Applications of Localized Frames to Galerkin-like Representations of Operators

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The mathematical concept of *frames* [4, 3] is an active field of mathematical research and has found many applications e.g. in signal processing, quantum mechanics and acoustics. A sequence $(\psi_k)_{k\in K}$ in the Hilbert space $\mathcal{H}$ is a *frame* for $\mathcal{H}$, for the discrete index set K, if there exist positive constants A and B (called lower and upper frame bound, respectively) that satisfy

$$A\|f\|^2 \leq \sum_k |\langle f, \psi_k\rangle|^2 \leq B\|f\|^2 \quad \forall f \in \mathcal{H}. \tag{1}$$

So a frame allows and stable and stably invertible analysis of the function f. Frames can be seen as generalizations of bases: Contrary to bases, frames allow redundant representations. Finding and constructing frames given certain *a-priori* properties, is more facile than for bases. This is advantageous for many applications in audio signal processing, but also for numerical solution for operator equations, see e.g. [10].

The numerical treatment of operator equations, $Of = g$, requires a discrete formulation, $M\vec{f} = \vec{g}$. This is often done with a so-called Galerkin scheme [9]. In the Finite Element Method and the Boundary Element Method usually spline-like bases are used. More recently, wavelet bases and frames [10, 8] have been applied.

On an abstract level, it is well known that for orthonormal sequences operators can be uniquely described by a matrix representation. An analogous result holds for frames and their duals [1]. For a bounded, linear operator O, define the infinite matrix $\mathcal{M}_{(\Psi,\Phi)}(O) = \langle O\phi_l, \psi_k\rangle$, rewriting the operator equation by setting $\vec{f} = C_{\widetilde{\Psi}} f$ and $\vec{g} = C_\Phi g$. Here $M = \mathcal{M}(O)^{(\Psi,\Phi)}$ is called the *stiffness matrix*.

In the active field of frame theory, there have been formulated many more general approaches, like continuous frames, fusion frames, semi-frames and reproducing pairs. The great usefulness of frame theory for applied topics, in particular for signal processing, results from the applicability for signal analysis for particular structured frames, like Gabor systems (the time-frequency shifts of a window) [5] and wavelet systems (the dilated shifts of a mother wavelet). Somewhere between such structured frames and 'abstract frames' lies the concept of localized frames, which share important properties of more structured sequence.

A frame in an Hilbert space is called $\mathcal{A}$-localized, if its Gram matrix, i.e. the matrix with entries $G_{k,l} = \langle psi_l, \psi_k\rangle$, is in a given solid spectral matrix algebra $\mathcal{A}$. We denote $\Phi \sim_{\mathcal{A}} \Psi$ if the cross-Gramian is in this algebra.

Localized frames are "good" frames. More precisely, the concept of localized frames was introduced in [6] in an attempt to understand which properties render a frame useful. Whereas an abstract frame can be viewed as a flexible coordinate system for a Hilbert space — and only for one Hilbert space! — localized frames go beyond Hilbert spaces and yield a description and characterization of a whole class of associated Banach spaces $\mathcal{H}_w^p$. Roughly speaking, the associated Banach space $\mathcal{H}_w^p$ is the coorbit space of all elements f such that the sequence $\langle f, \psi_k \rangle_k$ belongs to the weighted ℓ^p-space. Indeed, the success of the structured frames mentioned above is built on their capacity to describe certain function spaces.

Another very nice property of localized frames is that the possess nice dual frames. Technically speaking, the canonical dual frame possesses the same localization. The time-frequency analysis of pseudodifferential operators by means of Gabor frames is a particularly successful example of the application of localized frames: certain symbol classes containing the Hörmander class $S_{0,0}^0$ can be completely characterized by the off-diagonal decay of the associated matrix [7], indicating that localized frames are also very useful for a Galerkin-type approach.

In this talk we are going to present results from [2]. We investigate the representation of operators using localized frames in a Galerkin-type scheme. We show how the boundedness and the invertibility of matrices and operators are linked and give some sufficient and necessary conditions for the boundedness of operators between the associated Banach spaces.

This includes results like the following: For two localized frames Ψ and Φ with $\Phi \sim_{\mathcal{A}} \Psi$ an operator is bounded from $\mathcal{H}^1_{w^{(1)}}$ into $\mathcal{H}^\infty_{w^{(2)}}$ if and only if its matrix $\mathcal{M}_{(\Psi,\Phi)}(O)$ is in $\ell^{\infty,\infty}_{1/w^{(2)} \otimes w^{(1)}}$, where the discrete mixed norm spaces $\ell_w^{p_1,p_2}$ are defined in a natural way.

Acknowledgment
The first author was in part supported by the START project FLAME Y551-N13 of the Austrian Science Fund (FWF) and the DACH project BIOTOP I-1018-N25 of Austrian Science Fund (FWF). The second author acknowledges the support of the FWF-project P 26273-N25.

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