Robustness Considerations based on $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

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OVERVIEW

We will concentrate on the setting of the LCA group $G = \mathbb{R}^d$, although all the results are valid in the setting of general **locally compact Abelian groups** as promoted by **A. Weil**.
Occasionally the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ is used and its dual $\mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions (e.g. for PDE and the *kernel theorem*, identifying operators from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ with their distributional kernels in $\mathcal{S}'(\mathbb{R}^{2d})$).
In the last 2-3 decades the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (equal to the modulation space $(\boldsymbol{M}^1(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^1})$) and its dual, $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$ or $\boldsymbol{M}^\infty(\mathbb{R}^d)$ have gained importance for many questions of Gabor analysis or time-frequency analysis.

OVERVIEW II

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions (requiring to work with the space $\boldsymbol{L}^1(\mathbb{R}^d) \cap \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$), and many other spaces.

Within the family there are two subfamilies, namely the *Wiener amalgam spaces* and the so-called *modulation spaces*, among them the Segal algebra $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ or Wiener's algebra $\left(\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{W}}\right)$.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t)\,.$$

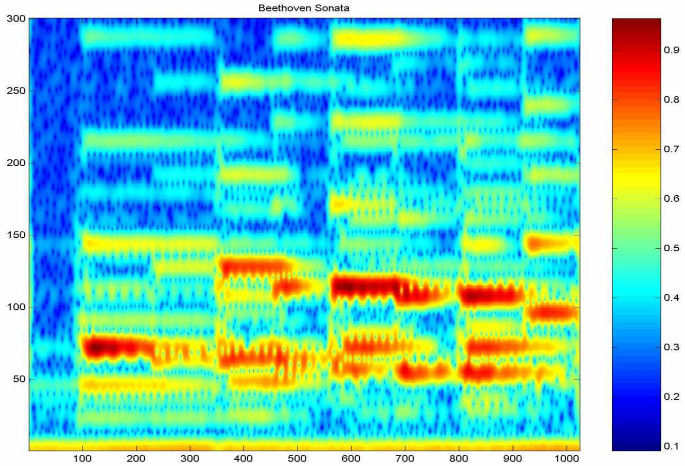
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x}\hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

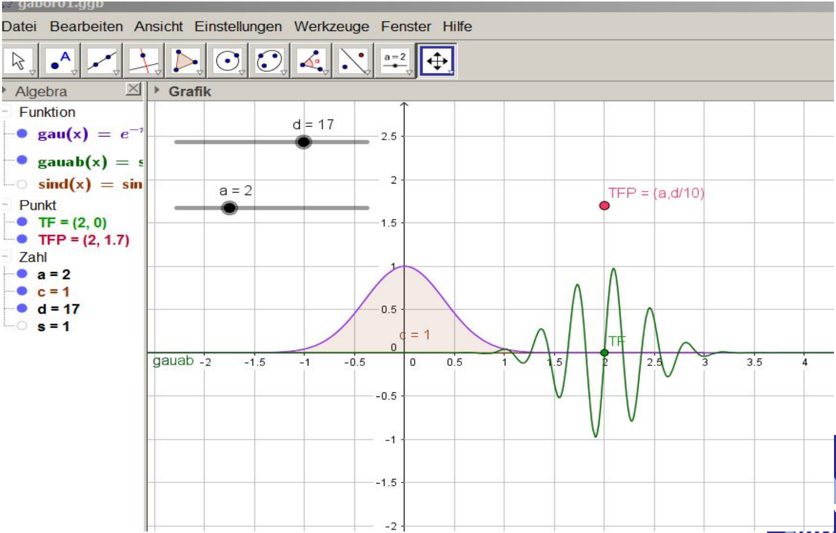
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g\rangle = \langle f, \pi(\lambda) g\rangle = \langle f, g_\lambda\rangle,\ \ \lambda = (t,\omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. Plancherel's Theorem gives

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2 = 1$ makes $f \mapsto V_g(f)$ isometric, hence we request this from now on. Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because $\forall h \in \mathcal{H}_1$

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \tag{1}$$

and thus by the *polarization principle* $T^*T = Id$.

In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \tag{2}$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \tag{3}$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g(f)(\lambda)\cdot\overline{V_g(h)(\lambda)}\, d\lambda. \quad (4)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f\in \boldsymbol{L}^2(\mathbb{R}^d). \quad (5)$$

This is quite analogous to the situation of the Fourier transform

$$f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, \quad f\in \boldsymbol{L}^2(\mathbb{R}^d), \quad (6)$$

with $\chi_s(t) = exp(2\pi i\langle s, t\rangle)$, $t, s\in\mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Discretizing the continuous reconstruction formula

Note the crucial difference between the classical formula (6) (Fourier inversion) and the new formula formula (5). The building blocks g_λ belong to the Hilbert space $\boldsymbol{L}^2(\mathbb{R}^d)$, in contrast to the characters $\chi_s \notin \boldsymbol{L}^2(\mathbb{R}^d)$. Hence finite partial sums cannot approximate the functions $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ in the Fourier case, but they can (and in fact do) approximate f in the $\boldsymbol{L}^2(\mathbb{R}^d)$-sense.
The continuous reconstruction formula suggests that sufficiently fine (and extended) Riemannian-sum-type expressions approximate f. This is a valid view-point, at least for nice windows g (any Schwartz function, or any classical summability kernel is OK: see [F. Weisz] Inversion of the short-time Fourier transform using Riemannian sums for example [6]).

Modulation spaces, in particular $S_0(\mathbb{R}^d)$ and $S_0'(\mathbb{R}^d)$

The reconstruction of f from its STFT (Short-time Fourier Transform) suggests that at least for "good windows" g one can control the smoothness (and/or decay) of a function or distribution by controlling the decay of $V_g(f)$ in the frequency resp. the time direction.

A polynomial weight depending on the frequency variable only can be used to describe Sobolev spaces, and (weighted) mixed-norm conditions can be used to define the (now classical) **modulation spaces** $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right)$.

We will put particular emphasis on the modulation spaces $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^{1,1} = \boldsymbol{M}^1$, characterized by the membership of $V_g(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$ and $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^{\infty,\infty} = \boldsymbol{M}^\infty$, with uniform convergence describing norm convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$, while pointwise convergence corresponds to the w^*-convergence in $\boldsymbol{S}_0'(\mathbb{R}^d)$.

Just as an alternative let us remind of the following situation concerning Gabor frames:

Theorem

Assume that (g, Λ) generators a Gabor frame with generator $g \in \boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$, with dual Gabor atom $\widetilde{g}$. Then $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$ belongs to $\boldsymbol{M}^p(\mathbb{R}^d)$ if and only if one of the following expressions (equivalent norms) are finite:

1. $|V_g(f)|_\Lambda\|_{\ell^p}$;
2. $\|V_{gd}(f)|_\Lambda\|_{\ell^p}$.

Alternatively, $f \in \boldsymbol{M}^p(\mathbb{R}^d)$ if and only if it has an atomic representation of the form $\sum_{\lambda \in \Lambda} c_\lambda \pi(\lambda) g$, with $\mathbf{c} = (c_\lambda)_{\lambda \in \Lambda} \in \ell^p(\Lambda)$.

Tensor products

Given two functions f^1 and f^2 on $\mathbb{R}^d$ respectively, we set $f^1 \otimes f^2$

$$f^1 \otimes f^2(x_1, x_2) = f^1(x_1) f^2(x_2), \;\; x_i \in \mathbb{R}^d, i = 1, 2.$$

For distributions this definition can be extended by taking w^*-limits or by duality, just like $\mu_1 \otimes \mu_2$ is defined, for two bounded measures $\mu_1, \mu_2 \in \boldsymbol{M}_b(\mathbb{R}^d)$.
It is important to know that we have $\sigma_1 \otimes \sigma_2 \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ for any pair of distributions $\sigma_1, \sigma_2 \in \boldsymbol{S}_0'(\mathbb{R}^d)$.
In particular $\boldsymbol{S}_0'(\mathbb{R}^d) \widehat{\otimes} \boldsymbol{S}_0'(\mathbb{R}^d)$ is well defined and a (proper) subspace of $\boldsymbol{S}_0'(\mathbb{R}^d)$.

The KERNEL THEOREM for *ScRd*

The *kernel theorem* for the Schwartz space can be read as follows:

Theorem

For every continuous linear mapping T from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$ there exists a unique tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ such that

$$T(f)(g) = \sigma(f \otimes g), \quad f, g \in \mathcal{S}(\mathbb{R}^d). \tag{7}$$

Conversely, any such $\sigma \in \mathcal{S}'(\mathbb{R}^{2d})$ induces a (unique) operator T such that (7) holds.

The proof of this theorem is based on the fact that $\mathcal{S}(\mathbb{R}^d)$ is a *nuclear Frechet space*, i.e. has the topology generated by a sequence of semi-norms, can be described by a metric which turns $\mathcal{S}(\mathbb{R}^d)$ into a complete metric space.

The KERNEL THEOREM for S_0 I

Tensor products are also most suitable in order to describe the set of all operators with certain mapping properties. The backbone of the corresponding theorems are the *kernel-theorem* which reads as follows (!! despite the fact that $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is NOT a *nuclear Frechet space*)
One of the corner stones for the kernel theorem is: One of the most important properties of $\boldsymbol{S}_0(\mathbb{R}^d)$ (leading to a characterization given by V. Losert from 1980) is the tensor-product factorization:

Lemma

$$\boldsymbol{S}_0(\mathbb{R}^k)\hat{\otimes}\boldsymbol{S}_0(\mathbb{R}^n) \cong \boldsymbol{S}_0(\mathbb{R}^{k+n}), \tag{8}$$

with equivalence of the corresponding norms.

The KERNEL THEOREM for S_0 II

The **Kernel Theorem** for general operators in $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$:

Theorem

If K is a bounded operator from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$, then there exists a unique kernel $k \in \boldsymbol{S}_0'(\mathbb{R}^{2d})$ such that $\langle Kf, g\rangle = \langle k, g \otimes f\rangle$ for $f, g \in \boldsymbol{S}_0(\mathbb{R}^d)$, where $g \otimes f(x,y) = g(x)f(y)$.

Formally sometimes one writes by "abuse of language"

$$Kf(x) = \int_{\mathbb{R}^d} k(x,y)f(y)dy$$

with the understanding that one can define the action of the functional $Kf \in \boldsymbol{S}_0'(\mathbb{R}^d)$ as

$$Kf(g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)f(y)dy\, g(x)dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} k(x,y)g(x)f(y)dxdy.$$

The KERNEL THEOREM for S_0 III

The kernel theorem as well as many other important properties and linear correspondences within Fourier and Time-frequency analysis can be *nicely described* by means of the **Banach Gelfand Triple** $(\mathbf{S}_0, \mathbf{L}^2, \mathbf{S}_0')(\mathbb{R}^d)$.

We will not make extensive use of this fact, although in the long run it is a very important and compact way of describing many of these correspondences (say integral kernel of the linear operator or spreading resp. Kohn-Nirenberg symbol of the linear operator).

For example, the kernel theorem as described above "outer shell" of the Gelfand triple isomorphism. The "middle = Hilbert" shell which corresponds to the well-known result that Hilbert Schmidt operators on $\mathbf{L}^2(\mathbb{R}^d)$ are just those compact operators which arise as integral operators with $\mathbf{L}^2(\mathbb{R}^{2d})$-kernels.

The KERNEL THEOREM for S_0 IV

Theorem

The classical kernel theorem for Hilbert Schmidt operators is unitary at the Hilbert spaces level, with $\langle T, S\rangle_{\mathcal{HS}} = trace(T * S')$ as scalar product on $\mathcal{HS}$ and the usual Hilbert space structure on $\boldsymbol{L}^2(\mathbb{R}^{2d})$ on the kernels.
Moreover, such an operator has a kernel in $\boldsymbol{S}_0(\mathbb{R}^{2d})$ if and only if the corresponding operator K maps $\boldsymbol{S}_0'(\mathbb{R}^d)$ into $\boldsymbol{S}_0(\mathbb{R}^d)$, but not only in a bounded way, but also continuously from w^*-topology into the norm topology of $\boldsymbol{S}_0(\mathbb{R}^d)$.

In analogy to the matrix case, where the entries of the matrix

$$a_{k,,j} = T(\mathbf{e}_j)_k = \langle T(\mathbf{e}_j), \mathbf{e}_k\rangle$$

we have for $K \in \boldsymbol{S}_0$ the continuous version of this principle:

$$K(x,y) = \delta_x(T(\delta_y), \quad x, y \in \mathbb{R}^d.$$

The Kernel Theorem as a BGT isomorphism

The different version of the kernel theorem for operators between $\boldsymbol{S}_0$ and $\boldsymbol{S}_0'$ can be summarized using the terminology of Banach Gelfand Triples (BGTR) as follows.

Theorem

There is a unique Banach Gelfand Triple isomorphism between the Banach Gelfand triple of kernels $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^{2d})$ *and the operator Gelfand triple around the Hilbert space* $\mathcal{HS}$ *of Hilbert Schmidt operators, namely* $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$, *where the first set is understood as the* w^* *to norm continuous operators from* $\boldsymbol{S}_0'(\mathbb{R}^d)$ *to* $\boldsymbol{S}_0(\mathbb{R}^d)$, *the so-called regularizing operators.*

Advantages over Schwartz Theory

- $\left(\mathbf{S}_0(G), \|\cdot\|_{\mathbf{S}_0}\right)$ is defined on LCA groups
- $\left(\mathbf{S}_0(G), \|\cdot\|_{\mathbf{S}_0}\right)$ is a Banach space, not just a *nuclear Frechet space* with a rich family of semi-norms;
- w^*-convergence in $\mathbf{S}_0'(\mathbb{R}^d)$ is useful and easy to explain (uniform convergence of $V_g(\sigma_n) \to V_g(\sigma_0)$);
- $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ plays a *universal* role for many specific questions in Fourier analysis (Gabor analysis, classical summability, etc.);
- there is a long list of equivalent characterizations;
- there are many sufficient conditions;
- sampling and periodization are unproblematic.d

Disadvantages over Schwartz Theory

- It is *not possible* to treat PDEs, because functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ need not be differentiable, e.g. the triangular function is compactly supported and has integrable Fourier transform, hence belongs to $\boldsymbol{S}_0(\mathbb{R}) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R})$.
- $\boldsymbol{S}_0'(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d)$, but sometimes the smallness is even an advantage;
- more?

Poisson's Formula

For general lattices (discrete, co-compact subgroups) Λ within any LCA groups G the following is true. Denoting by $\Lambda^\perp$ the *orthogonal lattice*, given by

$$\Lambda^\perp := \{\chi \in \widehat{G} \,|\, \chi(\lambda) \equiv 1 \ \forall \lambda \in \Lambda\}$$

[4]

Theorem

For $f \in \mathbf{S}_0(\mathbb{R}^d)$ one has

$$\sum_{k \in \mathbb{Z}^d} f(k) = \sum_{n \in \mathbb{Z}^d} \hat{f}(n), \tag{9}$$

the sum being absolutely convergent on both sides.

Reservoir of Classical Summability kernels

Recall that I like to denot the $\boldsymbol{L}^1$-normalized dilation operator by St_ρ, which applied to $\boldsymbol{L}^1(\mathbb{R}^d)$-functions via

$$[\mathrm{St}_\rho g](z) = 1/\rho^d f(x/\rho), \quad \rho > 0, x \in \mathbb{R}^d,$$

$$\text{satisfying} \quad \|\mathrm{St}_\rho(g)\|_{\boldsymbol{L}^1(\mathbb{R}^d)} = \|g\|_{\boldsymbol{L}^1(\mathbb{R}^d)}.$$

On the Fourier transform side it goes into value-preserving dilation:

$$[\mathrm{D}_\rho h](z) = h(\rho z), \quad \rho > 0, z \in \mathbb{R}^d.$$

Classical Summability II

Summability kernels allow to recover an $\boldsymbol{L}^1(\mathbb{R}^d)$-function (equiv. class of measurable functions) by applying the *inverse Fourier integral*

$$\int_{\mathbb{R}^d} h(s) e^{2\pi i s t} ds$$

to the Fourier transform $\widehat{f}$ (for some given $f \in \left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$), *multiplied with* $\mathrm{D}_\rho h$, for some $h \in \boldsymbol{S}_0(\mathbb{R}^d)$ with $h(0) = 1$ (resp. $h = \widehat{g}$, for some $g \in \boldsymbol{L}^1(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} g(x)dx = 1$).
Since the pointwise product $\mathrm{D}_\rho h \cdot \widehat{f}$ corresponds on the time-side to the convolution product $\mathrm{St}_\rho g * f$ we only have to verify that for any $f \in \boldsymbol{L}^1(\mathbb{R}^d)$ we have $\lim_{\rho \to 0} \mathrm{St}_\rho g * f = f$! Since $\mathrm{D}_\rho h \in \boldsymbol{S}_0(\mathbb{R}^d)$ is is clear that $\mathrm{D}_\rho h \cdot \widehat{f}$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$, hence the ordinary Fourier inversion theorem can be applied (for any fixed $\rho > 0$).

Classical Summability III

The ingredients for this argument are

- On the FT side: $\widehat{f} \cdot \mathrm{D}_\rho g \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d) \cdot \boldsymbol{S}_0(\mathbb{R}^d) \subset \boldsymbol{S}_0(\mathbb{R}^d)$;
- On the time-side: continuous shift, i.e. $\|T_x f - f\|_{\boldsymbol{L}^1(\mathbb{R}^d)} \to 0$ for $|z| \to 0$, because this implies

$$\|\mathrm{St}_\rho g * f - f\|_{\boldsymbol{B}} \to 0, \quad \text{for } \rho \to 0.$$

Thus the same argument is valid for any (!) Segal algebra $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ (in the sense of H. Reiter), because they all share these properties, and some of them still do not satisfy $\mathcal{F}\boldsymbol{B} \subset \boldsymbol{L}^1(\mathbb{R}^d)$.

How to COMPUTE approximate duals

The idea behind many approximation procedures is to have a *STRUCTURAL PRESERVING* approximation. In our case we want to reduce, up to some *approximation error* the computation of Gabor coefficients of a given function with respect to a given Gabor family $\mathcal{G}(g, a, b)$ to the (numerical exact or approximate) computation of appropriate sets of coefficients. Note that for the case of an irrational quotient a/b (eccentricity) no pair of integer lattice constants will have exactly that *same* eccentricity, so some approximations are needed.
We restrict our attention here to the *separable case*, being aware that also the separable case (e.g. hexagonal lattices) deserve equal attention nowadays!

Similar time-frequency lattices

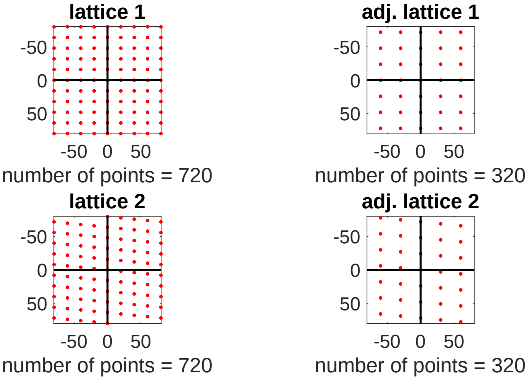
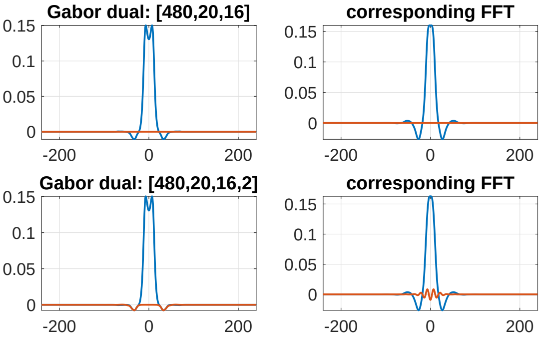


Figure: complatt2s.eps

... generate similar dual atoms



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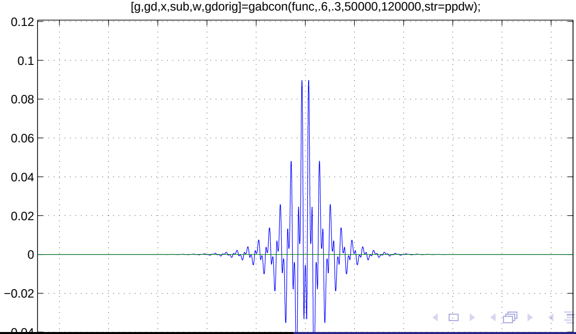
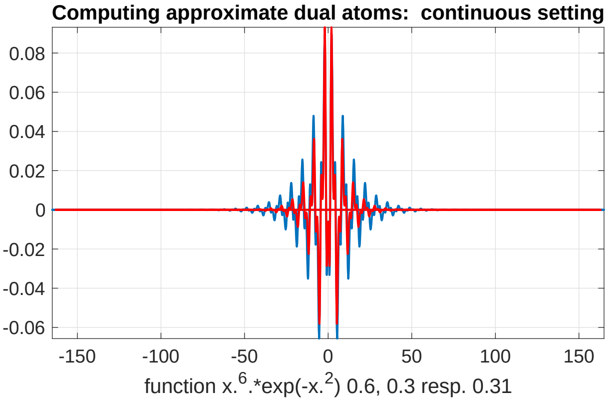
... generate similar dual atoms





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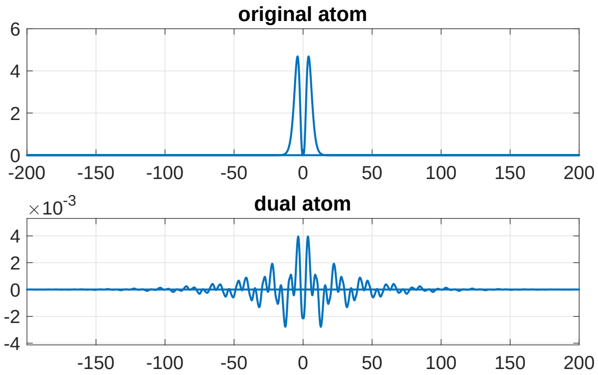

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Families of Gabor families

As a first step towards the question of "varying the lattice constants" (or more generally varying the lattice) one has to ask, whether the Bessel property, namely the estimate

$$\sum_{\lambda \in \Lambda} |V_g(f)(\lambda)|^2 \leq C\|f\|_2^2, \quad \forall f \in \mathcal{H} = \boldsymbol{L}^2(\mathbb{R}^d)$$

is valid for any given (decent) family of lattices Λ, say $\Lambda = a\mathbb{Z}^d \times b\mathbb{Z}^d$, for $a, b \in [\gamma, 1]$ for some $\gamma > 0$.
The answer is again: aside from more complicated but hardly much larger spaces that universal answer (even in the context of LCA groups) is: Assume that the window is in $\boldsymbol{S}_0(\mathbb{R}^d)$!
For details see [1].

Varying the TF-lattice Λ

The key-result of [1] describes the fact, that the set of all lattices Λ, such that $\mathcal{G}(g,\Lambda)$ gives rise to a *Gabor frame* is an open subset of the product domain, with atoms takein in $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and lattices described by suitable $2d \times 2d$, non-singular matrices, i.e. $\Lambda = \boldsymbol{A} * \mathbb{Z}^{2d}$, for $\det(\boldsymbol{A}) \neq 0$.
Moreover, the dual atom depends continuously, in the sense of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$, on the ingredients. In particular, a small change in the matrix results only in a small change of the dual window $\widetilde{g}$ (which depends on $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $\boldsymbol{A}$.

Jitter error stability

The result just mentioned is remarkable in the sense that it is not just a simple consequence of the fact that similar Gabor families create similar (with respect to the operator norm on $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}))$ Gabor frame operators.
Such an argument is only valid for a fixed TF-lattice Λ, whenever the atom g is replaced by a similiar (e.g. compactly supported one) in $\boldsymbol{S}_0(\mathbb{R}^d)$. In contrast, different lattices create operators, which have a large deviation from the original Gabor frame operator, when considered in the operator norm over $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$ or even just $(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2)$!
Small perturbations (*jitter error*) however are valid for the case of $\boldsymbol{S}_0(\mathbb{R}^d)$-atoms and are verified by the usual perturbation argument applied within the Banach algebra of invertible operators on $(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0})$.

Approximate reconstruction with good windows

Moyal's equality, which can be expressed as

$$\|V_g(f)\|_{\mathbf{L}^2(\mathbb{R}^d\times\widehat{\mathbb{R}}^d)} = \|g\|_{\mathbf{L}^2(\mathbb{R}^d)}\|f\|_{\mathbf{L}^2(\mathbb{R}^d)}, \quad f,g\in\mathbf{L}^2(\mathbb{R}^d),$$

hence $f\mapsto V_g(f)$ is an isometric linear embedding of $\big(\mathbf{L}^2(\mathbb{R}^d), \|\cdot\|_2\big)$ into $\big(\mathbf{L}^2(\mathbb{R}^{2d}), \|\cdot\|_2\big)$ as long as $\|g\|_2 = 1$. Therefore V_g^* is the inverse of V_g on its range, or in other words we have the *continuous reconstruction formula*

$$f = \int_{\mathbb{R}^d\times\widehat{\mathbb{R}}^d} V_g(f)(\lambda)\pi(\lambda)g\, d\lambda.$$

It is therefore natural to assume that it can be approximated (for any given $f\in\mathbf{L}^2(\mathbb{R}^d)$) by corresponding Riemannian sums! (see [6]).

The Fundamental Formula for Gabor Analysis

The so-called FIGA is discussed in great detail in the paper [3]
It relies on the application of the Poisson formula for the *symplectic Fourier transform*.
Even if one is interested in $\boldsymbol{L}^2$-windows it is important to make use of the fact, that for $g \in \boldsymbol{S}_0(\mathbb{R}^d)$ and $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ the STFT $V_g(f)$ belongs to the Wiener amalgam space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2)(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.
Hence the pointwise product of two such short-time Fourier transform (as it is needed in the evaluation of the scalar products in $\ell^2(\Lambda)$) involves functions in

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2) \cdot \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^2) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d) = \boldsymbol{S}_0(\mathbb{R}^d)$$

hence samples are in $\ell^1(\Lambda)$ and Poisson's formale applies!!

Approximate dual Gabor atoms

Let us shortly mention here, why it is important to find approximate dual atoms which are close to the true (canonical) dual atom $\widetilde{g}$ or at least close to some (valid) dual atom which guarantees perfect reconstruction **in the $S_0(\mathbb{R}^d)$-sense!**
This again has to do with the possibility to estimate the error on the Bessel bound of the *synthesis operator*. Assume again, we are only interested in Gabor analysis for signals in $L^2(\mathbb{R}^d)$ (a narrow-minded view-point anyway).
Then, assuming we have only an approximation to $\widetilde{g}$ in the sense of the $L^2(\mathbb{R}^d)$-norm, we could only argue that the reconstruction procedure, starting from the true samples of $V_g f$ over Λ are given, we would use the synthesis with respect to the replacement of $\widetilde{g}$, close to $\widetilde{g}$ in the L^2-sense. What comes out is that one would be able only to estimate the S_0'-error in the reconstruction.

Quasi-interpolation in $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$

An important result concerning discretization resp. approximtion is the result with Kaiblinger about quasi-interpolation in $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$. Let us discuss the most simple case, which is *piecewise linear interpolation* in $\left(S_0(\mathbb{R}), \|\cdot\|_{S_0}\right)$.
The typical first application of this principle is the approximate factorization of the *Fourier transform* (given on $S_0(\mathbb{R}^d)$ as *integral transform*) by the FFT, applied to samples of the function over a sufficiently wide range, at a sufficiently high sampling rate!
Results in this direction have been given in the paper with N. Kaiblinger (see [2]).

Robustness for Banach Gelfand Triples

It is no surprise that practically all the robustness considerations formulated so far concering $\boldsymbol{S}_0$-atoms, or approximatin in the $\boldsymbol{S}_0(\mathbb{R}^d)$-sense, provide not only stability and robustness (e.g. with respect to the choice of the lattice, etc.) in the operator norm on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$, but also for the space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ and its dual space $(\boldsymbol{S}_0'(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0'})$, which together form the **Banach Gelfand Triple** $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Gabor Frames without Inequalities

The triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')$ also allows to describe the usual properties of a set of vectors in a finite dimensional Hilbert spaces, at least concerning Gabor frames, see Gröchenig's paper: Gabor frames without inequalities, [5].
Frames are a strong for of "generating systems of vectors", coming with a control on the set of coefficients. This can be expressed equivalently at the level of $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ with ℓ^1-coefficients.
In the same way, the Riesz property (for the adjoint case) can be formulated as injectivity problem, and this should be considered for the pair $\ell^\infty(\Lambda)$ and $\boldsymbol{S}_0'(\mathbb{R}^d)$, according to [5].

Further information, LINKS

A lot of further material can be found through the NuHAG web-page, in particular at

www.nuhag.eu/talks

E.g. selecting one the following filters:

- BanGelTriples
- FeiTalks
- FeiConcept

or one of the (drafts of) lecture notes found at

http://www.univie.ac.at/nuhag-php/home/skripten.php

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