NuHAG Numerical Harmonic Analysis Group

Modulation Spaces from the View-point of Coorbit Theory

Hans G. Feichtinger, Univ. Vienna & TUM
hans.feichtinger@univie.ac.at
www.nuhag.eu

Luigi Rodino 70th Birthday Conference

This talk will be mostly about motivations, connections and background information, and a bit of history of modulation spaces, from a rather personal view-point.

Given the fact that on the one hand Torino is one of the international hot-spots of modulation space theory, and on the other hand the fact that I realize occasionally that my original motivations cannot be read from the published papers (sometimes because the ideas have not been made explicit at that time, or because the results are widely spread in the literature), this talk appears as a good opportunity to me to explain such things.

Generalities and View-points

The talk will also shed some light on the strategies behind the various constructions (Wiener amalgams, modulation spaces, coorbit spaces, double module spaces, Banach Gelfand triples). Many results have not been published explicitly because they arise as special cases of results of a more general nature.
But I admit that one needs a guidance and detailed explanations to understand the situation. So, for example, duality and pointwise multiplier results on Wiener amalgam spaces (as introduced in [6]) have been only given in the framework of decomposition spaces ([9]).

The personal view on modulation spaces

The theory of modulation spaces has been developed in the early 1980, culminating in the well-known technical 1983 report on **Modulation spaces on locally compact Abelian groups**, and the first "public appearance" of modulation spaces at the conference in Kiew: *A new family of functional spaces on the Euclidean n-space,* in the same year.
They have been first designed as Wiener amalgam spaces on the Fourier transform side, using BUPUs, but soon the connection to the STFT and the Heisenberg group began to play a role.
Around 1986-1989 the appearance of wavelets suggested to look for a unified theory of wavelet analysis and time-frequency analysis, based on the common group-theoretical basis. The results have been published under the name of coorbit theory with K. Gröchenig in 1988/89.

Basic Facts about Wiener Amalgams

Wiener amalgam spaces, as the name says, had their origin in the work of Norbert Wiener, mostly in connection with his investigations around the Tauberian theorem (see [19]).

The so-called *Wiener algebra* $\boldsymbol{W}(\mathbb{R}^d)$, according to current systematic conventions $\boldsymbol{W}(\boldsymbol{C}_0, \ell^1)(\mathbb{R}^d)$, was given as an interesting example of a so-called *Segal algebra* in Hans Reiter's book [18], see [3].

At that time J. Fournier and J. Stewart (see [12]) gave a nice survey on the role of the spaces they called $\ell^q(\boldsymbol{L}^p)$, while Busby and Smith observed the convolution properties of the classical amalgam space ([1]).

Advantages of the family $\ell^q(L^p)$

One of the draw-backs of the classical Banach spaces $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p \leq \infty$ is the fact that there are no inclusion relations between any two of these spaces. However, the obstacles are if a different nature.

If $p_1 < p_2$ there are functions (locally like $x^{-\alpha}$, for a suitable value of $\alpha > 0$) which are locally in L^{p_1} but not in L^{p_2}.
In contrast, for $p_1 < p_2$ there are (step) functions in $L^{p_1} \setminus L^{p_2}$.
For Wiener amalgams the situation is quite easy:

$$W(L^{p_1}, \ell^{q_1}) \subset W(L^{p_2}, \ell^{q_2}) \quad \Leftrightarrow \quad p_2 \leq p_1 \text{ and } q_1 \leq q_1.$$

Hence $W(L^\infty, \ell^1)$ is the smallest space in this family (with $W(\mathbb{R}^d)$ as the closure of the test functions), while $W(L^1, \ell^\infty)$ is the largest, closed in the dual of $W(\mathbb{R}^d)$.

The Magic Square for Wiener Amalgams

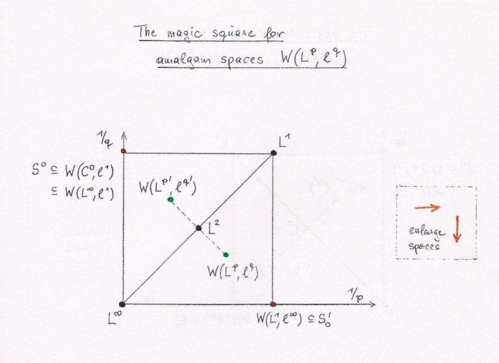


Figure: The inclusion relations: magic square

BUT overall classical Wiener Amalgams do not behave well under the Fourier transform!

The Hausdorff-Young Result for Amalgams

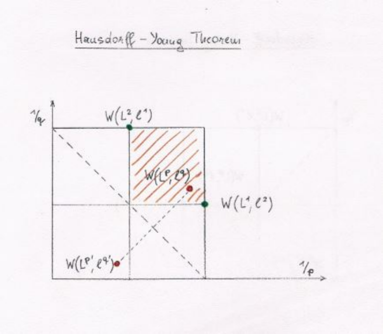


Figure: Hausdorff-Young theorem for Wiener amalgams

Besov space (J. Peetre, H. Triebel)

Working with function space over LCA groups I was looking for a construction of smoothness spaces and thought that one possibility is to use (replacing dyadic intervals by uniform ones) spaces such as $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)$ "on the Fourier transform side", and then by "pulling them back to the time-side. *THIS was the original idea for modulation spaces*. This was the original idea for the definition of $\boldsymbol{M}^{p,q}$ (the unweighted modulation spaces).

Especially the space $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$ (introduced as Segal algebra $(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0})$ in 1979, see [4]) appears as an interesting special case, among others because it is *invariant under the Fourier transform*, i.e. the group FT maps $\boldsymbol{S}_0(G)$ onto $\boldsymbol{S}_0(\widehat{\mathcal{G}})$.
Since I wanted to avoid the use of distribution theory (over LCA groups one has to use the Schwartz-Bruhat theory, which is quite involved), I choose a different way.

BUPUs, discrete versus continuous norms

An important step taken during the study of Wiener amalgams (see [6]) was the demonstration, that one obtains in full generality (for arbitrary global components) two types of characterizations:

1. "discrete" characterizations using BUPUs (bounded uniform partitions, e.g. in $(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1})$), or
2. continuous norms (using a continuous control function) using a "moving window" function g.

Of course, one has to show that different BUPUs (or localization functions) define the same spaces and *equivalent norms*.

The name MODULATION spaces

Using the continuous version of the Wiener amalgam norm one finds that over $\mathbb{R}^d$ the modulation space $\boldsymbol{M}^{p,q}(\mathbb{R}^d) = \mathcal{F}^{-1}(\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q))$ can be characterized as the subspace of $f \in \boldsymbol{\mathcal{S}}'(\mathbb{R}^d)$ with the following finite norm:

$$\left(\int_{\mathbb{R}^d} \| M_s g * f \|_p^q \right)^{1/q} < \infty. \tag{1}$$

Here g is the window function (typically $0 \neq g \in \boldsymbol{\mathcal{S}}(\mathbb{R}^d)$) and M_s is the *modulation operator*

$$[M_s g](x) = e^{2\pi i s \cdot x} f(x), \;\; s, x \in \mathbb{R}^d.$$

The name MODULATION spaces II

Recalling that the Riemann-Lebesgue Lemma shows that the Fourier transform of $\boldsymbol{L}^1(\mathbb{R}^d)$ functions tends to zero at infinity it is clear that one has essentially

$$M_s g * f(x) \to 0 \quad \text{for } s \to \infty.$$

So in this sense modulation spaces capture the smoothness by quantifying the decay of the expression $M_s g * f(x)$, resp. the convolution of the signal f with the modulated window g (as a function of x and $s \in \mathbb{R}^d$) by certain integrability conditions.

Note that in communication theory *amplitude modulation* was used to modulate a pure frequency $e^{2\pi isx}$ by the amplitude of the function g to be transmitted!

Compactness in Modulation Spaces

A number of results have been immediately available at the time of the introduction of modulation spaces, because they had been proved already in “full generality” before.
For example, all the modulation spaces are carrying two module structures: one with respect to $\boldsymbol{L}^1(\mathbb{R}^d)$-convolution, the other with respect to pointwise multiplication of $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$.
Hence, whenever $p, q < \infty$ (resp. whenever $\boldsymbol{S}_0(\mathbb{R}^d)$ is dense in $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$) one has the usual characterization of *compact sets*: A bounded and closed set $S \subset \boldsymbol{M}^{p,q}(\mathbb{R}^d)$ is compact if and only if it is *equicontinuous* and *tight*.

Double module structures

Of course modulation spaces are also special cases of Banach spaces with a double module structures, as studied in [8]. In particular, one can ask the question about the so-called *main diagram* for these spaces.

One of the key points is the following one (even valid for general modulation spaces): Any such space contains the smallest space with this $\boldsymbol{L}^1/\mathcal{F}\boldsymbol{L}^1$-double module structure, namely $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ and is contained in its dual space. The test functions are dense (i.e. the space is minimal) if and only if translation and modulation are a strongly continuous (!) group of isometries on these spaces. It is a dual space if and only if w^*-limits (in the sense of $\boldsymbol{S}_0'$) of bounded nets belong to the Banach space itself. Finally, the space if *reflexive* if and only if both the space and its dual are minimal and maximal.

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t - x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

Behavior under Fourier transform

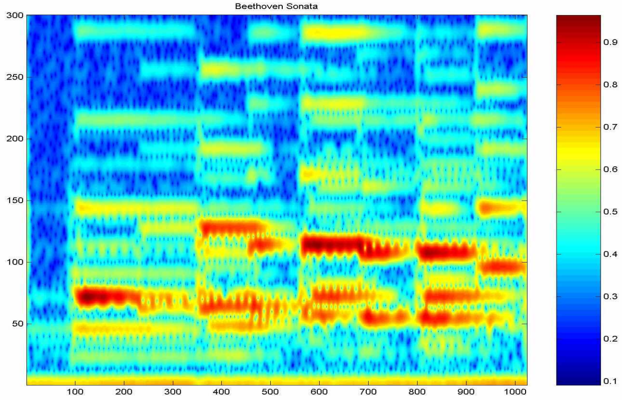
$$(T_x f)\widehat{\ } = M_{-x} \hat{f} \qquad (M_\omega f)\widehat{\ } = T_\omega \hat{f}$$

The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT

A typical waterfall melody (Beethoven piano sonata) depicted using the spectrogram, displaying the energy distribution in the TF = time-frequency plan:



A Banach Space of Test Functions (Fei 1979)

A function in $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is in the subspace $\boldsymbol{S}_0(\mathbb{R}^d)$ if for some non-zero g (called the "window") in the Schwartz space $\mathcal{S}(\mathbb{R}^d)$

$$\|f\|_{S_0} := \|V_g f\|_{\boldsymbol{L}^1} = \iint_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} |V_g f(x,\omega)| dx d\omega < \infty.$$

The space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is a Banach space, for any fixed, non-zero $g \in \boldsymbol{S}_0(\mathbb{R}^d)$), and different windows g define the same space and equivalent norms. Since $\boldsymbol{S}_0(\mathbb{R}^d)$ contains the Schwartz space $\mathcal{S}(\mathbb{R}^d)$, any Schwartz function is suitable, but also compactly supported functions having an integrable Fourier transform (such as a trapezoidal or triangular function) are suitable. It is convenient to use the Gaussian as a window.

Basic properties of $M^1 = S_0(\mathbb{R}^d)$

Lemma

Let $f \in \boldsymbol{S}_0(\mathbb{R}^d)$, then the following holds:

(1) $\pi(u,\eta)f \in \boldsymbol{S}_0(\mathbb{R}^d)$ *for* $(u,\eta) \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d$, *and* $\|\pi(u,\eta)f\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

(2) $\hat{f} \in \boldsymbol{S}_0(\mathbb{R}^d)$, *and* $\|\hat{f}\|_{\boldsymbol{S}_0} = \|f\|_{\boldsymbol{S}_0}$.

In fact, $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ is the smallest non-trivial Banach space with this property, and therefore contained in any of the $\boldsymbol{L}^p$-spaces (and their Fourier images).

Various Function Spaces

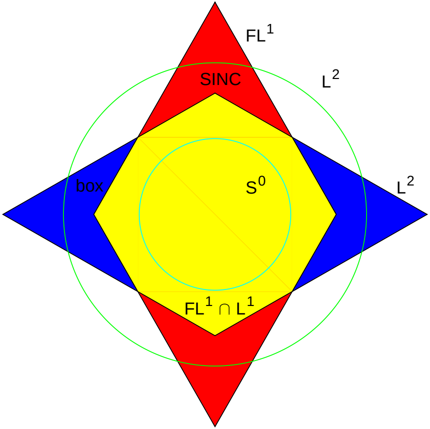


Figure: The usual Lebesgues space, the Fourier algebra, and the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d)$ inside all these spaces

BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $(\mathbf{B}, \|\cdot\|_{\mathbf{B}})$, which is densely embedded into some Hilbert space $\mathcal{H}$, which in turn is contained in $\mathbf{B}'$ is called a Banach Gelfand triple.

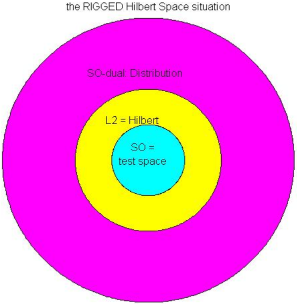
Definition

If $(\mathbf{B}_1, \mathcal{H}_1, \mathbf{B}_1')$ and $(\mathbf{B}_2, \mathcal{H}_2, \mathbf{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\mathbf{B}_1$ and $\mathbf{B}_2$.
2. A is [unitary] isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to a weak* isomorphism as well as a norm-to-norm continuous isomorphism between $\mathbf{B}_1'$ and $\mathbf{B}_2'$.

A schematic description: the simplified setting

In our picture this simple means that the inner “kernel” is mapped into the ”kernel”, the Hilbert space to the Hilbert space, and at the outer level two types of continuity are valid (norm and w^*)!



The Fourier transform as BGT automorphism

The Fourier transform $\mathcal{F}$ on $\mathbb{R}^d$ has the following properties:

1. $\mathcal{F}$ is an isomorphism from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\widehat{\mathbb{R}}^d)$,
2. $\mathcal{F}$ is a unitary map between $\boldsymbol{L}^2(\mathbb{R}^d)$ and $\boldsymbol{L}^2(\widehat{\mathbb{R}}^d)$,
3. $\mathcal{F}$ is a weak* (and norm-to-norm) continuous bijection from $\boldsymbol{S}_0'(\mathbb{R}^d)$ onto $\boldsymbol{S}_0'(\widehat{\mathbb{R}}^d)$.

Furthermore, we have that Parseval's formula

$$\langle f, g \rangle = \langle \widehat{f}, \widehat{g} \rangle \qquad (2)$$

is valid for $(f, g) \in \boldsymbol{S}_0(\mathbb{R}^d) \times \boldsymbol{S}_0'(\mathbb{R}^d)$, and therefore on each level of the Gelfand triple $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d)$.

Inclusion relations

The family of modulation spaces $\boldsymbol{M}^{p,q}(\mathbb{R}^d)$ show a very similar behaviour compared to ordinary Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p,\ell^q)(\mathbb{R}^d)$. Different parameters define different spaces, and inclusion mappings are always automatically continuous.
Furthermore any automorphism (e.g. rotation or scaling operators) leave these spaces invariant, not always isometrical, of course, as a *simple consequence of the fact that different windows define the same space* (up to equivalence of norms).
Some inclusions go in the opposite direction, because the Fourier algebra $\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ is contained in $\boldsymbol{L}^2(\mathbb{R}^d)$ which in turn is contained in $\boldsymbol{L}^1(\mathbb{R}^d)$ (locally!), hence within $\mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$. Thus:

$$\boldsymbol{M}^{p_1,q_1} \subset \boldsymbol{M}^{p_2,q_2} \Leftrightarrow p_1 \le p_2, q_1 \le q_2.$$

An inclusion diagram

The fact that there are clear inclusions in both families (Wiener amalgams resp. modulation spaces), but also a smallest and a largest space in each of these two families, with the inclusions (we have $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) = \boldsymbol{M}^{1,1} = \boldsymbol{M}^1$ and $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty) = \boldsymbol{M}^{\infty,\infty}$):

$$\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1) \subset \boldsymbol{W}(\boldsymbol{C}_0, \ell^1) \subset \boldsymbol{L}^2 \subset \boldsymbol{W}(\boldsymbol{L}^1, \ell^\infty) \subset \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^\infty, \ell^\infty). \tag{3}$$

Hence for a typical space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ one can ask *what is set of all parameters* (p, q) such that

$$\boldsymbol{M}^{p,q} \subseteq \boldsymbol{B} \qquad \textit{or} \qquad \boldsymbol{B} \subseteq \boldsymbol{M}^{p,q}$$

respectively

$$\boldsymbol{W}(\boldsymbol{L}^p, \ell^q) \subseteq \boldsymbol{B} \qquad \textit{or} \quad \boldsymbol{B} \subseteq \boldsymbol{W}(\boldsymbol{L}^p, \ell^q).$$

Key aspects of my talk

1. What is the setting of coorbit theory?
2. In which sense are modulation spaces coorbit spaces?
3. Which results on coorbit theory had been influenced by modulation space theory?
4. Which results about modulation spaces are implicit consequences of coorbit theory?

The setting of **Coorbit Theory**

Coorbit theory has been developed by myself together with Karlheinz Gröchenig as a reaction to the first publications on wavelet theory (autumn 1986) by Yves Meyer, see[17, 16], and earlier A. Grossmann and J. Morlet (see [14], [15]).
The summer school with E. Stein and R. Howe in Germany organized by D. Poguntke clarified to us (we both took part) that the STFT (the function that had been used to provide the continuous description of modulation spaces) had a lot to do with the Schrödinger representation of the reduced Heisenberg group, while the CWT (continuous wavelet transform) was just a *representation coefficient* of the affine group, the so-called "$ax + b$"-group.

Modulation Spaces as Coorbit Spaces

As already indicated modulation spaces, e.g. the by now *classical modulation spaces* $\boldsymbol{M}^s_{p,q}(\mathbb{R}^d)$ can be viewed as coorbit spaces, by relating the usual definition of the short-time Fourier transform a function on the reduced Heisenberg group (see [10] for details of this transition).
There are two possible view-points: Group representation theory suggest to talk about the so-called Schrödinger representation of the reduced Heisenberg group, $\mathbb{H}^d = \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{T}$, OR (taking a more practical approach): The collection of all unitary operators which are scalar multiples (scalars from the torus) of time-frequency shifts.

Coorbit Results of Modulation Spaces

There is a number of results following from the generalities of coorbit theory, which have not been formulated before only for the time-frequency context. We give only a short summary:

Theorem

Irregular Sampling of the STFT: *Given* $0 \neq g \in \boldsymbol{S}_0(\mathbb{R}^d)$ *there exists* $\delta > 0$ *such that for any* δ*-dense family* $(x_i)_{i \in I}$ *there is a stable linear reconstruction of any* $f \in \boldsymbol{L}^2(\mathbb{R}^d)$ *from the samples of* $(V_g f(x_i))_{i \in I}$ *in the form*

$$f = \sum_{i \in I} V_g f(x_i) \widetilde{g}_i.$$

The convergence is unconditional in $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ *for any* $f \in \boldsymbol{L}^2(\mathbb{R}^d)$*, and absolute in* $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$ *for* $f \in \boldsymbol{S}_0(\mathbb{R}^d)$*, and at least* w^**-convergent for* $f \in \boldsymbol{S}_0'(\mathbb{R}^d)$*.*

Modulation Spaces inspiring Coorbit Theory

In the development of coorbit theory essentially the *unification aspect* for three situations had been dominant (with further generalizations imminent):

1. the wavelet case;
2. the Gabor (time-frequency) case;
3. Möbius invariant function spaces on the disc

We will concentrate on the comparison of the first two cases.

The Foundations of Coorbit Theory

Coorbit Theory is based on the following assumptions:

1. There is an *irreducible unitary representation* π of some locally compact group $\mathcal{G}$ on some Hilbert space $\mathcal{H}$;
2. For so-called *admissible elements* φ (in the domain of a densely defined possibly unbounded operator $\boldsymbol{A}$) one can define the continuous *voice transform* on $\mathcal{H}$:

$$V_\varphi(f)(x) = \langle f, \pi(x)\varphi\rangle, \quad f \in \mathcal{H}.$$

3. Given a *solid, translation invariant Banach space* of $(\boldsymbol{Y}, \|\cdot\|_{\boldsymbol{Y}})$ on $\mathcal{G}$ one defines
4. $\boldsymbol{\mathcal{C}o}(\boldsymbol{Y}) : \{f \mid V_\varphi(f) \in \boldsymbol{Y}\}$, with $\|f\|_{\boldsymbol{\mathcal{C}o}(\boldsymbol{Y})} := \|V_\varphi(f)\|_{\boldsymbol{Y}}$.

The Foundations of Coorbit Theory II

An important asset for the derivation of the basic properties of coorbit spaces are the following two consequences of the square integrability of the representation.

- For suitably normalized (admissible) atoms/windows one has an isometric embedding of $\mathcal{H}$ into $\left(\boldsymbol{L}^2(G), \|\cdot\|_2\right)$, i.e.

$$\|V_\varphi(f)\|_2 = \|f\|_{\mathcal{H}}, \quad f \in \mathcal{H}.$$

- The range of V_φ within $\boldsymbol{L}^2(G)$ is characterized by:

$$V_\varphi(f) * V_\varphi(\varphi) = V_\varphi(f)$$

  where "$*$" denotes convolution of functions on $\mathcal{G}$;
- The inverse of V_φ on the range of is just V_φ^*, resp. one has the reproducing formula

$$f = \int_G V_\varphi(f)(x)\,\pi(x)\varphi\,dx, \quad f \in \mathcal{H},$$

  which is understood (first!) in the weak sense.

Modulation Spaces inspiring Coorbit Theory

What are Modulation Spaces?

During the preparation of the article [7] the question arose: *What are modulation spaces?*
The answer that came our finally: Modulation spaces are coorbit spaces arising from the Schrödinger representation of the reduced Heisenberg group, resp. these are Banach spaces of distributions characterized by the behaviour of the STFT (cf. [11]).
Thus it is not so much the particular use of (weighted) mixed-norm spaces, or the particular order in which these norms are taken.
In this sense the *generalized Wiener amalgam spaces* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ are just other (general) modulation spaces.
One can define modulation spaces also with other function spaces, such as weighted Lorentz or Orlicz spaces, even the coordinate system is chosen differently. Then we would describe images of $\boldsymbol{M}^s_{p,q}$-spaces under the *Fractional Fourier transform*.

Key aspects of my talk

1. B

[13], [5], [6]
CORDERO papers related to modulation spaces:
optimal dilations: [2]

R. C. Busby and H. A. Smith.
Product-convolution operators and mixed-norm spaces.
Trans. Amer. Math. Soc., 263:309–341, 1981.

E. Cordero and F. Nicola.
Sharpness of some properties of Wiener amalgam and modulation spaces.
Bull. Austral. Math. Soc., 80(1):105–116, 2009.

H. G. Feichtinger.
A characterization of Wiener's algebra on locally compact groups.
Arch. Math. (Basel), 29:136–140, 1977.

H. G. Feichtinger.
On a new Segal algebra.
Monatsh. Math., 92:269–289, 1981.

H. G. Feichtinger.
A new family of functional spaces on the Euclidean n-space.
In *Proc.Conf. on Theory of Approximation of Functions*, Teor. Priblizh., 1983.

H. G. Feichtinger.
Banach convolution algebras of Wiener type.
In *Proc. Conf. on Functions, Series, Operators, Budapest 1980*, volume 35 of *Colloq. Math. Soc. Janos Bolyai*, pages 509–524. North-Holland, Amsterdam, Eds. B. Sz.-Nagy and J. Szabados. edition, 1983.

H. G. Feichtinger.
Modulation Spaces: Looking Back and Ahead.
Sampl. Theory Signal Image Process., 5(2):109–140, 2006.

H. G. Feichtinger and W. Braun.
Banach spaces of distributions with double module structure and twisted convolution.
In *Proc. Alfred Haar Memorial Conf.*, Coll. J. Bolyai Soc. North Holland Publ. Comp., 1985.

H. G. Feichtinger and P. Gröbner.
Banach spaces of distributions defined by decomposition methods. I.
Math. Nachr., 123:97–120, 1985.

H. G. Feichtinger and K. Gröchenig.
Gabor wavelets and the Heisenberg group: Gabor expansions and short time Fourier transform from the group theoretical point of view.
In C. K. Chui, editor, *Wavelets :a tutorial in theory and applications*, volume 2 of *Wavelet Anal. Appl.*, pages 359–397. Academic Press, Boston, 1992.

H. G. Feichtinger and K. Gröchenig.
Irregular sampling theorems and series expansions of band-limited functions.
J. Math. Anal. Appl., 167(2):530–556, 1992.

J. J. F. Fournier and J. Stewart.
Amalgams of L^p and ℓ^q.
Bull. Amer. Math. Soc. (N.S.), 13:1–21, 1985.

K. Gröchenig.
Foundations of Time-Frequency Analysis.
Appl. Numer. Harmon. Anal. Birkhäuser, Boston, MA, 2001.

A. Grossmann and J. Morlet.
Decomposition of Hardy functions into square integrable wavelets of constant shape.
SIAM J. Math. Anal., 15:723–736, 1984.

A. Grossmann, J. Morlet, and T. Paul.
Transforms associated to square integrable group representations. II: Examples.
Ann. Inst. H. Poincar'e, 45(3):293–309, 1986.

P. G. Lemarié and Y. Meyer.
Ondelettes et bases hilbertiennes. (Wavelets and Hilbert bases).
Rev. Mat. Iberoam., 2:1–18, 1986.

Y. Meyer.
De la recherche petrolière la géometrie des espaces de Banach en passant par les paraproduite. (From petroleum research to Banach space geometry by way of paraproducts).
In *Séminaire sur les équations aux dérivées partielles, 1985 - 1986, Exp. No. I*, page 11. Ecole Polytechnique, Palaiseau, 1986.

H. Reiter.
Classical Harmonic Analysis and Locally Compact Groups.
Clarendon Press, Oxford, 1968.

N. Wiener.
The Fourier Integral and certain of its Applications.
Cambridge University Press, Cambridge, 1933.