Banach Spaces of Ultra-Distributions over Locally Compact Abelian Groups

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submitted ABSTRACT

It is clear that the theory of tempered distributions is in some sense limited, because it does not allow to define the Fourier transform for objects which have exponential growth. On the other hand the Schwartz-Bruhat theory, which is the analogue of L.Schwartz theory of tempered distributions for general LCA (locally compact Abelian) groups is already quite cumbersome. Since the borderline between Banach spaces of test functions which are smaller and smaller and the trivial one (the useless space, consisting only of the zero-function) is quite thin it is interesting to take a look at the relevant condition, which is formulated in the classical Acta Mathematica paper by Yngve Domar entitled Harmonic analysis based on certain commutative Banach algebras. There the so-called **Beurling-Domar condition** (BD) is formulated, see Hans Reiter's book ([23, 26]).

We will discuss various aspects of this approach and to which extent it allows to build Banach spaces of test functions (and corresponding distributions), and even finally Frechet spaces of this type, over general LCA groups. We hope to indicate that this view-point allows certain unifications and technical simplifications compared to the technical details needed either in the Schwartz-Bruhat or in the now classical theory of ultra-distributions over the Euclidean space.

My Overall Goals

Let me provide some personal background.
As a student of Hans Reiter ([23]) I was immersed into the area of Harmonic Analysis[1] from the beginning, in the spirit of Andre Weil ([30]): The natural setting for Fourier Analysis is the world of LCA groups.
Results for $\mathbb{R}^n$ should not be proved by induction, but viewing it as a generic LCA group. Moreover, results formulated for general LCA groups should not be mere transcriptions of well known results for the special case $\mathbf{G} = \mathbb{R}^n$, so ideally a result should be valid in full generality but also interesting for Euclidean spaces.
Influenced by the work of Hans Triebel (long list of books, beginning with [28, 29] and Elias Stein ([27]) tried to deal with function spaces on LC groups.

[1] or what is often called AHA: Abstract Harmonic Analysis, but he did not like this expression!

The setting of LCA Groups

Let us quickly recall that LCA (= locally compact Abelian) groups **G** are the natural framework for Fourier Analysis. It contains the family of elementary LCA groups, which are products of Euclidean spaces $\mathbb{R}^d$ and related groups relevant for classical Fourier Analysis: periodic functions living on the torus, discrete non-periodic signals living on $\mathbb{Z}$ and finite vectors representing periodic-discrete signals interpreted as functions on $\mathbb{Z}_n$ ($\mathbb{Z}/n\mathbb{Z}$ resp. the unit-roots of order n).
Nowadays it is even recognized by engineers (see [3]) that the view point of Abstract Harmonic Analysis (AHA) helps to take a unified approach to the different settings where the ordinary Fourier transform makes sense, which requires the (translation invariant) *Haar measure* on the group **G** and the *pure frequencies* alias *characters*, which constitute the (multiplicative) *dual group* $\widehat{\mathbf{G}}$.

The setting of LCA Groups II

More formally we require that **G** is a topological, locally compact group with a commutative group law, written usually as addition. In the case of $\mathbb{R}^d$ it is simply addition of vectors in $\mathbb{R}^d$.
A *character* is simply a homomorphism from **G** into the unit circle $\mathbb{U}$, resp. a continuous, complex-valued function $\chi : \mathbf{G} \mapsto \mathbb{C}$ with $|\chi(x)| = 1$ and $\chi(x+y) = \chi(x)\chi(y), \ \forall x, y\mathbf{G}$ (exponential law).
With the topology of uniform convergence over compact subsets of **G** this set of functions is again a LCA group, called the *dual group* resp. *frequency domain*. By *Pontrjagin's* theorem the natural embedding of **G** into the dual group $\widehat{\widehat{\mathbf{G}}}$ establishes an isomorphism.
In this sense every LCA is a dual group.
Using the (essentially unique) Haar measure on **G** one can define for any bounded measure μ on **G** the Fourier-(Stieltjes) transform via $\widehat{\mu}(\chi) = \mu(\overline{\chi})$.

The setting of LCA Groups III

For every such group one can either define $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$ with the help of the Haar measures, introducing convolution via

$$[f * g](z) = \int_{\mathbf{G}} g(x-y)f(y)dy$$

or more elegantly (avoiding integration theory, see [16]) by identifying the Banach space $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ of all *bounded measures* with the dual of $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$ on the one hand, and with the Banach algebra of translation invariant operators on $\left(\boldsymbol{C}_0(G), \|\cdot\|_\infty\right)$ on the other hand, thus defining a multiplication (called *convolution*) in $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$. This last identification is characterized by the fact that it identifies the Dirac measure δ_x with the shift operator T_x, with $[T_x f](z) = f(z-x)$.
In other words we require

$$\delta_x * f = T_x f, \quad f \in \boldsymbol{C}_0(G). \tag{1}$$

The setting of LCA Groups IV

The *convolution* $\mu_1 * \mu_2$ can be characterized by the composition law for the corresponding "linear translation invariant systems" (on **G**) but also by taking (in a suitable sense) limits of convolutions of discrete measures, based on the simple rule that follows from (1).

$$\delta_x * \delta_y = \delta_{x+y} = \delta_y * \delta_x, \quad \forall x, y \in \mathbf{G}. \tag{2}$$

The convolution theorem for discrete measures of the form $\mu = \sum_{k=1}^{\infty} c_k \delta_{x_k}$ with $\sum_{k=1}^{\infty} |c_k| < \infty$ follows directly from the exponential law, and by taking suitable limits (in the strong operator topology) one obtains the *convolution theorem*

$$\widehat{\mu_1 * \mu_2} = \widehat{\mu_1} \cdot \widehat{\mu_2}, \quad \mu_1, \mu_2 \in \boldsymbol{M}_b(G), \tag{3}$$

as well as injectivity of the mapping $\mu \mapsto \widehat{\mu}$.

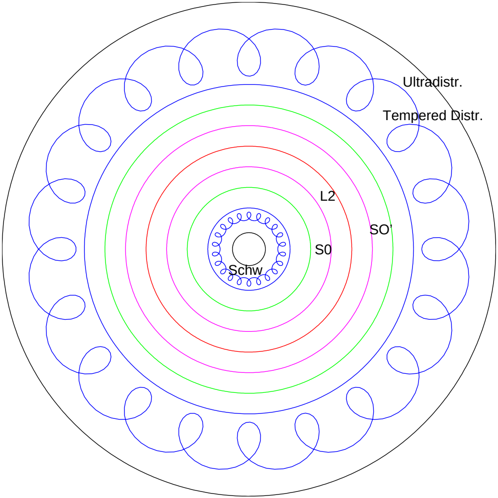
The setting of LCA Groups V

What we do not have (compared to the situation over $\mathbf{G} = \mathbb{R}^d$):

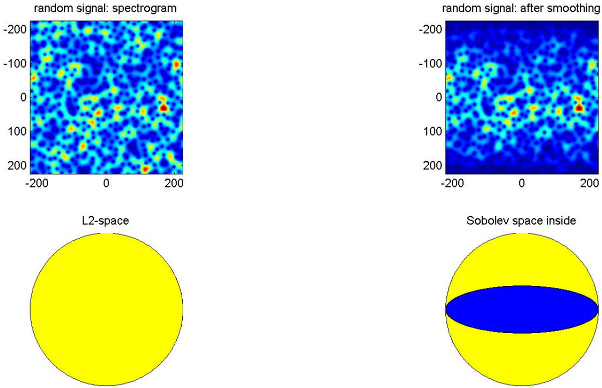
1. a general LCA does not have a differentiability structure (Lie group), so smoothness has to be described differntly
2. there is not a nice group of automorphism (dilations), which allows to produce bounded approximate units (Dirac sequences) or summability kernels (in $\mathcal{F}\boldsymbol{L}^1(\widehat{\mathbf{G}})$) by dilation;
3. there are not always arbitrary fine lattices Λ within $\mathbf{G}$, i.e. discrete, cocompact subgroups with small fundamental domain (like $\alpha\mathbb{Z}^d \lhd \mathbb{R}^d$, for small $\alpha > 0$).

A non-obvious example of a totally disconnected LCA group is the group of p-adic numbers $\mathbb{Q}_p$ (a completion of $\mathbb{Q}$ with respect to a non-Euclidian distance, useful for number theoretic considerations).

Ultradistributions and the Fourier Transform



Spectrogram of functions in Sobolev Spaces



Sobolev Embedding and $(S_0(\mathbb{R}^d), \|\cdot\|_{S_0})$

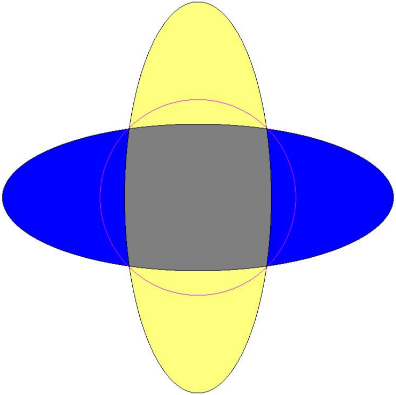


Figure: blue = Sobolev space, yellow = weighted L^2

Sobolev Embedding and $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ II

We will denote (for now) by L^2_s the weighted L^2-space with weight $v_s(t) = (1+|t|^2)^{s/2}$, for $s \in \mathbb{R}$. Then the Sobolev space $(\mathcal{H}_s(\mathbb{R}^d), \|\cdot\|_{\mathcal{H}_s})$ is defined as the Fourier inverse image of $L^2_s(\mathbb{R}^d)$ (with natural norm).

Theorem

For $s > d$ one has

$$Q_s(\mathbb{R}^d) := H^s(\mathbb{R}^d) \cap L^2_s \subset S_0(\mathbb{R}^d),$$

with continuous embedding with respect to the natural norms.

Beurling Algebras

For me the study of function spaces started with *Segal algebras* (cf. [24]) and *Beurling algebras* as described in Reiter's book ([23]).

The first family of Banach spaces $(\boldsymbol{S}, \|\cdot\|_{\boldsymbol{S}})$ can be described (in a more modern terminology) as *dense Banach ideals* within the Banach convolution algebra $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$, with the extra property that the usual approximate units (which may be chosen from the dense subset $S \subset \boldsymbol{L}^1(G)$) satisfy $\|f - e_\alpha * f\|_{\boldsymbol{S}} = 0$ for all $f \in \boldsymbol{S}$. Among all these Segal algebras the so-called Wiener algebra $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$ (which can be defined over LCA groups) and the Segal algebra $\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right)$ play an important role.

Both are the smallest elements within subfamilies of Segal algebras, with the additional pointwise Banach module structure over $\boldsymbol{C}_0(G)$ or $\mathcal{F}\boldsymbol{L}^1(G)$ respectively.

Double Module Structures

Any Segal algebra $\boldsymbol{S} \subset \boldsymbol{L}^1(G)$ has by definition an isometrically translation invariant norm and in fact the property

$$\|\mu * f\|_{\boldsymbol{S}} \leq \|\mu\|_{\boldsymbol{M}_b} \|f\|_{\boldsymbol{S}} \quad \forall \mu \in \boldsymbol{M}_b(G), f \in \boldsymbol{S}.$$

By the convolution theorem its Fourier transform image $\mathcal{F}\boldsymbol{S}$ has a pointwise module structure over $\mathcal{F}\boldsymbol{L}^1(\widehat{\mathbf{G}})$, but on the other hand many Segal algebras are pointwise $(\boldsymbol{C}_0(G), \|\cdot\|_\infty)$-modules (the Wiener algebra being the smallest) and even more are pointwise $\mathcal{F}\boldsymbol{L}^1(\mathbf{G})$ modules (the Fourier image of $\boldsymbol{L}^1(\widehat{\mathbf{G}})$, via Pontrjagin), and there is a smallest one, $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbf{G})$, defined for any LCA group, also denoted as $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ (see [12], [19]).

Work of Arne Beurling

Let us recall the definition of so-called Beurling weights, resp. sub-multiplicative weight functions, i.e. satisfying

$$w(x+y) \leq w(x)w(y), \quad \forall x, y \in \mathbf{G}. \tag{4}$$

Without loss of generality we will restrict our attention to continuous weights satisfying $w(x) \geq w(0) = 1$.

Definition

A weight function w on a locally compact abelian group G is said to satisfy the condition of Beurling-Domar if the following condition is satisfied for every $x \in \mathbf{G}$:

$$\sum_{n \geq 1} \frac{\log w(x^n)}{n^2} < \infty. \tag{5}$$

(In case of addditive notation is used x^n is replaced by nx.)

Beurling continued

In fact, in his address at the 9^{th} Congr. Math. Scand. in 1938 Beurling assumes that the weight function should satisfy the extra condition that $w(\rho x) \geq w(x)$, for $x \in \mathbb{R}, \rho > 1$. We refer mostly to Domar's paper [7] and Reiter's book [23].

The relevant term is the concept of *quasi-analyticity* which is treated in great detail in the books of P. Koosis ([21, 20]) or the reference to the Denjoy-Carleman Theorem (Thm.1.3.8) in L. Hörmander's book [18] (see for a first summary published in 1979, [10, 9], and a more recent from 2007: [17]).

EXAMPLES:

- polynomial weights of the form $(1 + |x|)^s, s \geq 0$; or
- subexponential weights with $\alpha \geq 0$ and $\beta \in [0, 1)$, of the form

$$w(x) = e^{\alpha|x|^\beta}.$$

.

The Relevance of the (BD)-condition

It is clear that stronger weights make the space $\boldsymbol{L}^1_w(G)$ smaller. Its members have to satisfy stronger decay conditions at infinity, and by consequence the Fourier transform will show a higher degree of smoothness.

However it is not clear whether $\mathcal{F}\boldsymbol{L}^1_w(G)$ still contains compactly supported functions (even over the real line this question is meaningful), and it is exactly here where the (BD)-condition plays the decisive role. It distinguishes the *quasi-analytic* (too strong) weights from the "good ones" (the ones satisfying (BD)), or the *non-quasi-analytic* class.

If (and only) if (BD) is satisfied the Fourier image of $\boldsymbol{L}^1_w(G)$ is a regular algebra, meaning that it has non-zero functions of arbitrary small support and many other good properties which we are used from the ordinary Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.

Wiener Amalgam Spaces I

For the definition of new function spaces from old ones the method of *Wiener amalgam spaces* has turned out to be very versatile and useful, and allows to generate many interesting Banach spaces (of test functions or by duality distributions).
The general theory of amalgam spaces (as developed in [13]) is based on the use of so-called BUPUs (bounded uniform partitions of unity), which allow to produce an "amalgam", generated from some *local component* (describing the local behaviour of a function or distribution), and its *global comportment* (decay or summability properties of the local property, usually expressed by means of (mixed norm) weighted L^q-spaces).

The classical "Wiener-type spaces"

The fact, that there are no inclusions between any two of the spaces $(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p)$, with $1 \leq p \leq \infty$ has global (in one direction) and local (in the other direction) reasons.
$\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$
The Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ allow to get rid of these restrictions, because they behave locally like $\boldsymbol{L}^p$ while globally their behaviour is that of ℓ^q. The family of these spaces is (more or less) closed under duality, under complex interpolation, but also pointwise multiplications and convolutions respect the local and the global component independently!
See [13]

Recalling the concept of Wiener Amalgam Spaces

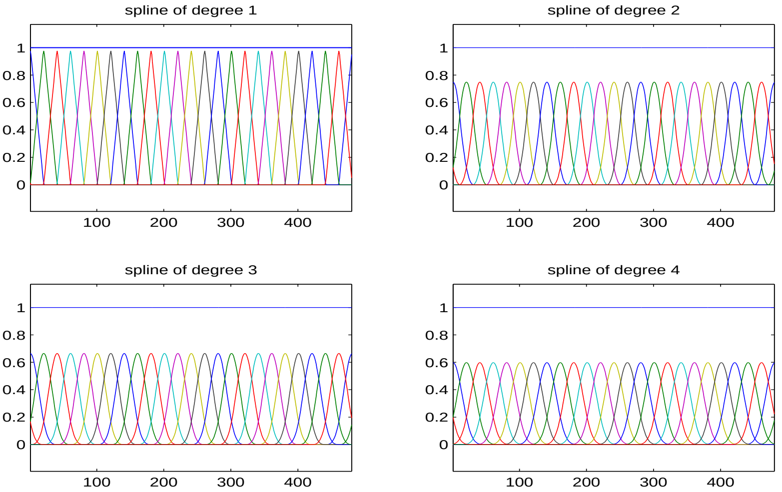
Wiener amalgam spaces are a generally useful family of spaces with a wide range of applications in analysis. The main motivation for the introduction of these spaces came from the observations that the non-inclusion results between spaes $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ for different values of p are either of *local* or of *global* nature. Hence it makes sense to separate these to properties using BUPUs.

Definition

A bounded family $\Psi = (\psi_n)_{n\in\mathbb{Z}^d}$ in some Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ of continuous functions on $\mathbb{R}^d$ is called a regular **Uniform Partition of Unity** if $\psi_n = T_{\alpha n}\psi_0, n \in \mathbb{Z}^d$, $0 \leq \psi_0 \leq 1$, for some ψ_0 with compact support, and

$$\sum_{n\in\mathbb{Z}^d} \psi_n(x) = \sum_{n\in\mathbb{Z}^d} \psi(x - \alpha n) = 1 \quad \text{for all} \quad x \in \mathbb{R}^d.$$

Illustration of the B-splines providing BUPUs



Recalling the concept of Wiener Amalgam Spaces II

Note that one can **define the Wiener amalgam space $W(B, \ell^q)$** by the condition that the sequence $\|f\psi_n\|_B$ belongs to $\ell^q(\mathbb{Z}^d)$ and its norm is one of the (many equivalent) norms on this space.

Different BUPUs define the same space and equivalent norms. Moreover, for $1 \leq q \leq \infty$ one has Banach spaces, with natural inclusion, duality and interpolation properties.
Many known function spaces are also Wiener amalgam spaces:

- $L^p(\mathbb{R}^d) = W(L^p, \ell^p)$, same for weighted spaces;
- $\mathcal{H}_s(\mathbb{R}^d)$ (the Sobolev space) satisfies the so-called ℓ^2-puzzle condition (P. Tchamitchian): $\mathcal{H}_s(\mathbb{R}^d) = W(\mathcal{H}_s, \ell^2)$, and consequently for $s > d/2$ (Sobolev embedding) the pointwise multipliers (V. Mazya) equal $W(\mathcal{H}_s, \ell^\infty)$.

Minimality of Wiener's algebra

The Wiener amalgam spaces are essentially a generalization of the original family $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)$, with local component $\boldsymbol{L}^p$ and global q-summability of the sequence of local $\boldsymbol{L}^p$ norms.
In contrast to the "scale" of spaces $\left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right), 1 \leq p \leq \infty$ which do *not allow for any non-trivial inclusion relations* we have nice (and strict) inclusion relations for $p_1 \geq p_2$ and $q_1 \leq q_2$:

$$\boldsymbol{W}(\boldsymbol{L}^{p_1}, \ell^{q_1}) \subset \boldsymbol{W}(\boldsymbol{L}^{p_2}, \ell^{q_2}).$$

Hence $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)$ is the smallest among them, and $\boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty)$ is the largest among them. The closure of the space of test functions, or also of $\boldsymbol{C}_c(\mathbb{R}^d)$ in $\boldsymbol{W}(\boldsymbol{L}^\infty, \ell^1)$ is just *Wiener's algebra* $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$, which was one of Hans Reiter's list *Segal algebras*. It can also be characterized as the smallest of all *solid* Segal algebras.

Introducing Modulation Spaces

Having the possibility to define Wiener amalgam spaces with $\mathcal{F}\boldsymbol{L}^p(\mathbb{R}^d)$ (the Fourier image of $\boldsymbol{L}^p(\mathbb{R}^d)$ in the sense of distributions) as a local component allowed to introduce **modulation spaces** in analogy to *Besov spaces*, replacing more or less the dyadic decompositions on the Fourier transform side by uniform ones.
Formally one can define the (unweighted) modulation spaces as

$$\boldsymbol{M}^{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)\right). \tag{6}$$

or more generally the now classical modulation spaces

$$\boldsymbol{M}^s_{p,q}(\mathbb{R}^d) := \mathcal{F}^{-1}\left(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q_{v_s})\right). \tag{7}$$

Fourier invariant modulation spaces

It is an interesting variant of the classical Hausdorff-Young theorem to observe that one has

Theorem

- *For* $1 \leq r \leq p \leq \infty$ *one has*

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{F}^p, \ell^r)) \subseteq \boldsymbol{W}(\boldsymbol{F}^r, \ell^p);$$

- *and as a consequence for* $1 \leq p, q \leq 2$*:*

$$\mathcal{F}(\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)) \subseteq \boldsymbol{W}(\boldsymbol{L}^{q'}, \ell^{p'}).$$

General Wiener Amalgam Spaces II

The general Wiener Amalgam Spaces are described by $\boldsymbol{W}(\boldsymbol{B}, \boldsymbol{C})$ for general local $\boldsymbol{B}$ or global components $\boldsymbol{C}$.
An important subclass of this family are the function spaces which are both Wiener amalgams and modulation spaces, resp. Wiener amalgam spaces which are Fourier invariant (for $\mathbf{G} = \mathbb{R}^d$).

Here first $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)(\mathbb{R}^d)(\mathbb{R}^d)$ comes to mind, for $1 \leq p \leq \infty$, with the special case $\boldsymbol{L}^2(\mathbb{R}^d)$ for $p = 2$, and $\boldsymbol{S}_0(\mathbb{R}^d)$ for $p = 1$ resp. its dual $\boldsymbol{S}_0'(\mathbb{R}^d)$ corresponding to $p = \infty$.
More generally one can take weighted components both in the local component by considering e.g. $\mathcal{F}\boldsymbol{L}^p_w(\mathbb{R}^d)$ and the same in the global direction (see [15]).

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p_w, \ell^p_v), \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_w, \ell^1_v); \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1_v, \ell^1_w).$$

Wiener Amalgam Spaces & Modulation Spaces

The defintion of modulation spaces $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$ has been based on the idea that one decomposes the Fourier transform of a given $f \in \boldsymbol{L}^p(\mathbb{R}^d)$ and measures the ℓ^q-behavior of those local pieces, so essentially to define $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$ as the inverse Fourier transform of the Wiener amalgam $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)$, and the more general spaces $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ by "punishing" the contributions with a weight depending on the frequency, in the same way as one would do it for Sobolev spaces $\mathcal{H}_s(\mathbb{R}^d)$ resp. for the Besov spaces $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$.

Wiener Amalgam Spaces and Gabor Analysis

Both the Wiener Amalgam Spaces with local components of the form $\mathcal{F}\boldsymbol{L}^p_w$ and global component $\boldsymbol{L}^q_m$ and modulation spaces (which can be viewed as Fourier images of such a Wiener Amalgam) via Gabor Analysis which in turn allows to derive some of the mapping properties for these spaces.

The usual procedure is the verification of mapping properties for (Gabor) atoms, providing results for global (weighted) ℓ^1-spaces, then extend by duality and (complex) interpolation to the general results.

The classical case is the Fourier invariance of $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$. Here one starts with

$$\|f\|_{\boldsymbol{S}_0} = \sum_{n \in \mathbb{Z}^d} \|f\psi_n\|_{\mathcal{F}\boldsymbol{L}^1} < \infty.$$

Wiener Amalgam Spaces and Gabor Analysis II

This clearly implies that f, as an absolutely convergent series in $\left(\mathcal{F}\mathbf{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\mathbf{L}^1}\right)$ belongs to $\mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$, resp. $\widehat{f} \in \mathbf{L}^1(\mathbb{R}^d)$, but we would like to have $\widehat{f} \in \mathbf{S}_0(\mathbb{R}^d)$. Using the existence of some $h \in \mathcal{S}(\mathbb{R}^d) \subset \mathbf{S}_0(\mathbb{R}^d) \subset \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$ with $T_{x_n} h \cdot \psi_n = \psi_n$ (for x_n well chosen) we get, for some constant $C > 0$:

$$\|\widehat{f}\|_{\mathbf{S}_0} = \sum_{n \in \mathbb{Z}^d} \|\widehat{T_{x_n} h} * \widehat{f\psi_n}\|_1 \leq \sum_{n \in \mathbb{Z}^d} \|M_{-x_n} h\|_1 \|\widehat{f\psi_n}\|_{\mathbf{S}_0}$$

and consequently for all $f \in \mathbf{S}_0(\mathbb{R}^d)$:

$$\|\widehat{f}\|_{\mathbf{S}_0} \leq \sum_{n \in \mathbb{Z}^d} \|\widehat{h}\|_1 \|\widehat{f\psi_n}\|_1 = C\|h\|_{\mathcal{F}\mathbf{L}^1} \|f\|_{\mathbf{S}_0}.$$

Wiener Amalgam Spaces and Shubin Classes I

Among the Fourier invariant spaces those with $p = 2$ are of course the most natural ones, especially for equal weights in the time and in the frequency direction, of the form

$$w(x) = w_s(x) = \langle x \rangle^2 = (1 + |x|^2)^{s/2}.$$

We expect (correctly) that they have to satisfy a weighted $\boldsymbol{L}^2$-condition in the time direction and a similar one in the frequency direction, i.e. we have the intersection of some $\boldsymbol{L}^2_w(\mathbb{R}^d)$ with a Sobolev space $\mathcal{H}_s(\mathbb{R}^d)$.

Wiener Amalgam Spaces and Shubin Classes II

It is hence not surprising that the weight $v_s(x,y) = (1 + x^2 + y^2)^{s/2}$ allows to describe this intersection using the STFT (Short Time Fourier Transform), in other words, that we have $M^2_{v_s}(\mathbb{R}^d)$
The original definition describes Shubin spaces $\left(\boldsymbol{Q}_s(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{Q}_s}\right)$ in the context of the *harmonic oscillator*. They can also be characterized by weighted ℓ^2-conditions on the *Hermite coefficients*. Moreover they form a Fourier invariant (one by one) of Banach spaces, and in fact Banach algebras for $s > d$ (i.e. whenever they are embedded into the Banach algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$).
Similar results are valid for LCA groups!

Compactness in TMIBs

Let me mention that the setting of TMIBs also allows to extend the classical compactness criteria for $\boldsymbol{L}^p(\mathbb{R})$-spaces as described in [14].

Theorem

A bounded subset M of a Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is relatively compact if and only it is tight and equicontinuous, i.e. if and only if for $\varepsilon > 0$ there exist two functions, one in $g \in \boldsymbol{L}^1_\eta$ and another one in $h \in \boldsymbol{L}^1_\omega$ such that

1. $\|\hat{h} \cdot f - f\|_{\boldsymbol{B}} < \varepsilon \quad \forall f \in M;$
2. $\|g * f - f\|_{\boldsymbol{B}} < \varepsilon \quad \forall f \in M;$

Tauberian Consequences

By means of *Tauberian arguments for the Beurling algebras* in use (as described in [23]) in combination with an abstract version of the ideal theorem for Segal algebras (see e.g. [8]), or more directly) one can also show for any TMIB:

Theorem

For any TMIB (which by assumption contains $\boldsymbol{W}(\boldsymbol{L}^1_w, \ell^1_\eta)$ *as a dense subspace) for any Gauss-function the set finite linear combinations of translates forms a norm-dense subspace.*

Note/Problem: It might be of course a nice mathematical problem to find, for any given number $k \in \mathbb{N}$ the best approximation by such finite linear combinations in the given norm. Especially finding the optimal centers will be a challenge.

Double Module Structures

There is another paper (from 1983), jointly with W. Braun (which was written essentially in Heidelberg) on Banach spaces with double module structures, which found little attention so far ([1]); Essentially it answers a variety of questions (using the two module structures based on convolution and pointwise multiplication):

1. When is such a space a dual space?
2. Can one have different spaces with the "same norm" (like the sup-norm, on $\boldsymbol{C}_0(\mathbb{R}^d)$, $\boldsymbol{C}_{ub}(\mathbb{R}^d)$, $L^\infty(\mathbb{R}^d)$ etc.);
3. How can one determine the predual of a dual space?

For reflexive spaces we have exactly one space with this norm and the same is true for the dual space: "test functions" are dense in in both $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and $(\boldsymbol{B}', \|\cdot\|_{\boldsymbol{B}'})$.

Double Module Structures II

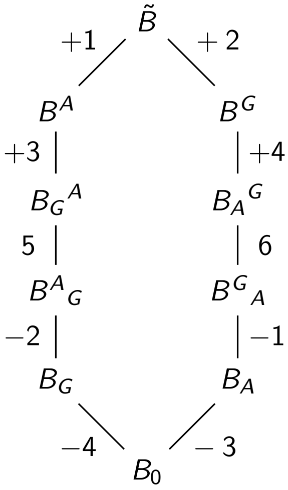


Figure: Figure caption

Double Module Structures: THE DIAGRAM

For $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ resp. $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$ we actually get 6 different spaces, all endowed with the sup-norm.
For Ł1RdN the diagram only has to examples, since it is the same as the the closure of the test functions. The upper/biggest space is simply the space of all bounded measures.
For many other concrete TMIBs we do not yet know how that diagram looks like.

Double Module Structures III

1. $$\lim_{\alpha} \|e_\alpha * f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{G}}$$

2. $$\lim_{\beta} \|h_\beta \cdot f - f\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad f \in \boldsymbol{B}_{\boldsymbol{A}}$$

3. $$\forall \beta \quad \lim_{\alpha} \|h_\beta \cdot (e_\alpha * f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}$$

4. $$\forall \beta \quad \lim_{\alpha} \|e_\alpha * (h_\beta \cdot f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad \boldsymbol{B}^{\boldsymbol{G}}_{\boldsymbol{A}}$$

5. $$\forall \alpha \quad \lim_{\alpha} \|e_\alpha * (h_\beta \cdot f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}_{\boldsymbol{G}}}^{\boldsymbol{A}}$$

6. $$\forall \alpha \quad \lim_{\alpha} \|e_\alpha * (h_\beta \cdot f - f)\|_{\boldsymbol{B}} = 0 \quad \Leftrightarrow \quad {\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}$$

Fourier Standard Spaces

There is work in progress concerning the rich structure of the so-called *FOURIER STANDARD SPACES*, which are by assumptions (to explain the Euclidean situation) Banach spaces of tempered distributions satisfying the following three assumptions:

1. $\mathcal{S}(\mathbb{R}^d) \subset \boldsymbol{B} \subset \mathcal{S}'(\mathbb{R}^d)$; (continuous embeddings);
2. $\|T_x f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \ \forall x \in \mathbb{R}^d, f \in \boldsymbol{B}$;
3. $\|M_s f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \ \forall s \in \mathbb{R}^d, f \in \boldsymbol{B}$;

and in fact the assumption (which follows automatically if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$):

$$\boldsymbol{L}^1(\mathbb{R}^d) * \boldsymbol{B} \subseteq \boldsymbol{B}, \quad \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{B} \subseteq \boldsymbol{B}.$$

A minimal TMIB

Using the notations in the papers by Dimovski, Pilipovic, Prangoski and Vindas we can show (cf. [11]):

Theorem

$\mathbf{W}(\mathcal{F}\mathbf{L}^1_{\nu_E}, \ell^1_{\omega_E})$ is the smalltes TMIB space within the family of all TMIBs which satisfy condition c) in the definition of TMIBs.

TMIB

A (B)-space E is said to be a translation-modulation invariant (B)-space of ultradistributions (in short: TMIB)) of class $*$ if it satisfies the following three conditions:

(a) The continuous and dense inclusions $\mathcal{S}^*(\mathbb{R}^d) \hookrightarrow E \hookrightarrow \mathcal{S}'^*(\mathbb{R}^d)$ hold.

(b) $T_x(E) \subseteq E$ and $M_\xi(E) \subseteq E$ for all $x, \xi \in \mathbb{R}^d$.

(c) There exist $\tau, C > 0$ (for every $\tau > 0$ there exists $C_\tau > 0$), such that

$$\omega_E(x) := \|T_x\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|x|)} \quad \text{and}$$
$$\nu_E(\xi) := \|M_{-\xi}\|_{\mathcal{L}_b(E)} \leq Ce^{M(\tau|\xi|)}. \quad (1)$$

There are clear connections to recent work of Dimovski, Pilipovic, Prangoski and Vindas (also presented at this conference), on TMIBs, translation and modulation invariant Banach spaces of ultra-distributions, see [5]. [4] and [6].

There is also the Schwartz-Bruhat space which is defined for general LCA ([2]), but is rather difficulat to handle (as can be seen in the work of H. Reiter on the metaplectic group, see e.g.[25]).

We think that the characterization given 1975 by Osborne ([22]) is quite interesting, as it makes only use of (polynomial type) decay conditions on both the "time" and the "frequency" side, avoiding the direct use of smoothness.

`https://www.univie.ac.at/nuhag-php/dateien/talks/3275_Belgrade17`

is my talk on **Fourier Standard space** given in Belgrad 2017.

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