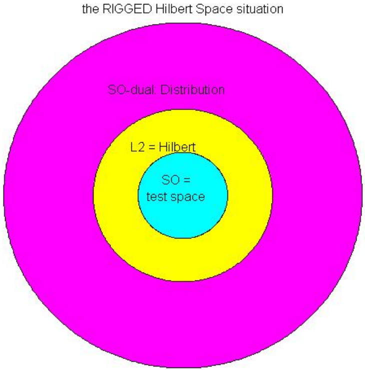
that turns $S_0(\mathbb{R}^d)$ into a Banach space. These norms are pairwise equivalent and we therefore allow ourselves to simply write $\|\cdot\|_{S_0}$ without specifying n g (see also [2]).

A schematic description: all the spaces



Banach Gelfand Triples: the simplified setting

Testfunctions $\subset$ Hilbert space $\subset$ Distributions, like $\mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$!



The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t).$$

Behavior under Fourier transform

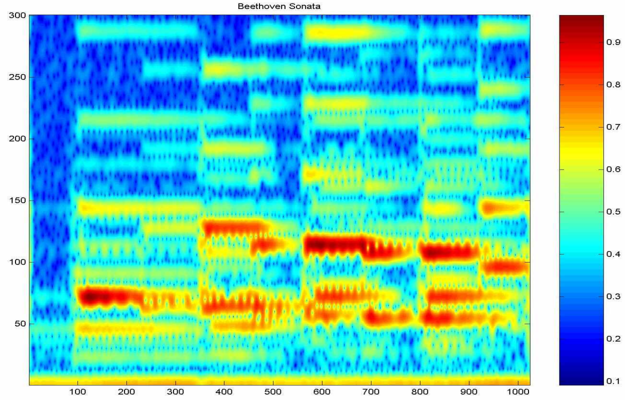
$$(T_x f)\widehat{} = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{} = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \ \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depictured using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\boldsymbol{\mathcal{S}}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in S_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in S_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{S_0} = \|f\|_{S_0}$.

(2) $\hat{f} \in S_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{S_0} = \|f\|_{S_0}$.

In fact, $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the L^p-spaces (and their Fourier images).

Various Function Spaces

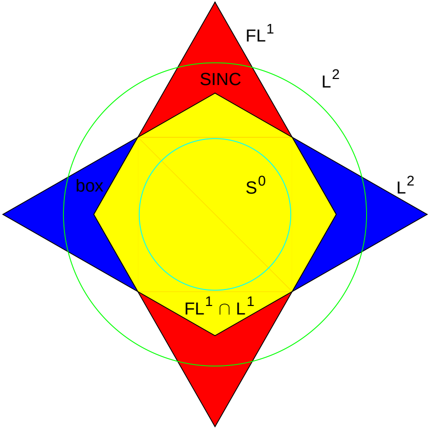


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

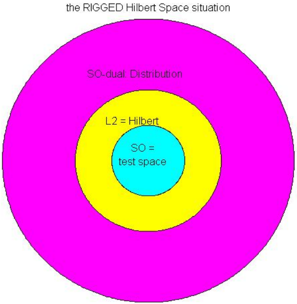
Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The prototypical examples over the torus

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{T})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{T}) = \boldsymbol{A}(\mathbb{T})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{T})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

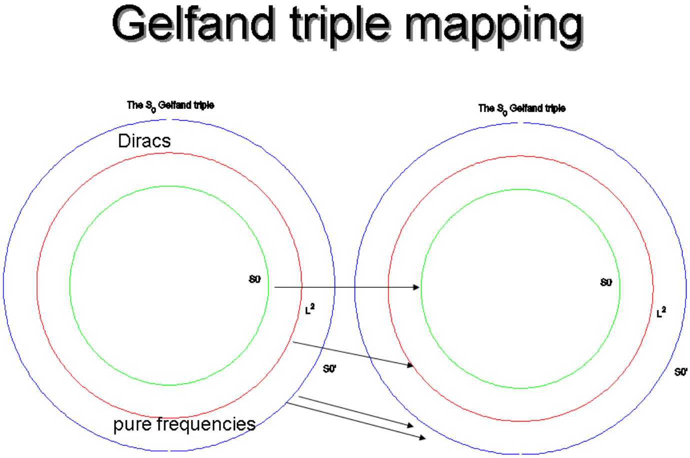
1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g\rangle = \langle \widehat{f}, \widehat{g}\rangle \tag{4}$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

A pictorial presentation of the BGTr morphism



Interpretation of key properties of the Fourier transform

Engineers and theoretical physicists tend to think of the Fourier transform as a change of basis, from the continuous, orthonormal system of Dirac measures $(\delta_x)_{x \in \mathbb{R}^d}$ to the CONB $(\chi_s)_{s \in \mathbb{R}^d}$. Books on quantum mechanics use such a terminology, admitting that these elements are "slightly outside the usual Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$", calling them "elements of the *physical Hilbert space*" (see e.g. R. Shankar's book on Quantum Physics). Within the context of BGTs we can give such formal expressions a meaning: The Fourier transform maps pure frequencies to Dirac measures:

$$\widehat{\chi_s} = \delta_s \quad \text{and} \quad \widehat{\delta_x} = \chi_{-x}.$$

Given the w^*-totality if both of these systems within $\boldsymbol{S}_0'(\mathbb{R}^d)$ we can now claim: The Fourier transform is the *unique* BGT-automorphism for $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ with this property!

Some concrete computations (M.DeGosson: Wigner Transform)

For $\phi \in \mathcal{S}(\mathbb{R}^n)$ the short-time Fourier transform (STFT) V_ϕ with window ϕ is defined, for $\psi \in \mathcal{S}'(\mathbb{R}^n)$, by

$$V_\phi \psi(z) = \int_{\mathbb{R}^n} e^{-2\pi i p \cdot x'} \psi(x') \overline{\phi(x' - x)} \mathrm{d}x'. \qquad (5)$$

The STFT is related to a well-known object from quantum mechanics, the cross-Wigner transform $W(\psi, \phi)$, defined by

$$W(\psi, \phi)(z) = \left(\tfrac{1}{2\pi\hbar}\right)^n \int_{\mathbb{R}^n} e^{-\frac{i}{\hbar} p \cdot y} \psi(x + \tfrac{1}{2}y) \overline{\phi(x - \tfrac{1}{2}y)} \mathrm{d}y. \qquad (6)$$

In fact, a tedious but straightforward calculation shows that

$$W(\psi, \phi)(z) = \left(\tfrac{2}{\pi\hbar}\right)^{n/2} e^{\frac{2i}{\hbar} p \cdot x} V_{\phi^\vee_{\sqrt{2\pi\hbar}}} \psi_{\sqrt{2\pi\hbar}}\left(z \sqrt{\tfrac{2}{\pi\hbar}}\right) \qquad (7)$$

where $\psi_{\sqrt{2\pi\hbar}}(x) = \psi(x\sqrt{2\pi\hbar})$ and $\phi^\vee(x) = \phi(-x)$;

This formula can be reversed to yield:

$$V_\phi\psi(z) = \left(\frac{2}{\pi\hbar}\right)^{-n/2} e^{-i\pi p\cdot x} W(\psi_{1/\sqrt{2\pi\hbar}}, \phi^\vee_{1/\sqrt{2\pi\hbar}})(z\sqrt{\frac{\pi\hbar}{2}}). \quad (8)$$

In particular, taking $\psi = \phi$ one gets the following formula for the usual Wigner transform:

$$W\psi(z) = \left(\frac{2}{\pi\hbar}\right)^{n/2} e^{\frac{2i}{\hbar}p\cdot x} V_{\psi_1}(\psi_2)(z\sqrt{\frac{2}{\pi\hbar}}).$$

with $\psi_1 = \psi^\vee_{\sqrt{2\pi\hbar}}$ and $\psi_2 = \psi_{\sqrt{2\pi\hbar}}$.

Another reference is the book of K. Gröchenig [6], which contains (in the terminology used there) in Lemma 4.3.1 the following formula, using the convention $g^{\checkmark}(x) = g(-x)$:

$$W(f,g)(x,\omega) = 2^d e^{4\pi i x\omega} V_{g^{\checkmark}} f(2x, 2\omega). \tag{9}$$

Charly (in [6]) also provides the folloing *covariance property*

$$W(T_u M_\eta f) = Wf(x-u, \omega-\eta). \tag{10}$$

$$W(\hat{f},\hat{g})(x,\omega) = W(f,g)(-\omega, x). \tag{11}$$

Usefulness of $S_0(\mathbb{R}^d)$ in Fourier Analysis

Most consequences result form the following inclusion relations:

$$L^1(\mathbb{R}^d) * S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d); \tag{12}$$

$$\mathcal{F}L^1(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d); \tag{13}$$

$$(S_0'(\mathbb{R}^d) * S_0(\mathbb{R}^d)) \cdot S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d); \tag{14}$$

$$(S_0'(\mathbb{R}^d) \cdot S_0(\mathbb{R}^d)) * S_0(\mathbb{R}^d) \subseteq S_0(\mathbb{R}^d); \tag{15}$$

$$S_0(\mathbb{R}^d) \widehat{\otimes} S_0(\mathbb{R}^d) = S_0(\mathbb{R}^{2d}). \tag{16}$$

1. $S_0(\mathbb{R}^d)$ is a valid domain of Poisson's formula;
2. all the classical Fourier summability kernels are in $S_0(\mathbb{R}^d)$;
3. the elements $g \in S_0(\mathbb{R}^d)$ are the natural building blocks for Gabor expansions;

The Banach Gelfand Triple

The Banach Gelfand Triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ is for many applications in theoretical physics and engineering, but also for *Abstract Harmonic Analysis* a good replacement for the Schwartz Gelfand Triple $(\boldsymbol{\mathcal{S}}, \boldsymbol{L}^2, \boldsymbol{\mathcal{S}}')$.

Lemma

$$(\boldsymbol{S}_0' * \boldsymbol{S}_0) \cdot \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0, \quad (\boldsymbol{S}_0' \cdot \boldsymbol{S}_0) * \boldsymbol{S}_0 \subseteq \boldsymbol{S}_0, \tag{17}$$

Clearly $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space and NOT a *nuclear Frechet* space, but still there is a kernel theorem!
The main exception are applications to PDE where $\boldsymbol{S}_0(\mathbb{R}^d)$ is not well suited, but there is a family of so-called *modulation spaces* which allows also to overcome this problem, and even go for the theory of *ultra-distributions*, putting weighted $\boldsymbol{L}^1$-norms on the STFT (see [6] for a first glimpse!).

A large variety of characterizations

There is a large variety of characterizations of $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ and $(\mathbf{S}_0'(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0'})$ (see e.g. [7]).
For example, a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ belongs to $\mathbf{S}_0'(\mathbb{R}^d)$ if and only if its STFT (well defined for $g \in \mathcal{S}(\mathbb{R}^d)$!) is a bounded function. Norm convergence is equivalent to uniform convergence of spectrograms, while w^*-convergence (!very important) corresponds to *uniform convergence over compact sets* of the corresponding STFTs. It is again independent of the choice of the window, even any non-zero $g \in \mathbf{S}_0(\mathbb{R}^d)$ can be used here.
There are *atomic characterizations*, or characterizations via *Wiener amalgam spaces*, for example

$$\mathbf{S}_0(\mathbb{R}^d) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d).$$

Modern Viewpoint I

Todays Rules of the Game

Choose a good *window* or *Gabor atom* (any $g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$ will do) and try to find out, for which lattices $\Lambda \in \mathbb{R}^{2d}$ the signal f resp. its STFT (with that window) can recovered in a STABLE way from the samples, i.e. from the values $\langle f, \pi(\lambda)g \rangle$.
We speak of *tight Gabor frames* (g_λ) if we can even have the expansion (for some constant $A > 0$)

$$f = A \cdot \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda, \quad \forall \quad f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Note that in general *tight frames* can be characterized as orthogonal projections of orthonormal bases of larger spaces!!!

Modern Viewpoint II

Another basic fact is that for each $g \in \mathcal{S}(\mathbb{R}^d)$ one can find, if Λ is dense enough (e.g. $a\mathbb{Z} \times b\mathbb{Z} \subset \mathbb{R}^d$ for $ab < 1$ in the Gaussian case) a *dual Gabor window* $\widetilde{g}$ such that one has at least

$$f = \sum_{\lambda \in \Lambda} \langle f, \widetilde{g}_\lambda \rangle g_\lambda = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle \widetilde{g}_\lambda \quad (18)$$

$\widetilde{g}$ can be found as the solution of the (positive definite) linear system $S\widetilde{g} = g$, where $Sf = \sum_{\lambda \in \Lambda} \langle f, g_\lambda \rangle g_\lambda$, so using $\widetilde{g}$ instead of g for analysis or synthesis corrects for the deviation from the identity operator. An important fact is the commutation relation $S \circ \pi(\lambda) = \pi(\lambda) \circ S$, for all $\lambda \in \Lambda$.
Thus (18) is just $S \circ S^{-1} = Id = S^{-1} \circ S$ in disguise!).

Modern Viewpoint III

The possibility of having such *tight Gabor frames* is resulting from the continuous reconstruction formula, valid for arbitrary $\boldsymbol{L}^2$-atoms g. Writing again for $\lambda = (t, \omega)$ and $\pi(\lambda) = M_\omega T_t$, and furthermore $g_\lambda = \pi(\lambda)g$ we have in fact for any $g \in \boldsymbol{L}^2(\mathbb{R}^d)$ with $\|g\|_2 = 1$:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda \rangle g_\lambda \, d\lambda.$$

It follows from **Moyal's formula** (energy preservation):

$$\|V_g(f)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)} = \|g\|_2 \|f\|_2, \quad f, g \in \boldsymbol{L}^2. \tag{19}$$

This setting is well known under the name of coherent frames when $g = g_0$, the Gauss function. Its range is the *Fock space*.

Modern Viewpoint IV

There is a similar representation formula at the level of operators, where we also have a continuous representation formula, valid in a strict sense for *regularizing operators*, which map w^*-convergent sequences in $\boldsymbol{S}_0'(\mathbb{R}^d)$ into norm convergent sequences in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle T, \pi(\lambda) \rangle_{\mathcal{HS}} \pi(\lambda) d\lambda. \qquad (20)$$

It establishes an isometry for Hilbert-Schmidt operators:

$$\|T\|_{\mathcal{HS}} = \|\eta(T)\|_{\boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)}, \quad T \in \mathcal{HS},$$

where $\eta T = \langle T, \pi(\lambda) \rangle_{\mathcal{HS}}$ is the *spreading function* of the operator T. The proof is similar to the proof of Plancherel's theorem.

Gabor Riesz bases and Mobile communication

Another usefulness of "sparsely distributed" Gabor systems comes from mobile communication:

1. Mobile channels can be modelled as slowly varying, or underspread operators (small support in spreading domain);
2. TF-shifted Gaussians are joint approximate eigenvectors to such systems, i.e. pass through was some attenuation only;
3. underspread operators can also be identified from transmitted pilot tones;
4. Communication should allow large capacity at high reliability.

The audio-engineer's work: Gabor multipliers

Fourier Transforms of Distributions in $S_0'(\mathbb{R}^d)$

The Fourier transform $\hat{\sigma}$ of $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ is defined by the simple relation

$$\hat{\sigma}(f) := \sigma(\widehat{f}), \quad f \in \mathcal{S}(\mathbb{R}^d).$$

His construction *vastly extends the domain of the Fourier transform* and allows even polynomials to have a Fourier tranform. Among the objects which can now be treated are also the Dirac measures δ_x, as well as Dirac combs $\sqcup\!\sqcup = \sum_{k \in \mathbb{Z}^d} \delta_k$.
It is the only w^*-w^*--continuous extension of the "ordinary FT".
Poisson's formula, which expresses that one has for $f \in \mathcal{S}(\mathbb{R}^d)$

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \widehat{f}(n), \tag{21}$$

can now be recast in the form

$$\widehat{\sqcup\!\sqcup} = \sqcup\!\sqcup.$$

Sampling and Periodization on the FT side

The convolution theorem, can then be used to show that sampling corresponds to periodization on the Fourier transform side, with the interpretaton that

$$\sqcup\!\sqcup \cdot f = \sum_{k\in\mathbb{Z}^d} f(k)\delta_k, \quad f \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d).$$

In fact, we have

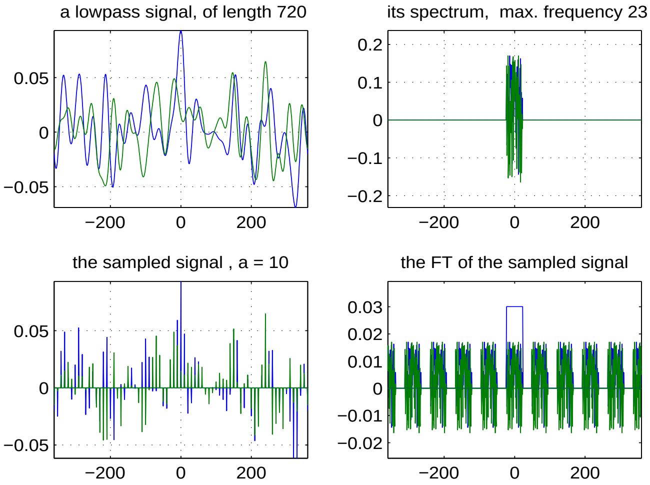
$$\widehat{\sqcup\!\sqcup \cdot f} = \widehat{\sqcup\!\sqcup} * \widehat{f} = \sqcup\!\sqcup * \widehat{f}.$$

This result is the key to prove **Shannon's Sampling Theorem** which is usually considered as the fundamental fact of digital signal processing (Claude Shannon: 1916 - 2001).

Recovery from Samples

If we try to recover a real function $f : \mathbb{R} \to \mathbb{R}$ from samples, i.e. from a sequence of values $(f(x_n))_{n \in I}$, where I is a finite or (countable) infinite set, we cannot expect perfect reconstruction. In the setting of $\left(\boldsymbol{L}^2(\mathbb{R}), \|\cdot\|_2\right)$ any sequence constitutes only set of measure zero, so knowing the sampling values provides *zero information* without side-information.
On the other hand it is clear the for a *(uniformly) continuous* function, so e.g. a continuous function supported on $[-K, K]$ for some $K > 0$ piecewise linear interpolation (this is what MATLAB does automatically when we use the PLOT-routine) is providing a good (in the uniform sense) approximation to the given function f as long as the maximal distance between the sampling points around the interval $[-K, K]$ is small enough.
Shannon's Theorem says that one can have **perfect reconstruction** for band-limited functions.

A Visual Proof of Shannon's Theorem



Shannon's Sampling Theorem

It is kind of clear from this picture that one can recover the spectrogram of the original function by isolating the central copy of the periodized version of $\widehat{f}$ by multiplying with some function $\hat{g}$, with g such that $\hat{g}(x) = 1$ on spec(f) and $\hat{g}(x) = 0$ at the shifted copies of $\widehat{f}$. This is of course only possible if these shifted copies of spec(f) do not overlap, resp. if the sampling is dense enough (and correspondingly the periodization of $\widehat{f}$ is a course one. This conditions is known as the *Nyquist criterion*. If it is satisfied, or supp(f) $\subset [-1/\alpha, 1/\alpha]$, then

$$f(t) = \sum_{k \in \mathbb{Z}^d} f(\alpha k) T_{\alpha k} g(x), \quad x \in \mathbb{R}^d.$$

Matrix-representation and kernels

We know also from linear algebra, that any linear mapping can be expressed by a matrix (once two bases are fixed). We have a similar situation through the so-called kernel theorem.
Naively the operator has a representation as an integral operator: $f \mapsto Tf$, with

$$Tf(x) = \int_{\mathbb{R}^d} K(x,y) f(y) dy, \;\; x, y \in \mathbb{R}^d.$$

But clearly no multiplication operator can be represented in this way (not even identity), for any locally integrable function $K(x,y)$. But we can reformulate the connection distributionally, as

$$\langle Tf, g \rangle = \langle K, f \otimes g \rangle,$$

and still call K (the uniquely determined) distributional kernel (on $\mathbb{R}^{2d}$) corresponding to T (and vice versa).

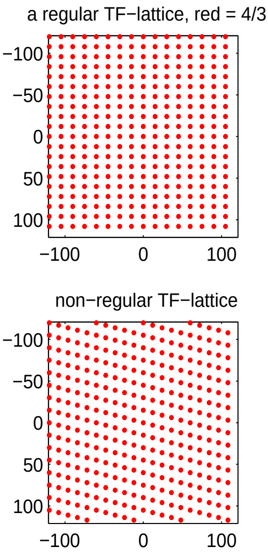
The Kernel Theorem in the S_0-setting

We will use $\mathbf{B} = \mathcal{L}(\mathbf{S}_0', \mathbf{S}_0)$ and observe and $\mathbf{B}'$ coincides with $\mathcal{L}(\mathbf{S}_0, \mathbf{S}_0')$ (correctly: the linear operators which are w^* to norm continuous!), using the scalar product of Hilbert Schmidt operators: $\langle T, S \rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*), \; T, S \in \mathcal{HS}$.

Theorem

There is a natural BGT-isomorphism between $(\mathbf{B}, \mathcal{H}, \mathbf{B}')$ *and* $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^{2d})$.
This in turn is isomorphic via the spreading and the Kohn-Nirenberg symbol to $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
Moreover, the spreading mapping is uniquely determined as the BGT-isomorphism, which established a correspondence between TF-shift operators $\pi(\lambda)$ *and the corresponding point masses* δ_λ.

Phase space lattices/ time-frequency plane



The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

We will call $(\pi(\lambda)g)_{\lambda\in\Lambda}$ a Gabor family with Gabor atom g.

Theorem

Given $g \in \boldsymbol{S}_0(\mathbb{R}^d)$. Then there exists $\gamma > 0$ such that any γ-dense lattice Λ (i.e. with $\cup_{\lambda\in\Lambda} B_\gamma(\lambda) = \mathbb{R}^d$) the Gabor family $(\pi(\lambda)g)_{\lambda\in\Lambda}$ is a Gabor frame. Hence there exists a linear mapping (the unique MNLSQ solution) $f \mapsto (c_\lambda) = \langle f, \widetilde{g}_\lambda\rangle, \lambda\in\Lambda$, for a uniquely determined function $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$, thus

$$f = \sum_{\lambda\in\Lambda} \langle f, \widetilde{g}_\lambda\rangle g_\lambda, \quad \forall f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

In other words, the minimal norm representation of any $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ can be obtained by just sampling the STFT with respect to the *dual window* $\widetilde{g}$.

The role of $S_0(\mathbb{R}^d)$ for Gabor Analysis

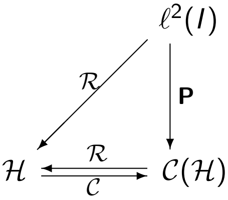
The dual Gabor atom $\widetilde{g} \in \boldsymbol{S}_0(\mathbb{R}^d)$ provides not only the minimal norm coefficients, but also $\ell^1(\Lambda)$-coefficients for $f \in \boldsymbol{S}_0(\mathbb{R}^d)$ and is well defined on $\boldsymbol{S}_0$, $\sigma \mapsto \sigma(\widetilde{g}_\lambda)$ and defines representation coefficients in $\ell^\infty(\Lambda)$.
So in fact $f \mapsto (\langle f, \widetilde{g}_\lambda \rangle)$ defines a Banach Gelfand triple morphism from the triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$ to $(\ell^1, \ell^2, \ell^\infty)$. The (left) inverse mapping is the synthesis mapping

$$(c_\lambda) \mapsto \sum_{\lambda \in \Lambda} c_\lambda g_\lambda,$$

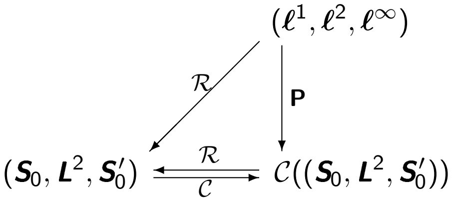
with norm convergence for $c \in \ell^1$ or ℓ^2, and still w^*-sense in $\boldsymbol{S}_0'(\mathbb{R}^d)$ for $c \in \ell^\infty(\Lambda)$.

Frames described by a diagram

Similar to the situation for matrices of maximal rank (with row and column space, null-space of $\boldsymbol{A}$ and $\boldsymbol{A}'$) we have:
$\mathbf{P} = \mathcal{C} \circ \mathcal{R}$ is a projection in $\boldsymbol{Y}$ onto the range $\boldsymbol{Y}_0$ of $\mathcal{C}$, thus we have the following commutative diagram.



The frame diagram for Gelfand triples $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$:



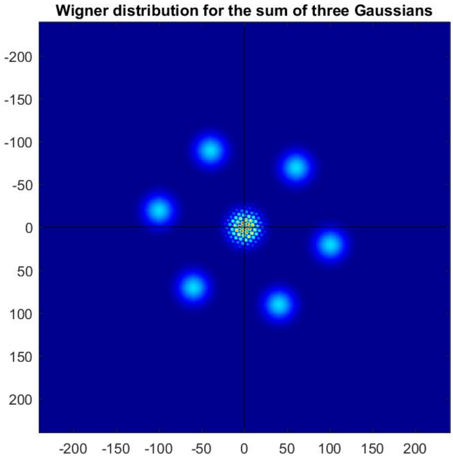


Figure: Wigner002.jpg

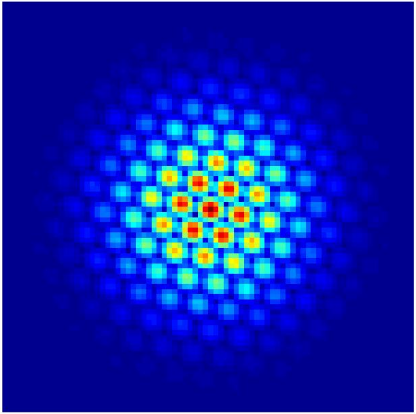
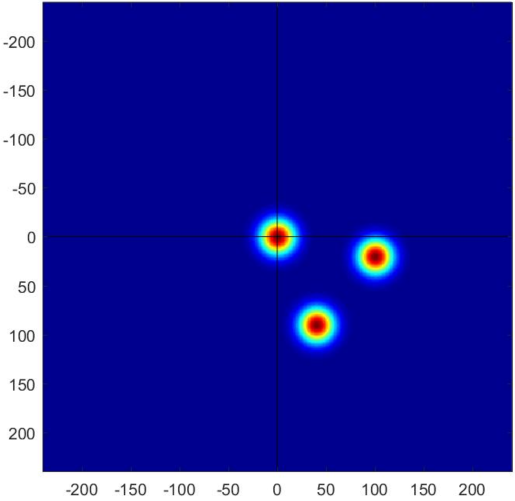


Figure: Wigner001.jpg

spectrogram of three Gaussian bumps



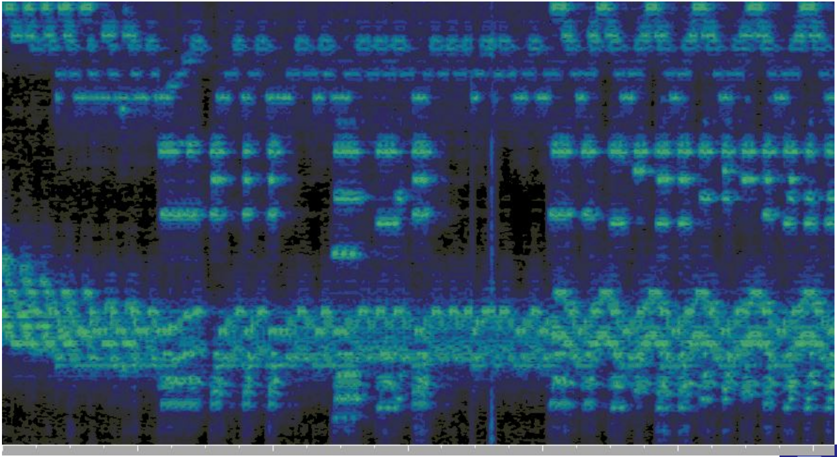


Figure: Playing at home on my Roland Stage Piano

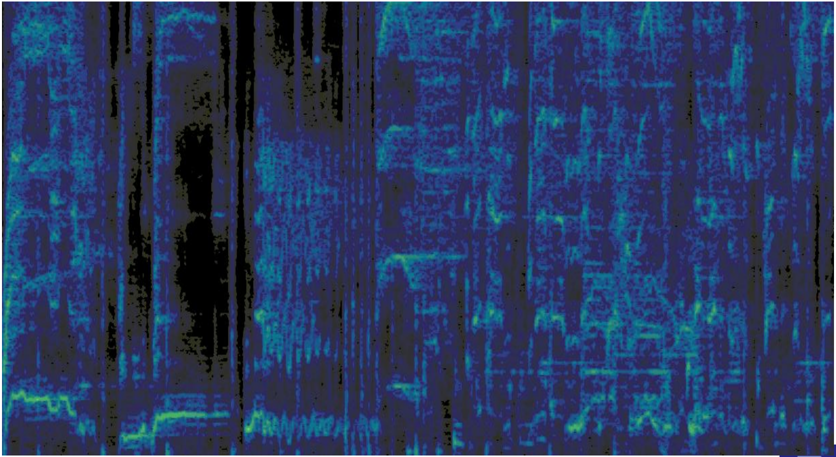
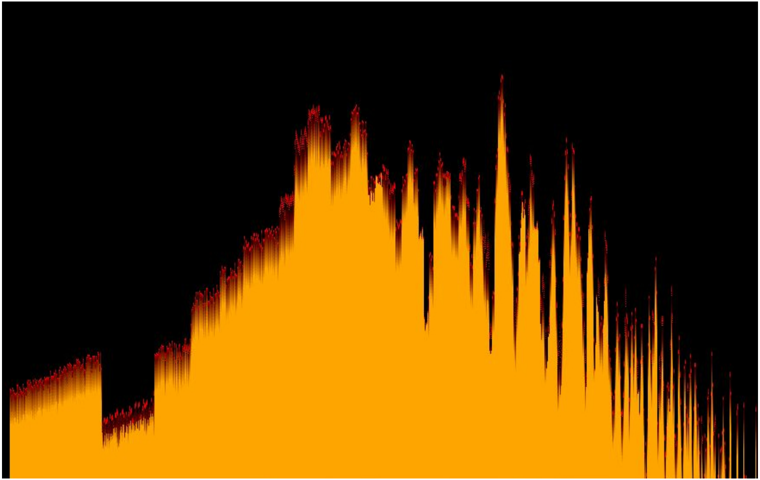


Figure: Janis Joplin "I need a man"

TF-analysis within the Windows Media Player



Using this approach one can SAVE statements such as...

The **sifting property**,

$$\int_{-\infty}^{\infty} f(x)\delta(x-\xi)dx = f(\xi), \tag{22}$$

The identity

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(k-l)x}dx = \delta(k-l) \tag{23}$$

Examples of "incorrect" statements

Sifting property of the Delta Dirac

$$\psi(x) = \int_{-\infty}^{\infty} \delta(x-y)\psi(y)dy$$

or the integration of the pure frequencies adding up to a Dirac:

$$\int_{-\infty}^{\infty} e^{2\pi isx} ds = \delta(x)$$

One can use a combination of both statements in order to derive a "highly formal" version of the Fourier inversion theorem.

Turning inaccurate formula into correct statements

In the setting of *tempered distributions* one can rewrite the first equation as

$$\psi = \psi * \delta$$

resp.

$$\mathcal{F}^{-1}(\mathbf{1}) = \delta,$$

or equivalently giving a "meaning" to the formula (see WIKIPEDIA)

$$\int_{-\infty}^{\infty} 1 \cdot e^{2\pi i x \xi} d\xi = \delta(x). \quad (24)$$

Strange formulas in WIKIPEDIA (2018)

WIKIPEDIA contains (p.4 on the **Dirac Delta function**)

$$\int_{-\infty}^{\infty} \delta(\xi - x)\delta(x - \eta)dx = \delta(\xi - \eta). \tag{25}$$

This is pretty confusing (to a mathematician). You have to first multiply one delta-function with another (is this possible?) and then even integrate out, with a result which is not a number but another Dirac function.
For us the “underlying” statement will become

$$\delta_0 * \delta_\eta = \delta_\eta$$

which is just a simple special case of the general rule

$$\delta_x * \delta_y = \delta_{x+y} = \delta_y * \delta_x, \quad x, y \in \mathbb{R}^d;$$

It can be seen as a special case of convolution of two measures.

References

E. Cordero, H. G. Feichtinger, and F. Luef.
Banach Gelfand triples for Gabor analysis.
In *Pseudo-differential Operators*, volume 1949 of *Lecture Notes in Mathematics*, pages 1–33. Springer, Berlin, 2008.

M. De Gosson.
Symplectic Methods in Harmonic Analysis and in Mathematical Physics, volume 7 of *Pseudo-Differential Operators. Theory and Applications*.
Birkhäuser/Springer Basel AG, Basel, 2011.

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

H. G. Feichtinger and T. Strohmer.
Gabor Analysis and Algorithms. Theory and Applications.
Birkhäuser, Boston, 1998.

D. Gabor.
Theory of communication.
J. IEE, 93(26):429–457, 1946.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

M. S. Jakobsen.
On a (No Longer) New Segal Algebra: A Review of the Feichtinger Algebra.
J. Fourier Anal. Appl., pages 1 – 82, 2019.

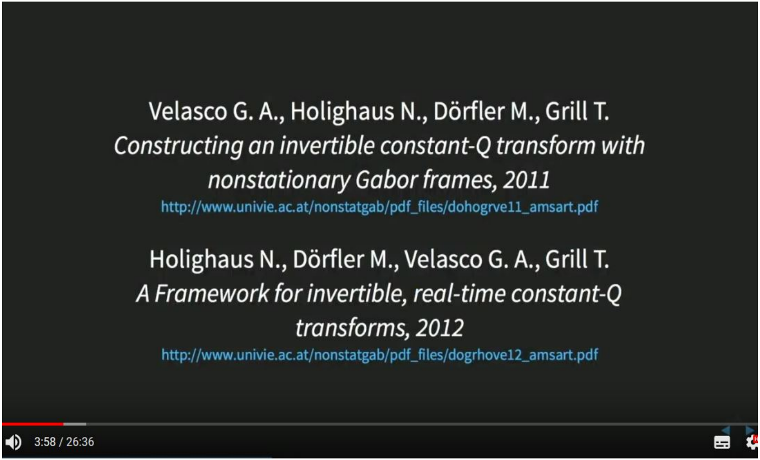
Further Resources

There are various interesting links.
For example the (OCTAVE/MATLAB based) LTFAT, the *Large Time Frequency Analysis Toolbox* (hosted by ARI, the Acoustic Research Institute of the Austrian Academy of Sciences OEAW, under Peter Balazs).
Or the GABORATOR (at www.gaborator.com), which allows even to upload a WAV-file and see the spectrogram while the music is replayed.

A screenshot from the Gaborator

Figure: A piano piece by Ferenc Liszt, please guess!

A screenshow from the Gaborator



Further information, reading material

The NuHAG webpage offers a large amount of further information, including talks and MATLAB code:

`www.nuhag.eu`
`www.nuhag.eu/bibtex (all papers)`
`www.nuhag.eu/talks (all talks)`
`www.nuhag.eu/matlab (MATLAB code)`
`www.nuhag.eu/skripten (lecture notes)`

Enjoy the material!!

Thanks for your attention!

Comment on the Physics Nobel Prize 2017

Time-Frequency Analysis and Black Holes

Breaking News of Oct. 3rd, 2017

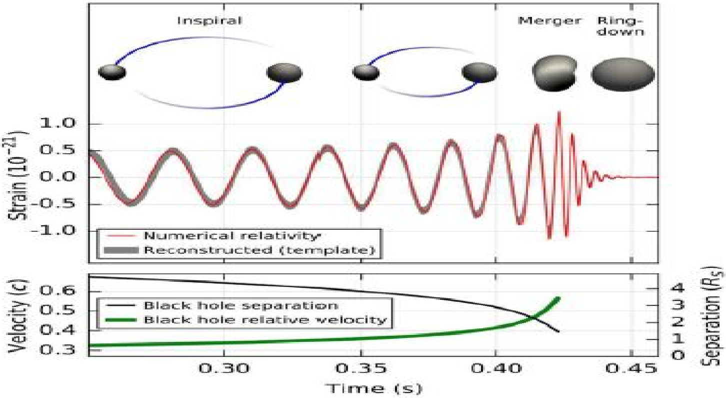
On Oct. 3rd, 2017 the **Nobel Prize in Physics** was awarded to three physicists who have been key figure for the **LIGO Experiment** which led last year to the detection of **Gravitational Waves** as predicted 100 years ago by Albert Einstein!
The Prize-Winners are

Rainer Weiss, Barry Barish und Kip Thorne.

They have supplied the key ideas to the so-called LIGO experiment which has meanwhile 4-times verified the existence of Gravitational waves by means of a huge laser-inferometric setup. The first detection took place in September 2016.

The shape of gravitational waves

Einstein had predicted, that the shape of the gravitaional wave of two collapsing black holes would be a chirp-like function, depending on the masses of the two objects.



A story on Wilson Bases

In 1991 Daubechies, Jaffard and Journe [DJJ91] followed an idea of Wilson in their construction of an orthonormal basis from a Gabor system $\mathcal{G}(g,\Lambda)$ of $\boldsymbol{L}^2(\mathbb{R}^d)$. Wilson suggested that the building blocks $\pi(x,\omega)g$ of an orthonormal basis of $\boldsymbol{L}^2(\mathbb{R}^d)$ should be symmetric in ω and should be concentrated at ω and $-\omega$.

Definition

To $g \in \boldsymbol{L}^2(\mathbb{R})$ we associate the Wilson system $\mathcal{W}(g)$

$$\psi_{k,n} = c_n T_{\frac{k}{2}} \left[M_n + (-1)^{k+n} M_{-n} \right] g, \quad (k,n) \in \mathbb{Z}^d \times \mathbb{N}_0;$$

$c_0 = \frac{1}{2}; c_n = \frac{1}{\sqrt{2}}$ for $n \geq 1$, $\psi_{k,0} = T_k g$; $\psi_{2k+1,0} = 0$ for $k \in \mathbb{Z}$.

Illustration of TF-concentration of Wilson bases

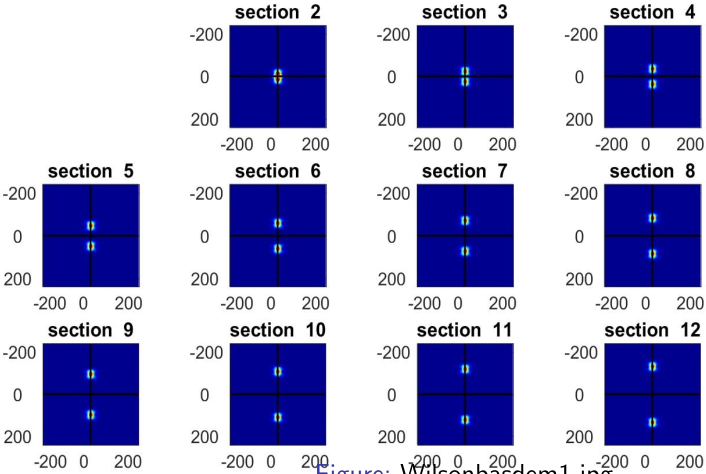


Figure: Wilsonbasdem1.jpg

Gravitational waves and Wilson bases

There is not enough time to explain the details of the huge signal processing task behind these findings, the literal “needle in the haystack”.
There had been two strategies:

- Searching for 2500 explicitely determined wave-forms;
- Using a family of 14 orthonormal Wilson bases in order to detect the gravitational waves.

The very **first** was detected by the second strategy, because the masses had been out of the expected range of the predetermined wave-forms.
NOTE: Wilson bases are cooked up from tight Gabor frames of redundancy 2 by pairing them, like $\cos(x)$ and $\sin(x)$ using Euler's formula (in a smart, woven way).

THANK YOU

Thank you for your attention

More at `www.nuhag.eu` or `www.nuhag.eu/talks`

The Banach space $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

Without differentiability there is a *minimal, Fourier and isometrically translation invariant Banach space* (called $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ or $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1}))$, which will serve our purpose. Its dual space $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ is correspondingly the largest among all Fourier invariant and isometrically translation invariant "objects" (in fact so-called local pseudo-measures or quasimeasures, orginally introduced in order to describe translation invariant systems as convolution operators).
Although there is a rich zoo of Banach spaces around (one can choose such a family, the so-called Shubin classes - to intersect in the Schwartz class and their union is corresondingly $\mathcal{S}'(\mathbb{R}^d)$), we will restrict ourselves to the situation of Banach Gelfand Triples, mostly related to $(S_0, L^2, S_0')(\mathbb{R}^d)$.